Third Memoir on Reciprocants. By L. J. ROGERS, B.A.

[Read November 11th, 1886.]

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I.

During the publication of my memoir on Homographic and Circular Reciprocants, I lighted upon a very simple geometrical interpretation of the latter. It was seen (Vol. xvii., p. 231) that the integration of circular reciprocant equations led to complete primitives, which remained unaltered after making certain transformations in the variables. These two transformations were such that any expression of the form $A(x^2 + y^2) + Bx + Cy$ assumed the form $\frac{C_1}{C_2}$, where $C_1 = 0$, $C_2 = 0$ were equations to circles in the new variables. It must be observed that after the first transformation the numerator of the fraction substituted for y was linear in form, but after the second or correlative transformation the numerator becomes circular.

Hence both x and y , after both transformations, yield expressions of the form $\frac{C_1}{C_0}$, $\frac{C_2}{C_0}$, respectively; $C_1 = 0$, $C_2 = 0$ being equations to circles, and $C_0 = 0$ being a circle of null radius.

This transformation contains 10 constants, and since, in the imaginary form of the transformation, there are only 6, we must evidently have four equations connecting these constants.

Let the transformation be conducted as follows:—

Replace x, y by X, Y , where

$$\left. \begin{aligned} X &= \kappa \frac{(x-\alpha)^2 + (y-\beta)^2 - \rho^2}{(x-\mu)^2 + (y-\nu)^2} \\ Y &= \lambda \frac{(x-\alpha')^2 + (y-\beta')^2 - \rho'^2}{(x-\mu)^2 + (y-\nu)^2} \end{aligned} \right\} \dots\dots\dots(1).$$

Now $X + Yi$, expressed in terms of x and y , was seen to be linear both in numerator and denominator. There must be, therefore, a

common factor in numerator and denominator, since at present, by (1), both are of the second degree. The same would be true if we change x into $x + \mu$, and y into $y + \nu$, *i.e.*,

$$\kappa(x + \mu - \alpha)^2 + \kappa(y + \nu - \beta)^2 - \kappa\rho^2 + \lambda i(x + \mu - \alpha')^2 + \lambda i(y + \nu - \beta')^2 - \lambda i\rho'^2 \dots\dots\dots(2)$$

is divisible by $y + xi$, leaving the factor $x + yi$ in the denominator.

This being true, the absolute term in (2) must vanish. That is,

$$\kappa(\mu - \alpha)^2 + \kappa(\nu - \beta)^2 - \kappa\rho^2 = -\lambda i(\mu - \alpha')^2 - \lambda i(\nu - \beta')^2 - \lambda i\rho'^2 \dots(3).$$

Treating $Y + Xi$ the same way, we get the same equation as (2), with every term on the right positive instead of negative.

We get then

$$\left. \begin{aligned} (\mu - \alpha)^2 + (\nu - \beta)^2 &= \rho^2 \\ (\mu - \alpha')^2 + (\nu - \beta')^2 &= \rho'^2 \end{aligned} \right\} \dots\dots\dots(4).$$

Moreover, from (2), we also get

$$\kappa(\mu - \alpha) + \lambda(\mu - \alpha') = \kappa(\nu - \beta) - \lambda(\nu - \beta');$$

and from a similar equation, got by treating $Y + Xi$, we get

$$\left. \begin{aligned} \kappa(\mu - \alpha) &= -\lambda(\nu - \beta') \\ \lambda(\mu - \alpha') &= \kappa(\nu - \beta) \end{aligned} \right\} \dots\dots\dots(5),$$

making up, in (4) and (5), the requisite number of relations among the constants.

From (5) we deduce

$$(\mu - \alpha)(\mu - \alpha') + (\nu - \beta)(\nu - \beta') = 0 \dots\dots\dots(6).$$

From (4) and (5) together we see that

$$\kappa\rho = \lambda\rho' \dots\dots\dots(7).$$

By means of these relations we may deduce an expression for $X^2 + Y^2$, which, involving mere algebraical work, it is scarcely worth while to trace out step by step. We have

$$X^2 + Y^2 = (\kappa^2 + \lambda^2) \frac{\left(x + \mu - 2\frac{\alpha\rho^2 + \alpha'\rho'^2}{\rho^2 + \rho'^2}\right)^2 + \left(y + \nu - 2\frac{\beta\rho'^2 + \beta'\rho^2}{\rho^2 + \rho'^2}\right)^2}{(x - \mu)^2 + (y - \nu)^2} \dots(8),$$

so that $X^2 + Y^2$ comes into the form $\frac{C_4}{C}$, where $C_4 = 0$ is a circle of

null radius. This can be easily verified from the original imaginary transformation.

We now proceed to interpret these results geometrically.

Let (μ, ν) be the Cartesian coordinates of a point I , A the centre of the circle $(x-\alpha)^2 + (y-\beta)^2 = \rho^2$, B the centre of $(x-\alpha')^2 + (y-\beta')^2 = \rho'^2$.

Then, by (4), we see that I is one of the intersections of the two circles. Let J be the other.

By (6), we see that these circles cut orthogonally.

Let (x, y) be the coordinates of any point P , and (X, Y) those of a point Q dependent on P .

It is required to show geometrically how Q may be evolved from P .

For simplicity, let us take IA, IB as coordinate axes. Then we get

$$\left. \begin{aligned} X &= \kappa \frac{x^2 + y^2 - 2\rho x}{x^2 + y^2} = \kappa - \frac{2\rho\kappa x}{r^2} \\ Y &= \lambda \frac{x^2 + y^2 - 2\rho' y}{x^2 + y^2} = \lambda - \frac{2\rho'\lambda y}{r^2} \end{aligned} \right\} \dots\dots\dots(9),$$

whence $(X-\kappa)^2 + (Y-\lambda)^2 = \frac{4\rho^2\kappa^2}{r^2}$, because $\rho\kappa = \rho'\lambda$, by (7).

Hence (X, Y) or Q lies on a circle whose centre is (κ, λ) , i.e., whose centre lies on the line

$$Y = \frac{\rho}{\rho'} x,$$

that is, on IJ .

The actual position is arbitrary, dependent on the value chosen for κ .

Let this centre be C .

Again, we see that the radius of the circle on which Q lies is inversely proportional to r , that is, IP .

Further, we see that $\frac{Y-\lambda}{X-\kappa} = \frac{y}{x}$, so that CQ is parallel to IP .

Clearing the figure, therefore, entirely of all unnecessary parts, we reduce it to the following, and obtain the succeeding method of generating Q .

Let P be any point on a given curve.

Let I, C be any points. From C draw CQ inversely proportional

and parallel to IP ; Q will then have been deduced from P by my so-called circular transformation.

It is, in fact, the generalized form of inversion, or rather inversion followed by rectangular transformation. For O may just as well coincide with I at first, and afterwards be transferred to any required point together with the curve generated by Q .

If, then, the curve P lead to a circular reciprocant differential equation, the curve thus generated by Q will be a curve of the same family as P .

Circular reciprocant curves are therefore those which are their own inverses with respect to any point whatever.

II.

We can now proceed to integrate the differential equations formed by equating to zero the two simplest circular reciprocants after $(1+t^2)b-3a^2t$, or ϕ_2 .

These are $4\phi_2\phi_4-5\phi_3^2-4\phi_1^2\phi_3^2=0$ (1),

and $4\phi_6\phi_3^2-18\phi_4\phi_3\phi_2+4\phi_3\phi_2^2\phi_1^2-8\phi_2^4\phi_1+15\phi_3^3=0$ (2)

(see Vol. xvii., p. 230).

The simplest method will be to integrate first the corresponding homographic reciprocant equations, and then transform according to the method given on page 226, Vol. xvii.

The first equation has for its correlative

$$4M_2M_4-5M_3^2+M_1^2M_2^2=0$$
(3).

Let us find the differential equation of the homographically generalized form of the equation $y=x^n$.

Here

$$y=x^n,$$

$$t=nx^{n-1},$$

$$M_1=n(n-1)x^{n-2},$$

$$M_2=-\frac{1}{2}n^2(n-1)(n+1)x^{2n-4},$$

$$M_3=\frac{1}{3}n^3(n-1)(n+1)^2x^{3n-6},$$

$$M_4=-\frac{3}{4}n^4(n-1)(n+1)^3x^{4n-8};$$

whence $4M_2M_4-5M_3^2+M_1^2M_2^2=\frac{1}{2}n^4(n^2-1)^2(n^2+1)x^{6n-12}$.

We easily see, then, that (3) is satisfied if $n=0, \pm 1$, or $\pm\sqrt{-1}$. We discard the real values, since they make M_1, M_2 , &c. all vanish

identically, and see that

$$y = x^{\sqrt{-1}}$$

is a solution of (3).

We can easily reduce this to a rational form, for the equation may be written $\log y = i \log x$, or even $\log y = i \log \frac{1-xi}{1+xi}$, since the substitution for x is homographic.

But it is easy to show, by Trigonometry, that

$$i \log \frac{1-xi}{1+xi} = 2 \tan^{-1} x.$$

Hence $\log y = 2 \tan^{-1} x$ is a rational solution of (3), and, by generalizing homographically, we get

$$\kappa + \log \lambda' \frac{y+\mu'}{y+\nu'} = 2 \tan^{-1} \lambda \frac{x+\mu}{x+\nu}.$$

This equation contains seven arbitrary constants, but these may be reduced to the requisite number five by changing κ into $\kappa - \log \lambda' + 2 \tan^{-1} \frac{1-\lambda}{1+\lambda}$, when the equation takes the form

$$\kappa + \log \frac{y+M'}{y+N'} = 2 \tan^{-1} \frac{x+M}{x+N} \dots \dots \dots (4).$$

The general homographic differential equation to $y = x^n$ is

$$(4M_2M_4 - 5M_3^2 + M_1^2M_2^2)(n^2 - 1) + 4M_2^3(n^2 + 1) = 0,$$

which is deduced by eliminating x by means of the values found for $4M_2M_4 - 5M_3^2 + M_1^2M_2^2$ and M_2^3 in the preceding section.

Eliminating n , we get the third homographic protomorph

$$4M_2M_3^2 - 18M_4M_2M_3 + 15M_2^3 - M_2^2M_3^2M_4 + 2M_1M_2^4$$

equated to zero as the differential equation of

$$\frac{y+M'}{y+N'} = k \left(\frac{x+M}{x+N} \right)^n \dots \dots \dots (6).$$

III.

We can now deduce the solutions of equations (1) and (2) in §II.

Equation (1) gives

$$y + xi = (x + yi)^i \text{ as a primitive,}$$

or

$$i(x^2 + y^2) = (x + yi)^{i+1},$$

$$i^i(x^2 + y^2)^i = (y + xi)^{i+1};$$

therefore

$$i^{i-1}(x^2 + y^2)^{i-1} = \left(\frac{y + xi}{x + yi}\right)^{i+1}$$

Taking logarithms and using polar coordinates, we get

$$\begin{aligned} (i-1) \log i + 2(i-1) \log r &= (i+1) \log \frac{y + xi}{x + yi} \\ &= (i+1) \log i \frac{x - yi}{x + yi} \\ &= (i+1) \log i - 2i(i+1) \tan^{-1} \frac{y}{x}; \end{aligned}$$

therefore

$$(i-1) \log r = \log i - (i-1) \theta,$$

$$r = i^{i-1} e^{-\theta}.$$

The complete solution of $4\phi_2\phi_4 - 5\phi_3^2 - 4\phi_1^2\phi_2^2 = 0$ is therefore the result of generalizing by inversion the equiangular spiral

$$r = e^\theta \dots\dots\dots (3).$$

This result is easily verified by considering the intrinsic equation to the spiral, $s = e^{n\phi}$.

Here $n\phi_1 = s^{-1}$, $n\phi_2 = -s^{-2}$, $n\phi_3 = 2s^{-3}$, $n\phi_4 = -6s^{-4}$, so that

$$4\phi_2\phi_4 - 5\phi_3^2 - 4\phi_1^2\phi_2^2 = 4s^{-6} \left(\frac{1}{n^2} - \frac{1}{n^4} \right),$$

which vanishes when $n = \pm 1$.

Otherwise, for all other values of n ,

$$4\phi_2\phi_4 - 5\phi_3^2 - 4\phi_1^2\phi_2^2 + 4n^2 \left(\frac{1}{n^2} - \frac{1}{n^4} \right) \phi_2^2 = 0.$$

Eliminating n , we get equation (2) as the differential equation of the curve obtained by generalizing by inversion any equiangular spiral whatever.

IV.

The chief point of interest in the solution of the differential equations in the foregoing sections is their analogy to the projective differential equations discovered by M. Halphen, and integrated by him in his *Thèses* (*Sur les Invariants Différentiels*, No. 408). The

expression there written as Δ is equal, but for a numerical factor, to $6R_3R_5-7R_4^2-3R_5^2R_3$ [see Vol. xvii., p. 349 (8)], and the solution of this expression equated to zero may be obtained as follows:—

$$\text{Let } y = x^n,$$

$$t = nx^{n-1},$$

$$a = n(n-1)x^{n-2}, \text{ \&c.,}$$

so that

$$3R_3 = -n^3(n-1)^2(n-2)(2n-1)x^{2n-6},$$

$$9R_5 = 2n^3(n-1)^2(n-2)(2n-1)(n+1)x^{2n-8},$$

$$9R_4 = -2n^3(n-1)^2(n-2)(2n-1)(n+1)^2x^{4n-12},$$

$$27R_5 = 8n^3(n-1)^2(n-2)(2n-1)(n+1)^3x^{6n-15}.$$

These values of the protomorphs give

$$\begin{aligned} & 6R_3R_5 - 7R_4^2 - 3R_5^2R_3 \\ &= \frac{4}{27} n^3(n-1)^3(n-2)^2(2n-1)^2(n+1)^2(n^3-n+1)x^{8n-24} \dots (1). \end{aligned}$$

Equating to zero, we get

$$n^3 - n + 1 = 0,$$

so that the solution of

$$6R_3R_5 - 7R_4^2 - 3R_5^2R_3 = 0 \dots\dots\dots (2)$$

is

$$y = x^\omega,$$

where ω is one of the imaginary cube roots of -1 .

This equation, rationalized as in the preceding sections, gives the equiangular spiral

$$r = e^{\theta\omega/3}$$

(see M. Halphen's *Thèse*, §13).

This curve, generalized by projection, will be the complete primitive of (2).

Eliminating x from (1), and the value given for R_3 , we see that the general differential equation for $y = x^n$ is of the form

$$(6R_3R_5 - 7R_4^2 - 3R_5^2R_3)^3 + kR_3^3 = 0 \dots\dots\dots (3),$$

where k is a function of n .

Eliminating k , we get the next projective photomorph equation

$$18R_6R_3^2 - 72R_6R_4R_5 + 56R_4^3 + 6R_4R_5^2R_3 - 9R_3^4 = 0 \dots\dots\dots (4),$$

so that the complete primitive of (3) is the projective generalisation of $y = x^n$.

This equation is written by M. Halphen, $T = 0$.

The equation to cuspidal cubics comes under the head of (3). It will be found to be

$$2^9 \cdot 5^3 \cdot P_3^3 = 3^4 \cdot 7^3 \cdot P_3^8 \dots\dots\dots (5).$$

V.

Criterion for distinguishing Projective and Self-polar Reciprocants.

We have already seen that there exist a series of projective reciprocants which are also self-polar; that is, are also reciprocant in r and ρ , where

$$r = \int a^{\frac{1}{2}} dx,$$

and

$$\rho = \int \left(\frac{d^2 \eta}{d\xi^2} \right)^{\frac{1}{2}} d\xi,$$

(ξ, η) being Boothian coordinates corresponding to the Cartesian co-ordinates (x, y) .

These projective reciprocants are

$$R_3,$$

$$6R_3R_5 - 7R_4^2 + 3R_3^2R_7, \text{ \&c.,}$$

and are of weight 3, 8, 12, 16, &c.,

and are all positive in character, except R_3 , when treated as reciprocants in x and y .

But the same is not true when they are looked upon as reciprocants in r and ρ .

Let us call these projective protomorphs

$$P_3, 6P_8, P_{12}, \dots \text{ respectively,}$$

so that any isobaric function of the P 's is a projective reciprocant. The negative character of P_3 will never affect the result; for it must always enter as a fourth power, since all the suffixes are multiples of 4 except this first.

Treated as reciprocants in r and ρ ,

$$P_3 \text{ is negative,}$$

$$P_8 \text{ is positive,}$$

$$P_{12} \text{ is negative, \&c.,}$$

the sign alternating.

Hence we easily see that, if

$$F(P_3, P_8, P_{13}, P_{18}, P_{23}, \dots) = 0$$

be the reciprocant differential equation of a projective curve, then its polar reciprocal will be

$$F(-P_3, P_8, -P_{13}, P_{18}, -P_{23}, \dots) = 0,$$

and, if the same equation be obtained, the curve will also be self-polar.

As an example of a self-polar curve, we may cite the curve

$$y = x^n,$$

which yields, when projectively generalized, the equation

$$P_3^3 + kP_8^3 = 0$$

[see § IV., (3)], where k is a function of n ; or, eliminating n ,

$$P_{13} = 0$$

[see § IV., (4)], which equations both satisfy the self-polar criterion.

Hence $y = x^n$ is a self-polar curve, as is easily proved by finding its tangential equation. All cuspidal cubics come under this heading, n being equal to 3.

This criterion for distinguishing between projective and self-polar reciprocants is virtually given by M. Halphen in the above-mentioned *Thèse*, § 39, &c.

VI.

I conclude by remarking on a curious property of self-polar reciprocants, viz., that any self-polar reciprocant is also a reciprocant in x and t .

In other words, if we change $a, b, c \dots$ into $t, a, b \dots$ in any projective, we shall get a mixed homogeneous reciprocant.

The truth of this statement is apparent in the two best known cases, viz., a and $9a^2d - 45abc + 40b^3$, which become t and

$$9t(tc - 5ab) + 40a^3,$$

the latter being the sum of $9t$ times the post-Schwarzian and a^3 , both negative reciprocants of characteristic 9.

Let us now attempt to express

$$6R_3R_5 - 7R_4^2 - 3R_5^2R_3,$$

after the similar transformation in terms of the M -protomorphs. If the M -function is of even degree in every term, or of odd degree in every term, then we shall get a reciprocant.

$$\text{Let} \quad F_3 = tb - \frac{2}{3}a^3,$$

$$F_5 = t^3c - 5tab + \frac{4}{3}a^3, \quad \&c.;$$

in short, let R_m become F_m when $a, b \dots$ are changed into $t, a, b \dots$

$$\begin{aligned} \text{Then} \quad F_{m+1} &= \left(t\delta_x - \frac{4m}{3}a \right) F_m \\ &= \left(t\delta_x - \frac{3m}{2}a \right) F_m + \frac{m}{6}aF_m \\ &= G_m F_m + \frac{m}{6}aF_m. \end{aligned}$$

$$\begin{aligned} \text{Thus,} \quad F_3 &= tb - \frac{2}{3}a^3 - \frac{1}{6}a^3 = M_2 - \frac{1}{6}M_1^3, \\ F_5 &= G_m (M_3 - \frac{1}{6}M_1^3) + \frac{1}{3}aF_3 \\ &= M_3 - \frac{1}{3}M_1M_2 + \frac{1}{3}M_1M_3 - \frac{1}{18}M_1^3 \\ &= M_3 - \frac{1}{18}M_1^3, \end{aligned}$$

which is a reciprocant.

$$\begin{aligned} \text{Furthermore,} \quad F_4 &= G_m (M_3 - \frac{1}{18}M_1^3) + \frac{1}{2}aF_3 \\ &= M_4 - \frac{1}{6}M_1^2M_3 + \frac{1}{2}M_1M_3 - \frac{1}{36}M_1^4, \end{aligned}$$

$$\begin{aligned} F_6 &= G_m F_4 + \frac{2}{3}aF_4 \\ &= M_5 - \frac{1}{3}M_1M_2^2 + \frac{1}{6}M_1^2M_3 + \frac{1}{2}M_2M_3 + \frac{1}{6}M_1M_4 - \frac{2}{9}M_1^3M_3 - \frac{1}{54}M_1^5. \end{aligned}$$

Given these values, we have to see whether

$$6F_3F_5 - 7F_4^2 - 3F_5^2F_3 \dots\dots\dots(1)$$

be a reciprocant. Substituting, we get

$$\begin{aligned} &6(M_3 - \frac{1}{18}M_1^3)(M_5 - \frac{1}{3}M_1M_2^2 + \frac{1}{6}M_1^2M_3 + \frac{1}{2}M_2M_3 + \frac{1}{6}M_1M_4 - \frac{2}{9}M_1^3M_3 - \frac{1}{54}M_1^5) \\ &\quad - 7(M_4 + \frac{1}{2}M_1M_3 - \frac{1}{6}M_1^2M_3 - \frac{1}{36}M_1^4)^2 \\ &\quad - 3(M_3 - \frac{1}{18}M_1^3)(M_5 - \frac{1}{3}M_1M_2^2)^2 \dots\dots\dots(2), \end{aligned}$$

Now, let us portion off the terms of uneven degree. We get

$$\begin{aligned} & 6 (M_3 - \frac{1}{18} M_1^3) (\frac{1}{2} M_2 M_3 + \frac{1}{6} M_1 M_4 - \frac{2}{9} M_1^2 M_2) \\ & - 14 (M_4 - \frac{1}{6} M_1^2 M_2) (\frac{1}{2} M_1 M_3 - \frac{1}{36} M_1^4) \\ & - 3 M_2 (M_3 - \frac{1}{18} M_1^3)^2. \end{aligned}$$

Omitting the factor $M_3 - \frac{1}{18} M_1^3$ running through, we get

$$6 (\frac{1}{2} M_2 M_3 + \frac{1}{6} M_1 M_4 - \frac{2}{9} M_1^2 M_2) - 7 M_1 (M_4 - \frac{1}{6} M_1^2 M_2) - 3 M_2 (M_3 - \frac{1}{18} M_1^3),$$

which is easily seen to vanish identically. Hence (2) only contains terms of even degree, and we see thereby that

$$6F_3 F_5 - 7F_4^2 - 3F_2^2 F_7,$$

is a reciprocant (positive) of degree 8 and characteristic 24.

Now, from R_3 and $6R_3 R_5 - 7R_4^2 - 3R_2^2 R_7$, of weights 3 and 8, deduce another projective reciprocant of weight 12, by § 3 (2).

It is obvious, then, that from F_3 and $6F_3 F_5 - 7F_4^2 - 3F_2^2 F_7$, whose characteristics are in the ratio of 3 to 8, we shall deduce an F -function of exactly the same form as the R -function just obtained.

The one only differs by a power of a from

$$3R_3 \delta_x (6R_3 R_5 - 7R_4^2 - 3R_2^2 R_7) - 8 (6R_3 R_5 - 7R_4^2 - 3R_2^2 R_7) \delta_x R_3,$$

while the other is the same, with F instead of R , except for some power of t as a through factor. We get, then,

$$R_6 R_3^2 + \dots \text{ from one pair,}$$

and

$$F_6 F_3^2 + \dots \text{ from the other,}$$

since, instead of x , we may differentiate for a function f , given by the

$$\text{equation} \quad \frac{df}{dx} = t,$$

so that

$$\frac{d}{dx} = \frac{1}{t^2} \cdot \frac{d}{df}.$$

The weight of $F_6 F_3^2 + \dots$ bears the same ratio to that of F_3 as that of $R_6 R_3^2 + \dots$ does to that of R_3 . Hence with the latter two we deduce a new R -function, and from the former two we deduce the same F -function, just as in the last paragraph. We get then a series of F -functions, all reciprocants, of exactly the same form as the

similarly deduced series of R -functions. But these latter are the projective protomorphs from which all others can be formed. Hence from any projective R -function we can deduce an F -function which will be a reciprocant. For in both R -function and F -function the character is $(-1)^n$, and if the R -protomorphs be combined in any way so that every term is of the same character, so also will be the case with the same function of the F -protomorphs.

Hence the truth of the statement at the beginning of this section is established.

December 9th, 1886.

SIR JAMES COCKLE, F.R.S., President, in the Chair.

Mr. F. S. Macaulay was admitted into the Society.

Prof. Dairoku Y. Kikuchi, B.A. Cambridge, Professor of Mathematics at Tokio, was elected a member.

The following communications were made:—

The Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants: E. B. Elliott.

Circular Notes: R. Tucker.

The Problem of the Duration of Play: Capt. MacMahon.

Note on two Annihilators in the Theory of Elliptic Functions: J. Griffiths.

Mr. Hammond made some remarks on the communication read by Capt. MacMahon at the November meeting.

The following presents were made:—

"Physical Society of London—Proceedings," Vol. VIII., Part II.; Oct., 1886.

"Educational Times," for December.

"Johns Hopkins University Circulars," Vol. VI., No. 52; November, 1886.

"Proceedings of the Canadian Institute, Toronto," Vol. IV., Fasc. 1; Toronto, 1886.

"Bulletin des Sciences Mathématiques," Nov., 1886.

"Acta Mathematica," 9, 1.

"Journal für Mathematik," Crelle, Band C., Heft 3.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. II., Fasc. 8; Roma, 1886.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Nos. 21 and 22; Firenze, 1886.