The Transmission of Stress across a Plane of Discontinuity in an Isotropic Elastic Solid, and the Potential Solutions for a Plane Boundary. By J. H. MICHELL, M.A. Received May 30th, 1899. Read June 8th, 1899.

1. The volume equations of equilibrium and the surface conditions are expressed in a form which leads directly to the potential solutions* of Boussinesq and Cerruti. The method is applied to cases where different sets of conditions hold on different parts of the plane boundary. The method of images is shown to be applicable to the determination of the stress due to volume forces in the semi-infinite solid. Finally, by the use of the method of images, a potential solution is given for the transmission of stress across a plane of discontinuity (1) when two, elastically different, isotropic solids are soldered together over the plane, (2) when there is free slipping over the plane.

2. Let a semi-infinite solid, supposed isotropic, lie on the positive side of the plane $z = 0$. The volume equations to be solved are

$$(\lambda + \mu) \theta_z + \mu \nabla^2 u = -X, \quad \&c.$$

The sets of boundary conditions to which the methods apply are those treated by Boussinesq, viz.:

$$(1) \quad R, S, T;$$

$$(2) \quad R, u, v;$$

$$(3) \quad w, S, T;$$

$$\text{or} \quad (4) \quad w, u, v;$$

given over the plane $z = 0$.

We proceed to transform these equations and conditions.

If we write in $\mu \nabla^2 w + (\lambda + \mu) \theta_z = -Z$

$$w = -\frac{\lambda + \mu}{2\mu} z\theta + w',$$

* Todhunter and Pearson, *History*, II., § 1489 *et seq.*; Love, *Elasticity*, I., chap. ix.

we get $2\mu(\lambda+2\mu)\nabla^2 w' = 2\mu(\lambda+\mu)z\Lambda - 2(\lambda+2\mu)Z$;
and, in addition, we have the well known equations

$$(\lambda+2\mu)\nabla^2\theta = -\Lambda,$$

$$2\mu\nabla^2\varpi = X_y - Y_x,$$

where

$$\Lambda = X_x + Y_y + Z_z,$$

$$2\varpi = v_x - u_y.$$

Thus the Laplacians of θ , w' , ϖ are known throughout.

On $z = 0$, we have $R = \lambda\theta + 2\mu w_z$

$$= -\mu\theta + 2\mu w'_z,$$

$$R_z = \lambda\theta_z + 2\mu w_{zz}$$

$$= -(\lambda+2\mu)\theta_z + 2\mu w'_{zz}$$

$$= -T_x - S_y - Z,$$

$$2\mu\varpi_z = S_x - T_y,$$

$$\theta - w_z = \frac{\lambda+3\mu}{2\mu}\theta - w'_z$$

$$= u_x + v_y.$$

The sets of conditions (1)-(4) therefore give as follows:—

(1) R , S , T given involve

$$\left. \begin{aligned} \text{(i.)} \quad & -\mu\theta + 2\mu w'_z = R \\ & -(\lambda+2\mu)\theta_z + 2\mu w'_{zz} = -T_x - S_y - Z \\ & 2\mu\varpi_z = S_x - T_y \end{aligned} \right\}$$

given.

(2) R , u , v involve

$$\left. \begin{aligned} \text{(ii.)} \quad & -\mu\theta + 2\mu w'_z = R \\ & (\lambda+3\mu)\theta - 2\mu w'_z = 2\mu(u_x + v_y) \\ & 2\varpi = v_x - u_y \end{aligned} \right\}$$

given.

(3) w , S , T involve

$$\left. \begin{aligned} \text{(iii.)} \quad & w' = w \\ & -(\lambda+2\mu)\theta_z + 2\mu w'_{zz} = -T_x - S_y - Z \\ & 2\mu\varpi_z = S_x - T_y \end{aligned} \right\}$$

given.

And, finally,

(4) w, u, v involve

$$(iv.) \quad \left. \begin{aligned} w' &= w \\ (\lambda + 3\mu) \theta - 2\mu w'_z &= 2\mu (u_x + v_y) \\ 2\varpi &= v_x - u_y \end{aligned} \right\}$$

given.

Now the linear functions of θ, w' , and ϖ in these conditions (i.)-(iv.) are all functions whose Laplacians are known throughout the solid, and whose values can therefore be written down by means of the known solutions of the potential problem for a plane boundary.

For example, suppose there are no volume forces and consider conditions (i.). They give at once

$$\begin{aligned} -\mu\theta + 2\mu w'_z &= -\frac{1}{2\pi} \frac{d}{dz} \iint \frac{R}{r} dx' dy', \\ -(\lambda + 2\mu) \theta + 2\mu w'_z &= \frac{1}{2\pi} \iint (T_x + S_y) \frac{1}{r} dx' dy', \\ 2\mu\varpi &= -\frac{1}{2\pi} \iint (S_x - T_y) \frac{1}{r} dx' dy', \end{aligned}$$

where (x', y') is now a point on $z = 0$, and (x, y, z) is a point in the solid. Thus

$$\begin{aligned} -(\lambda + \mu) \theta &= \frac{1}{2\pi} \frac{d}{dz} \iint \frac{R}{r} dx' dy' + \frac{1}{2\pi} \iint (T_x + S_y) \frac{1}{r} dx' dy', \\ 2\mu(\lambda + \mu) w'_z &= -\frac{\lambda + 2\mu}{2\pi} \frac{d}{dz} \iint \frac{R}{r} dx' dy' - \frac{\mu}{2\pi} \iint (T_x + S_y) \frac{1}{r} dx' dy'. \end{aligned}$$

Integrating the second integral by parts, we arrive at the form of solution given by Boussinesq for this case. The values of u, v are found from those of

$$u_x + v_y = \theta - w_z \quad \text{and} \quad v_x - u_y = \varpi$$

(which are now known) by a well known process, quite independently of the boundary conditions. It need only be noted here that the logarithmic potentials of Boussinesq are introduced in order to bring the known values of $u_{xx} + u_{yy}$ and $v_{xx} + v_{yy}$ into the forms $U_{xx} + U_{yy}$, $V_{xx} + V_{yy}$, which are derived from terms containing the second differential coefficient with respect to z by the use of the equation

$$U_{zz} = -U_{xx} - U_{yy},$$

satisfied by a harmonic function. The values of u, v can then be written down by inspection.

The solutions for the other sets of conditions are obtained in exactly the same way and appear in the forms given by Boussinesq.

3. Suppose now that one set of conditions holds over part of the boundary, and another set over the rest, the volume forces being still omitted.

Taking the combinations in turn and using obvious symbolism,

$$(a) \quad \left. \begin{array}{l} w, u, v \\ w, S, T \end{array} \right\}$$

$$\text{give} \quad (a) \quad \left. \begin{array}{l} w', \quad (\lambda + 3\mu) \theta - 2\mu w'_z, \quad \varpi \\ w', \quad -(\lambda + 2\mu) \theta_z + 2\mu w'_{zz}, \quad \varpi_z \end{array} \right\}.$$

Here w' is known all over the boundary, and its value can therefore be written down for all points by the previous method.

$$\text{Further,} \quad w'_{zz} = -w'_{xx} - w'_{yy}$$

is known over the boundary, and therefore the set of conditions is reducible to

$$(a') \quad \left. \begin{array}{l} (\lambda + 3\mu) \theta - 2\mu w'_z, \quad \varpi \\ (\lambda + 3\mu) \theta_z - 2\mu w'_{zz}, \quad \varpi_z \end{array} \right\}.$$

The problem is therefore reduced to finding a harmonic function V , such that V has a given value over part of the plane $z = 0$ and V_z a given value over the rest. This is the same thing as finding the electric charge, on a disc coinciding with the first part of the plane, such that its potential together with the potential of the known distribution $-V_z/2\pi$ over the rest of the plane may have a given value over the disc. The solution is known for simple cases, such as circular or elliptic discs. It is not proposed to enter into details here. In this way $(\lambda + 3\mu) \theta - 2\mu w'_z$ and ϖ are found. Having thus found w', θ, ϖ , the solution is completed as in § 2.

$$\text{Now consider} \quad (b) \quad \left. \begin{array}{l} R, u, v \\ R, S, T \end{array} \right\},$$

$$\text{which give} \quad (\beta) \quad \left. \begin{array}{l} -\mu \theta + 2\mu w'_z, \quad (\lambda + 3\mu) \theta - 2\mu w'_z, \quad \varpi \\ -\mu \theta + 2\mu w'_z, \quad -(\lambda + 2\mu) \theta_z + 2\mu w'_{zz}, \quad \varpi_z \end{array} \right\}.$$

Here $-\mu\theta + 2\mu w'_z$ is determined throughout at once, and the conditions are reduced to

$$(\beta') \quad \left. \begin{aligned} &-(\lambda + 2\mu) \theta + 2\mu w'_z, \quad \varpi \\ &-(\lambda + 2\mu) \theta_z + 2\mu w'_{zz}, \quad \varpi_z \end{aligned} \right\},$$

the value of $-(\lambda + 2\mu) \theta + 2\mu w'_z$ being derived from

$$-\mu\theta + 2\mu w'_z \quad \text{and} \quad (\lambda + 3\mu) \theta - 2\mu w'_z.$$

The conditions (β') are treated in the same way as the conditions (α') :

$$(c) \quad \left. \begin{aligned} &w, \quad S, \quad T \\ &R, \quad S, \quad T \end{aligned} \right\}$$

$$\text{give} \quad (\gamma) \quad \left. \begin{aligned} &w', \quad -(\lambda + 2\mu) \theta_z + 2\mu w'_{zz}, \quad \varpi_z \\ &-\mu\theta + 2\mu w'_z, \quad -(\lambda + 2\mu) \theta_z + 2\mu w'_{zz}, \quad \varpi_z \end{aligned} \right\}.$$

Hence $-(\lambda + 2\mu) \theta + 2\mu w'_z$ and ϖ are determined at once, and the conditions reduce to

$$\left. \begin{aligned} &-\mu\theta_z + 2\mu w'_{zz} \\ &-\mu\theta + 2\mu w'_z \end{aligned} \right\},$$

remembering that w'_{zz} is known when w' is known. Boussinesq* has treated the particular case

$$\left. \begin{aligned} &w, \quad 0, \quad 0 \\ &0, \quad 0, \quad 0 \end{aligned} \right\}$$

for which $-(\lambda + 2\mu) \theta + 2\mu w'_z = 0, \quad \varpi = 0,$

so that the conditions reduce to

$$\left. \begin{aligned} &w' = w \\ &w'_z = 0 \end{aligned} \right\}$$

and

$$\begin{aligned} R &= -\mu\theta + 2\mu w'_z \\ &= \frac{2\mu(\lambda + \mu)}{\lambda + 2\mu} w'_z \end{aligned}$$

over

$$z = 0,$$

which gives at once Boussinesq's solution.

* Todhunter and Pearson, II., § 1510.

$$\begin{array}{l} \text{The alternative case} \quad \left. \begin{array}{l} 0, 0, 0 \\ R, 0, 0 \end{array} \right\} \end{array}$$

$$\text{reduces to} \quad \left. \begin{array}{l} (\gamma) \quad w' = 0 \\ w'_z = \frac{\lambda + 2\mu}{2\mu(\lambda + \mu)} R \end{array} \right\}.$$

It is clear that Boussinesq's case, in conjunction with Case (1) of § 2, can be made to cover the whole of this section.

$$\begin{array}{l} (d) \quad \left. \begin{array}{l} w, u, v \\ R, u, v \end{array} \right\} \end{array}$$

$$\text{give} \quad \begin{array}{l} (\delta) \quad \left. \begin{array}{l} w', \quad (\lambda + 3\mu) \theta - 2\mu w'_z, \varpi \\ -\mu \theta + 2\mu w'_z, \quad (\lambda + 3\mu) \theta - 2\mu w'_z, \varpi \end{array} \right\}, \end{array}$$

whence $(\lambda + 3\mu) \theta - 2\mu w'_z$ and ϖ are written down and the conditions reduce to

$$(\delta') \quad \left. \begin{array}{l} w' \\ w'_z \end{array} \right\}.$$

Boussinesq's solution can be made to cover this case also by use of Case (2), § 2.

$$\begin{array}{l} (e) \quad \left. \begin{array}{l} w, u, v \\ R, S, T \end{array} \right\} \end{array}$$

$$\text{give} \quad \begin{array}{l} (\epsilon) \quad \left. \begin{array}{l} w', \quad (\lambda + 3\mu) \theta - 2\mu w'_z, \varpi \\ -\mu \theta + 2\mu w'_z, -(\lambda + 2\mu) \theta_z + 2\mu w'_{zz}, \varpi_z \end{array} \right\}. \end{array}$$

This, perhaps the most interesting combination, unfortunately does not appear to admit of such a simple solution as the others.

4. Suppose now there are volume forces on the semi-infinite solid, the conditions at the boundary being one of the sets (1)–(4). By using the solutions of § 2 the given stresses or displacements over the boundary are reduced to zero, and this we suppose done in what follows.

Take, as an example, conditions (4), which are reduced to

$$u = v = w = 0,$$

and therefore give, by (iv.),

$$\left. \begin{aligned} w' &= 0 \\ (\lambda + 3\mu) \theta - 2\mu w'_z &= 0 \\ \varpi &= 0 \end{aligned} \right\}$$

Now, $2\mu (\lambda + 2\mu) \nabla^2 w' = -(\lambda + \mu) z \Lambda - 2(\lambda + 2\mu) Z,$

$(\lambda + 2\mu) \nabla^2 \{(\lambda + 3\mu) \theta - 2\mu w'_z\} = (\lambda + \mu) z \Lambda_z - 2\mu \Lambda + 2(\lambda + 2\mu) Z_z,$

$2\mu \nabla^2 \varpi = X_y - Y_x.$

Hence $w', (\lambda + 3\mu) \theta - 2\mu w'_z,$ and ϖ are the potentials of the distributions corresponding to the given values of their Laplacians in the semi-infinite solid together with the potentials of the negative images of those distributions in the plane $z = 0.$

Thus the values of $w, \theta,$ and ϖ are written down at once, and the solution is then completed, as before, by finding $u, v.$

This process is clearly applicable to each of the sets of conditions. Positive images will appear if the normal rate of change is given to vanish over $z = 0.$

5. Consider now the transmission of stress across a plane of discontinuity in an infinite solid. Let $z = 0$ be the plane, and let the constants referring to the region z positive be unaccented, those for the region z negative accented.

First, let there be complete union or a perfect solder over $z = 0,$ so that the interfacial conditions are—

R, S, T, u, v, w continuous across the plane.

These conditions, by the preceding work, lead to the continuity of

$$\left. \begin{aligned} -\mu \theta + 2\mu w'_z \\ -(\lambda + 2\mu) \theta_z + 2\mu w'_{zz} + Z \\ \mu \varpi_z \\ w' \\ \frac{\lambda + 3\mu}{2\mu} \theta - w'_z \\ \varpi \end{aligned} \right\},$$

where, of course, $w' = w + \frac{\lambda' + \mu'}{2\mu'} z \theta,$

in the region z negative.

Now, $\mu \nabla^2 w' = -Z,$

over $z = 0.$

Hence $w'_{zz} + Z/\mu = -(w'_{xz} + w'_{yy})$

is continuous, and therefore, if p, q are any numbers,

$$\left. \begin{aligned} & \left(-\mu + p \frac{\lambda + 3\mu}{2\mu} \right) \theta + (2\mu - p) w'_z \\ & - (\lambda + 2\mu) \theta_z + (2\mu + q) w'_{zz} + (1 + q/\mu) Z \end{aligned} \right\}$$

are continuous.

We proceed to choose p, q so that the first two terms of the latter expression may be a multiple of the z -rate of change of the former, for both z positive and z negative.

We assume accordingly

$$\frac{2\mu - p}{\mu - p(\lambda + 3\mu)/2\mu} = \frac{2\mu + q}{\lambda + 2\mu} = -\gamma \text{ (say)}$$

$$\text{and } \frac{2\mu' - p}{\mu' - p(\lambda' + 3\mu')/2\mu'} = \frac{2\mu' + q}{\lambda' + 2\mu'} = -\gamma' \text{ (say).}$$

Then, writing $\lambda + 3\mu = 2\mu\kappa,$

$$\lambda' + 3\mu' = 2\mu'\kappa',$$

$$\text{we have } \frac{2\mu' - p}{\mu' - p\kappa'} (2\kappa' - 1) \mu' - 2\mu' = q = \frac{2\mu - p}{\mu - p\kappa} (2\kappa - 1) \mu - 2\mu,$$

and this gives

$$\begin{aligned} & \gamma^2 (\kappa' \mu - \kappa \mu') - 4p \{ \kappa \kappa' (\mu'^2 - \mu^2) - \kappa \mu'^2 + \kappa' \mu^2 \} \\ & + 4 \{ (\kappa' - 1) \mu'^2 - (\kappa - 1) \mu \} \mu \mu' = 0. \end{aligned}$$

If $\kappa = \kappa'$, i.e., if Poisson's ratio is the same for the two parts, the equation takes the simpler form

$$\kappa p^2 + 4p\kappa(\kappa - 1)(\mu + \mu') - 4(\kappa - 1)\mu\mu' = 0.$$

Since $\kappa > 3/2$ for stability of the material, this equation has two real roots; but this is immaterial.

Let p_1, p_2 be the two roots, q_1, q_2 the corresponding values of q , γ_1, γ_2 those of γ .

Then we have

$$\left. \begin{aligned} (\kappa p_1 - \mu)(\theta + \gamma_1 w'_z) \\ (\lambda + 2\mu)(\theta_z + \gamma_1 w'_{zz}) - (1 + q_1/\mu) Z \end{aligned} \right\}$$

continuous, as well as the corresponding expressions for the second root.

$$\text{Put} \quad (\kappa p_1 - \mu)(\theta + \gamma_1 w'_z) = V$$

$$\text{and} \quad (\lambda + 2\mu) / (\kappa p_1 - \mu) = K$$

$$\text{then} \quad \left. \begin{aligned} V \\ K \frac{dV}{dz} - (1 + q_1/\mu) Z \end{aligned} \right\}$$

are continuous.

Now, from the volume-equations, we have

$$\nabla^2 V = -4\pi\rho,$$

where ρ is a known function of the given volume forces.

The method of images may now be applied to find V ; the problem is the same as that of the electric flow across a plane of discontinuity in a conducting solid, treated, *e.g.*, by Maxwell, *Electricity and Magnetism*, I., § 315.

The terms $-(1 + q_1/\mu) Z$, &c., of course, correspond to an interfacial layer for which the potential can be written down.

Hence, corresponding to the two values of p , we can write down the values of

$$\theta + \gamma_1 w'_z,$$

$$\theta + \gamma_2 w'_z,$$

throughout the solid.

These give θ and w'_z , and therefore also w by a simple integration with respect to z . There is no trouble in finding ϖ , for ϖ and $\mu\varpi_z$ are continuous across the interface, and the Laplacian of ϖ is known throughout; so that ϖ is found by the same method as V .

The solution is then completed by finding u, v as before.

Given discontinuities of force or displacement over the interface can plainly be treated by the same process.

6. Another interesting case is that in which there is free slipping at the interface, so that R , w are continuous, and

$$S = T = 0$$

over $z = 0$.

We then have $-(\lambda + 2\mu) \theta_z + 2\mu w'_{zz} + Z = 0$

over the whole plane, and this determines

$$-(\lambda + 2\mu) \theta + 2\mu w'_z$$

throughout. Also $-\mu \theta + 2\mu w'_z$

and $w'_{zz} + Z/\mu$

are continuous, and therefore also

$$\frac{\lambda + 2\mu}{\mu(\lambda + \mu)} (-\mu \theta_z + 2\mu w'_{zz}) + \frac{2\lambda + 3\mu}{\mu(\lambda + \mu)} Z.$$

Thus, writing $-\mu \theta + 2\mu w'_z = V$,

we have to find V from $\nabla^2 V$ given throughout, and

$$\left. \begin{aligned} &V \\ &\frac{\lambda + 2\mu}{\mu(\lambda + \mu)} \frac{dV}{dz} + \frac{2\lambda + 3\mu}{\mu(\lambda + \mu)} Z \end{aligned} \right\}$$

continuous across the interface.

This is the problem of § 5.

Further, the value of ϖ is written down from the condition that $\varpi_z = 0$ over the interface. Having determined θ , w and ϖ in this manner, the solution is completed as before.

Other cases, in which the continuity extends over only part of the plane, may be treated in a similar way.

In conclusion, it may be remarked that Betti's method,* derived from his reciprocal theorem, can, with obvious modifications, be applied to the case of a surface of discontinuity of any form in an infinite solid.

* Love, *Elasticity*, I., § 141.