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To cite this article: Major-General P. Anstruther C.B. (1861) The Trajectory Of Balls, Royal United Services Institution. Journal, 5:18, 309-321, DOI: [10.1080/03071846109416752](https://doi.org/10.1080/03071846109416752)

To link to this article: <http://dx.doi.org/10.1080/03071846109416752>

Published online: 25 Sep 2009.

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Friday, May 24th, 1861

Col. P. J. YORKE, F.R.S. in the Chair.

THE TRAJECTORY OF BALLS.

By Major-General P. ANSTRUTHER, C.B., H.M. Madras Artillery.

PREFACE.

Extract from "Straith's Memoir of Artillery," pp. 81, 82: "Dr. Gregory, in his Lectures upon Gunnery, observes, on the difference between the times employed by a ball in ascending and descending vertically through the same space: If a 24-pound iron ball were projected vertically upwards, with a velocity of 2000 feet per second, it would ascend to the height of 6424 feet before its upward motion was extinguished, and it would pass over that space in less than $9\frac{1}{2}$ seconds. (This is computed in 'Hutton's Mathematics,' vol. 3.) It might, on a cursory view of the subject, be supposed that the circumstances of the descent would be analogous to those of the ascent, but in an inverted order: and so they would in a non-resisting medium; but in the air the case is widely different. After the ball had descended 2700 feet, the resistance of the air would be equal to the weight of the ball, there would remain no further cause of acceleration, and the ball would descend uniformly with its *terminal* velocity (that is, the greatest velocity which a heavy body can acquire when falling in the air), which does not exceed 419 feet per second. It would require, there-

fore, $\frac{6424 - 2700}{419}$, or 6 seconds, to descend the remaining 3724 feet, in addition to the time, about 10 seconds, which had been occupied in descending through the first 2700 feet; so that, in this instance, the time of descent would be about double that of ascent. In all cases where the projectile velocity exceeds 300 or 400 feet, the time of descent will exceed that of ascent; and their difference is greater the more the initial velocity exceeds that limit."

1. As we are unable to procure a copy of Dr. Gregory's Lectures on Gunnery, we must content ourselves with an extract, quoted from Hector Straith's Memoir on Artillery, *vide* Preface.

Speaking of a ball fired vertically upwards, he says, after detailing the particulars of the initial velocity, time of flight, &c. &c. "so that, in this case, the time of descent would be about double that of ascent."

2. These are the words of a professor at the Military Seminary of Addiscombe, quoting those of the Professor of Mathematics at the Royal Military Academy of Woolwich; we may therefore safely infer that this was what the greater portion of Her Majesty's Artillery learned as part of the science of gunnery.

3. Believing that, as in the case supposed above, so in any case of a ball's being fired upwards, the time of ascent and the time of descent must of necessity be exactly equal, it is obviously our imperative duty to protest in the most public manner in our power against the continued training up of our young officers in a course of error.

4. By the first law of motion we learn that "A body in motion, not acted upon by any external force, will move in a straight line and with a uniform velocity."

Suppose a 24-pound iron ball projected vertically upwards with velocity V , it would, in any time T , ascend in a straight line to a distance equal to TV , the number of feet per second denoted by V multiplied by the number of seconds denoted by T , if it were not acted upon by any external force whatever. Let AX represent a horizontal plane, and suppose the 24-pounder gun placed at A and fired vertically upwards, the ball would, in the time T , go vertically upwards to a distance equal to TV ; let the line AZ forming a right angle with AX represent the measure and direction of the path of the ball if not acted upon by any external force.

5. But the ball does not go from A to Z in the time T ; it is seen to go part of the way and to return, striking the plane AX at the expiry of the time T . Dr. Gregory gives for the time of ascent "less than 9.5 seconds," and he gives a time of "about 10 seconds" for one part of the descent, and another of "6 seconds" for the other part of the descent; a total of 25.5 seconds for the whole time of flight of this 24-pound ball fired vertically upwards.

6. It follows, therefore, that the ball has been acted upon by some external force during its time of flight, and a very short reflection gives us the certainty that it has been acted upon by two, and only two, external forces; these are,

1. The resistance of the atmosphere, and,
2. The force of gravity.

These two we shall consider separately.

7. The first of the two, the resistance of the atmosphere, can only act by diminishing the length of AZ ; it cannot take it out of the line of direction given to it, nor can this force in any degree whatever tend to bring the ball back to A , as we see the ball does come back in 25.5 seconds. Therefore there is, somewhere or other on the line AZ , a point Y , at which the ball would have arrived in the time 25.5 seconds if acted upon by no other external force, but by this force alone; the portion of AZ which is between Z and Y is the measure of the retardation produced by the resistance of the atmosphere.

8. The remaining portion of AZ , which is between A and Y , is the measure of the effect of the force of gravity upon

Fig. 1.

Fig. 2.

this ball in motion from A towards Z during 25·5 seconds of time, and we proceed to consider this with the assistance of the second law of motion, from which we learn, that, "When any force acts upon a body in motion, the change of motion which it produces is the same in magnitude and direction as the effect of the force upon the body at rest." That is to say, when the force of gravity alone acts upon this 24-pound ball in motion from A towards Y, the change of motion which it produces is the same in magnitude and direction as the effect of the force of gravity upon the 24-pound ball at rest.

9. Now what is the effect of the force of gravity upon a 24-pound ball suffered to fall from a state of rest, during the given time, 25·5 seconds? What is the fall by gravity during 25·5 seconds? We can answer this readily; it is $25\cdot5^2 \times 13\cdot75 = 8940\cdot9375$ feet, say 8941 feet. Dr. Gregory seems to have thought that the ball was not opposed by gravity until it had culminated; but this is evidently an error. The ball will fall by gravity, from a state of rest, through 8941 feet in 25·5 seconds of time, acquiring thereby velocity V; therefore this is the exact measure of the height to which it would have ascended in 25·5 seconds of time, if fired with initial velocity V, and if not opposed by gravity. We must therefore now determine the value of V, which is not difficult.

10. If we deduct the fall for 25·49 seconds from that for 25·50, the difference will give the space described by the ball in the last hundredth part of a second of its fall. It is as follows, viz.:—

25·50 seconds, $25\cdot50^2 \times 13\cdot750 = 8940\cdot9375000$ feet.

25·49 " $25\cdot49^2 \times 13\cdot751 = 8934\cdot5761151$ feet.

A difference of 0·01 seconds and 6·3613849 feet,

Showing a velocity of 636·13849 feet per second = V.

11. If an initial velocity of 2000 feet is required to give 25·5 seconds time of flight to a ball fired vertically upwards, an initial velocity of $2000 \sqrt{2}$ must be necessary to give the same ball at 45° elevation the same time of flight, so that to range at 45°, with 25·5 seconds of flight, the initial velocity must be 2828·4. Now Sir Howard Douglas tells us that at Landguard Fort (in 1806) the shell of 200lbs. with a charge of 10lbs. did range 2960 yards in 25·3 seconds at 45° elevation. Did a charge of one-tenth of the shot's weight give a velocity of 2828·4 feet per second? Our calculation makes the range twenty yards longer in three-tenths of a second more time, and the velocity $636\cdot1385 \sqrt{2}$ or 900 feet per second. Is the velocity gained by using a charge of one-tenth of the shot's weight more likely to be 2828 feet, or 900 feet?

12. The Washington experiments, p. 47, prove that with a long gun, 32-pounder, the velocities gained were, with

0·333 of the shot's weight, 1691 feet per second.

0·25 " " 1570 " "

0·2 " " 1462 " "

0·166 " " 1396 " "

0·125 " " 1218 " "

Ours of 0·1 may be correct, as 900; but how can Dr. Gregory's of 2828?

13. Will any amount of powder give any cast-iron ball a velocity of 2828, or even of 2000 feet? The French table given in page 429 of the

Aide-Mémoire gives velocity 1798, 1794, and 1732, as the result of half the shot's weight, and the experiments show that a further increase of charge would not have given any perceptible increase of the velocity. That table makes one-tenth of the shot's weight give 1004 feet, 974·5 feet, or 1018 feet from guns; we say 900 feet from mortars.

14. We shall now leave Dr. Gregory, and turn to Dr. Cape, who in his "Course of Mathematics" gives the following example on the law of gravity, vol. 2, page 216:—

"An arrow shot perpendicularly upwards from a bow, returned again in 10 seconds; required the velocity of projection, and the height to what it rose?"

"Ans.—Velocity 161; height 402·5 feet."

15. *Negatur*. If the fall by gravity were, as held by Dr. Cape, in any time, t , equal to the square of the time multiplied by 16·1, then the height and velocity would be exactly double that given by Dr. Cape. The rising and falling of the ball would be, in that case, as is shown in the following table:—

Flight, 10 Seconds.

Seconds of Time.	Ascent.		Descent by Gravity.	Height at each period of Time.
	Gradual.	Cumulative.		
1	305·9	305·9	16·1	289·8
2	273·7	579·6	64·4	515·2
3	241·5	821·1	144·9	676·2
4	209·3	1030·4	257·6	772·8
5	177·1	1207·5	402·5	805·0
6	144·9	1352·4	579·6	772·8
7	112·7	1465·1	788·9	676·2
8	80·5	1545·6	1030·4	515·2
9	48·3	1593·9	1304·1	289·8
10	16·1	1610·0	1610·0	0·0

16. Of this any gentleman may convince himself by supposing the same time of flight, but 45° of elevation; then the range would be $t^2 \times 16·1 = 10^2 \times 16·1 = 1610$ feet; the velocity $20 \sqrt{2} \times 16·1$ or $20 \times 16·1 \sqrt{2} = 322 \sqrt{2}$; showing that the vertical velocity was $20 \times 16·1 = 322$ feet not 161—.

17. But, to draw the true trajectory, the above law of gravity must be altered as follows:—

Fig. 3.

*Time of Flight 10 seconds.**Elevation 45°* *Range 510 yards.**Initial Velocity 418.6 feet.**Scale, 400 feet = 1 inch.*

Seconds of Time.	Ascent.		Descent by Gravity.	Height at each Second of Time.
	Gradual.	Cumulative.		
1	282.6	282.6	16.2	266.4
2	255.4	538.0	61.4	473.6
3	227.6	765.6	144.0	621.6
4	199.2	964.8	254.4	710.4
5	170.2	1135.0	395.0	740.0
6	140.6	1275.6	565.2	710.4
7	110.4	1386.0	764.4	621.6
8	79.6	1465.6	992.0	473.6
9	48.2	1513.8	1247.4	266.4
10	16.2	1530.0	1530.0	0.0

18. In Figure 3 we show the true trajectory for 10 seconds time of flight, on a scale of 400 feet to an inch, the elevation being 45° . It is very simple, and obviously true. It is thus constructed: Draw a right-angled triangle, $\Delta B G$, in which the angles are 45° , 45° , and 90° . Then ΔG must represent the range, and $B G$ the fall by gravity for 10 seconds. On $B G$ mark off the fall for 9 seconds, 1247.4 feet, and draw a horizontal line. Its intersection with ΔG will mark the portion of ΔG through which, if not acted upon by gravity, the ball would have passed in the first of the ten seconds of its flight. Let fall a perpendicular, 16.1 feet in length, and its lower extremity will mark the place of the ball at the expiry of the first of the 10 seconds. In like manner we mark the fall for 8, 7, 6, 5, 4, 3, 2, and 1 seconds, and draw horizontal lines, which will divide ΔG in proportion, as shown in our para. 17. The column marked "Cumulative Ascent" shows the distance from Δ , measured vertically on ΔG . Therefore the distances are $282.6 \sqrt{2}$, $538 \sqrt{2}$, $765.6 \sqrt{2}$, $964.8 \sqrt{2}$, $1135 \sqrt{2}$, $1275.6 \sqrt{2}$, $1386 \sqrt{2}$, $1465.6 \sqrt{2}$, $1513.8 \sqrt{2}$, and $1530 \sqrt{2}$ —which is the whole length of ΔG .

19. This ascent would be the same at any other elevation as at 45° . If gravity were not acting upon the ball it would, at any elevation, go $1513.8 \sqrt{2}$ feet in 9 seconds. Now suppose that gravity does act, we see that the fall by gravity is 1247.4 feet for 9 seconds. Then we find the range thus: $\Delta B^2 - B G^2 = \Delta G^2$. But the portion of ΔB for 9 seconds is $1513.8 \sqrt{2}$, and $B G$ is 1247.4; therefore $\Delta G = 1739.87$ feet, the range for this time of flight, which is 70 yards further than the range at 45° . The elevation to give this is easily found, as $1513.8 \sqrt{2} : 90^\circ :: 1247.4 : 28^\circ 16' 10''$, the required elevation.

20. The initial velocity for 10 seconds of flight at 45° is $296 \sqrt{2}$, or 418.6 feet per second; its range 510 yards. The French Table, p. 422

of the *Aide-Mémoire*, gives for 9 seconds 437 yards; for $10\frac{1}{2}$ seconds, 537 yards; therefore for 10 seconds about 500 yards—ours being 510.

21. Supposing that, instead of firing the ball vertically upwards with initial velocity V , we were to give a small additional charge of powder, and elevation 85° , the additional charge of powder being exactly that which would preserve the same time of flight. Then, the time of flight being the same, the fall by gravity must of necessity be the same also; and the range, being measured, will enable us to calculate the exact height to which the ball would have ascended, if it had not been acted upon by gravity.

22. In Figure 4, let BG represent the fall by gravity in the time of flight, whether for $25\cdot5''$, as in Doctor Gregory's case, or for ten seconds, as in Doctor Cape's. Then, if we measure AG , the range at 85° , we can easily calculate the exact length of BG . For as $\sin. 5^\circ : AG :: \sin. 85^\circ : BG$. Supposing that we fired with a certain charge, and found that at 85° the range AG was 1106·24 feet, then we should know that $BG = 8940\cdot9375$ feet.

Fig. 4.

23. There is no reason why we should not fire the larger 10-inch howitzer of $6\frac{1}{4}$ tons weight with such a charge as would, at 85° elevation, give a range of 1106·24 feet, or 368·75 yards. Any range at this elevation would do as well; but suppose that, by chance, the range were to prove exactly 1106·24 feet, and that, firing the solid shot of 140lbs. weight, we made the charge of powder exactly such, that we could be certain of getting the same range, then if we were to lay the target of the electro-chronoscope level on the ground, covering the spot on which by experiment we knew the ball would fall, the ball would cut the electric wire, and we should know that the time of flight had been $25\cdot49997$ seconds, which we should call $25\cdot5$ seconds. We should then calculate the height which would give us the measure of the fall by gravity in $25\cdot5$ seconds, which we have hitherto never been able to determine.

24. Or if we fired with a very small charge at 85° , and finding the range to be 133·8676 feet, then placing the target of the electro-chronoscope on the spot struck, and firing again, suppose the wire to be cut in $9\cdot99997$ seconds, which we should call 10 seconds, then we should calculate BG by saying, as the $\sin. 5^\circ : 133\cdot8676 :: \sin. 85^\circ : 1529\cdot9997$, which we should call 1530, and determine thence that the fall by gravity for 10 seconds is 1530 feet, or $10^2 \times 15\cdot3$ feet.

25. A few experiments of this nature with the electro-chronoscope would settle all the undetermined questions in gunnery, and render our science as certain as any of the others which have been brought to such perfection by mathematical calculations.

26. The electro-chronoscope will for a thirtieth or fortieth part of the price of the ballistic pendulum give us far more accurate results; the expense, after the purchase of the instrument itself, is only the powder burnt, and we may now hope to know the real path of our projectiles.

27. No method appears to present itself of calculating exactly the range from the elevation, except at 45° , but we can calculate the range for given time, and then find the angle of elevation which would give it. For example, suppose the time 20.1 seconds. Then B G, the fall by gravity, is $20.1^2 \times 14.29 \text{ feet} = 5773.3029 \text{ feet}$. And A B we find thus:—From 25.5 deduct 20.1; the difference is 5.4.

From 8940.9375, the fall for 25.5 seconds.

Take 459.5616, the fall for 5.4.

It leaves us 8481.3759, the ascent for 20.1 seconds.

Then 8481.3759 $\sqrt{2}$ is the ascent at any angle in the time 20.1 seconds.

We now say, as $8481.3759 \sqrt{2} : 90^\circ :: 5773.3029$: the angle of elevation, which is $28^\circ 46' 20''$. Hence we find the range 10513.6 feet; or 3504.5 yards. In this manner we have calculated the table B, from which it will appear that with this velocity, 900 feet per second, the angle of elevation for greatest range is $28^\circ 32' 40''$ which throws two feet further than $28^\circ 46' 20''$. We showed, in a letter of the 23rd July last, addressed to the gentlemen of the Institution of Civil Engineers, that with velocity 1000 the greatest range was obtained at $28^\circ 53' 45''$, which indicates that the elevation for greatest range will always be about $28^\circ 30'$ to 29° .

28. Our limits do not permit of drawing the trajectory for 25.5 seconds time of flight, but a sketch of that for 10 seconds is herewith given.* It shows the height attained in a flight of 10 seconds at 45° to be 740 feet, as calculated in our paragraph 17, and this after travelling over a horizontal distance of 1185 feet; so that neither is the vertical height quite equal to half what it would have been if not acted upon by gravity, viz. 765 feet, nor is the horizontal distance of culmination quite equal to three-quarters of the range 1147.5 feet.

29. We will give one more example of the working of this problem. On the 24th of November, 1854, at Shoeburyness, the 10-inch howitzer of 125 cwt. is recorded to have thrown its shot of 140 lbs. to a distance of 4800 or 4900, say 4850 yards, at 30° elevation, the charge being 16 lbs. of powder. The initial velocity is not given; we will find it from the above data.

In the triangle A B G we have the three angles 30° , 60° , and 90° , and

Fig. 5.

we have A G, the range of 4850 yards or 14,550 feet; we, therefore, easily find

B G = 8400.4 feet and

A B = 16800.8 feet.

Here B G represents the fall by gravity, and we deduce the time of flight from the measure of the fall by gravity. In 24.64 seconds the fall by gravity is $24.64^2 \times 13.836 = 8400.2451456$ feet, therefore, 24.64 seconds

must have been the time of flight.

The report shows incidentally that there was no accurate instrument

* Vide Fig. 3, at paragraph 17.

employed to measure the true time of flight; mere ocular observation is not to be depended upon; we shall assume that 24.64 seconds is the time of falling 8400.4 feet, and, therefore, of the range now in question.

30. The next question is to find what velocity would carry a ball from A to B, 16800.8 feet, in 24.64 seconds of time, gravity not acting.

We find that 16800.8 divided by the square root of 2 gives us $\frac{16800.8}{\sqrt{2}} = 11880$, therefore the velocity required to throw a ball vertically upwards 11880 feet in 24.64 seconds of time, if not opposed by gravity, is one of the two equal components of the initial velocity sought.

Now, a ball falling from a height of

12454.7840768 feet will reach the ground in 30.68 seconds

572.6151936 being passed through in 6.04 seconds, so that

11882.1688832 feet were passed through in 24.64 seconds. This is only 2 feet more than we wished. But a ball falling by gravity from a height of feet

12447.6068937 will reach the ground in 30.67 seconds,

570.7570473 being passed through in 6.03 seconds, so that

11876.8498464 feet were passed through in 24.64 seconds, which is 3 feet less than we wished. Therefore 30.676 is the time in which a ball would, by falling, gain such velocity that in the last 24.64 seconds it should fall through 11880 feet, and the velocity it would thereby have acquired is easily found. From

12454.784, the fall for 30.68 seconds, deduct

12447.607, the fall for 30.67, and the difference, being

7.177 is the fall in one-hundredth of a second; then

717.7 is the velocity acquired by falling 30.676 seconds.

31. Then the resultant or equivalent of two equal velocities, one vertical, the other horizontal, each being 717.7 feet per second, will be $717.7 \sqrt{2}$, or 1015 feet, the true initial velocity which would give a range of 4850 yards at 30° elevation. And we have seen that the vertical or horizontal space passed over, if not acted on by gravity, would be for 45° elevation 12451.2 feet, or 4150.4 yards.

32. This agrees tolerably well with our former calculation, that velocity 1000 would range 3973 yards; it adds 177 yards of range for additional 15 feet of velocity.

33. It is to be regretted that artillery writers should always use such very great velocities in their arguments. It renders impossible any experimental test. Nothing above a velocity 1100 should ever be taken as matter for argument; that would be greater velocity than we ever have attained in practice. What is the use of talking of velocity 1250 feet, when it would at 45° elevation give a range of 5 miles 6 furlongs? Its time of flight would be 53 seconds; the fall by gravity in 53 seconds is $53^2 \times 11 = 30,899$ feet, or 10,300 yards; therefore the range at 45° is 10,300 yards, or 5 miles 6 furlongs.

34. We are now in a condition to give a very simple and easy solution of the question which so many have called difficult, viz. the measure of the retardation of the ball by the resistance of the atmosphere. The initial velocity of the ball which, when fired vertically upwards, returned to the ground in 25.5 seconds, we have shown to be 636.13849 feet per second. If wholly unopposed, this ball would have gone from A to Z,

Fig. 1, a distance equal to $T V$, *vide* paragraph 4. Hence $T = 25.5$, and $V = 636.13849$; therefore $T V = 16221.531496 = \Delta Z$.

If we deduct 8940.9375 the length of ΔY , we find the difference 7280.593995 feet, the length of $Y Z$, and the measure of the retardation by the resistance of the atmosphere, a ball travelling with velocity 636.13849 feet per second.

35. We have, in paragraph 13, expressed a doubt whether gunpowder can give to iron ball a velocity of $2000 \sqrt{2}$; we are by no means sure that it could give 2000 feet velocity; we are very sure that cast-iron guns are not to be trusted with any velocity approaching that. The 32 -pounder gun of 9 feet 7 inches, 63 cwt. which burst at Deal in 1837 , was only charged with $\frac{37.5}{1000}$ of its shot's weight; it was giving velocity less than 1000 feet per second when it yielded.

36. It is much to be regretted that all our artillery writers employ such enormous, such imaginary, velocities, when they adduce examples to illustrate their arguments. Captain Boxer, of the Royal Artillery, supposes an initial velocity of 1774 feet per second, when he means to prove that at 35° the range will be greater than at 45° , and the consequence is, neither he, nor any one else, can tell what either elevation would give as range for such velocity. It is much to be desired, that all should agree on some one velocity, say 1000 feet per second, and let all calculations have reference to that.

37. Captain Boxer quotes Hutton for various approximations to the elevation for giving maximum range, as given by Newton, Robins, Euler, and Robinson; and he shows Hutton's calculation for a 24 -pounder, with velocity 1640 feet, which, Hutton says, would at $34^\circ 15'$, throw 8336 feet, and this he fixes as the greatest range which that velocity could give to a 24 -pounder ball. It is true Captain Boxer does say that this range is "considerably less than that found by experiment," but he does not point out the fatal error which vitiates the whole of this calculation.

38. Hutton reasoned in the wrong direction, he could not possibly arrive at the truth. He fixes a theory of terminal velocity, and deduces ranges which every day's practice proves to be erroneous. Let us put the question as it should be put, reason from the supposed facts, and see the result. The given facts are the range 8336 feet, and the elevation $34^\circ 15'$. Required the initial velocity. Hutton ascribes this range and elevation to velocity 1640 feet per second. We shall prove that initial velocity 829.13

feet per second is the true measure. In the triangle $\Delta G B$, let $A G$ be the given range 8336 feet, the angle $G A B 34^\circ 15'$, the angle $G B A 55^\circ 45'$, the angle $B G A 90^\circ$, then the side $B G = 5675.78$,

and $A B = 10084.8$ feet.

In this triangle ΔG is the range, $B G$ the fall by gravity, $A B$ the ascent, if not deflected by gravity. Then, as the fall by gravity is

Fig. 6.

5675.78 feet, we may be sure the time of flight is 19.92 seconds, because $19.92^2 \times 14.308 = 5677.5$ feet, only 1.72 feet over, and near enough for our purpose.

39. It only remains to show the initial velocity which would carry a ball from A to B in 19.92 seconds, if not opposed by gravity. As it is seen to do this at $34^\circ 15'$, clearly it would do so at 45° or any other elevation, the resistance of the atmosphere being the same whatever angle we fire at; the velocity being the same, the distance traversed will be the same if gravity does not act. At any elevation the velocity may, with advantage, be considered as the resultant or equivalent of two velocities, one of them vertical, represented by the sine of the angle of elevation; the other horizontal, represented by the cosine of that angle. To calculate radius, let us suppose sine and cosine equal, that is to say, let us suppose elevation 45° , then the vertical velocity and horizontal velocity each will be represented by v , and their resultant or equivalent, $v\sqrt{2} = V$, the said V being the real initial velocity of the ball. Dividing AB by the square root of 2, we get the distance $\frac{1993.49}{1.41421356} = 7131.04$ feet, and we see that this is the vertical height to which, if not opposed by gravity, the vertical velocity v would carry the ball in 19.92 seconds.

40. To ascertain what v is, we must see what height a ball must fall from, to acquire by falling such velocity as to pass through 7131.04 feet in the last 19.92 seconds of its fall, and this, though tedious, is not difficult. We find after some few trials that if a ball be suffered to fall by gravity from a height of 7258.8140625 feet it will reach the ground at the end of 22.75 seconds of time, and in the earlier 2.83 seconds of this it will have passed through $(2.83^2 \times 16.017 \text{ feet}) = 128.27855112$ feet, so that in the remaining 19.92 seconds it would fall through 7130.5355112

feet, only 6 inches short of 7131.04, the desired length, or $\frac{AB}{\sqrt{2}}$. The velocity v , which a ball acquires by falling 22.75 seconds, is thus found:

$$22.75^2 \times 14.025 = 7258.8140625$$

$$22.74^2 \times 14.026 = 7252.9511976$$

a difference of $\frac{5.8628649}{14.025}$ feet in 0.01 second, showing a velocity, v , of 586.28619 feet per second.

41. Then as $(V = v\sqrt{2})$ the initial velocity, resultant or equivalent of two velocities of 586.28619 each, is equal to $586.28619\sqrt{2}$, it is 829.1343, as we said in para. 38 we should prove it to be.

42. We subjoin a table, C, by which any artillerist may amuse himself by finding the angle of elevation for any time of flight with this velocity, and we shall give three examples of

TABLE A.

Time of Flight 25.5 Seconds. Initial Velocity 636.1335 feet per Second.

Seconds of Time.	Vertical Ascent (in feet).		Fall by Gravity (in feet).	Actual Height attained in the whole Time.	Space passed through in each Half-Second.
	Gradual.	Cumulative.			
0.5	315.9375	315.9375	4.0625	311.875	311.875
1	311.5375	627.475	16.2	611.275	299.4
1.5	307.0625	934.5375	36.3375	898.2	286.925
2	302.5125	1237.05	64.4	1172.65	274.45
2.5	297.8875	1534.9375	100.3125	1434.625	261.975
3	293.1875	1828.125	144.	1684.125	249.5
3.5	288.4125	2116.5375	195.3875	1921.15	237.025
4	283.5625	2400.1	254.4	2145.7	224.55
4.5	278.6375	2678.7375	320.9625	2357.775	212.075
5	273.6375	2952.375	395.	2557.375	199.6
5.5	268.5625	3220.9375	476.4375	2744.5	187.125
6	263.4125	3484.35	565.2	2919.15	174.65
6.5	258.1875	3742.5375	661.2125	3081.325	162.175
7	252.8875	3995.425	764.4	3231.025	149.7
7.5	247.5125	4242.9375	874.6375	3368.25	137.225
8	242.0625	4485.	992.	3493.	124.75
8.5	236.5375	4721.5375	1116.2625	3605.275	112.275
9	230.9375	4952.475	1247.4	3705.075	99.8
9.5	225.2625	5177.7375	1385.3375	3792.4	87.325
10	219.5125	5397.25	1530.	3867.25	74.85
10.5	213.6875	5610.9375	1681.3125	3929.625	62.375
11	207.7875	5818.725	1839.2	3979.525	49.9
11.5	201.8125	6020.5375	2003.5375	4016.95	37.425
12	195.7625	6216.3	2174.4	4041.9	24.95
12.5	189.6375	6405.9375	2351.5625	4054.375	12.475
12.75	92.4984375	6498.4359375	2442.5015625	4055.934375	1.559375
13	90.9390625	6589.375	2535.	4054.375	1.559375
13.5	177.1625	6766.5375	2724.6375	4041.9	12.475
14	170.8125	6937.35	2920.4	4016.95	24.95
14.5	164.3875	7101.7375	3122.2125	3979.525	37.425
15	157.8875	7259.625	3330.	3929.625	49.9
15.5	151.3125	7410.9375	3543.6875	3867.25	62.375
16	144.6625	7555.6	3763.2	3792.4	74.85
16.5	137.9375	7693.5375	3988.4625	3705.075	87.325
17	131.1375	7824.675	4219.4	3605.275	99.8
17.5	124.2625	7948.9375	4455.9375	3493.	112.275
18	117.3125	8066.25	4698.	3368.25	124.75
18.5	110.2375	8176.5375	4945.5125	3231.025	137.225
19	103.1875	8279.725	5198.4	3081.325	149.7
19.5	96.0125	8375.7375	5456.5875	2919.15	162.175
20	88.7625	8464.5	5720.	2744.5	174.65
20.5	81.4375	8545.9375	5988.5625	2557.375	187.125
21	74.0375	8619.975	6262.2	2357.775	199.6
21.5	66.5625	8686.5375	6540.8375	2145.7	212.075
22	59.0125	8745.55	6824.4	1921.15	224.55
22.5	51.3375	8796.9375	7112.8125	1684.125	237.025
23	43.6875	8840.625	7406.	1434.625	249.5
23.5	35.9125	8876.5375	7703.8875	1172.65	261.975
24	28.0625	8904.6	8006.4	893.2	274.45
24.5	20.1375	8924.7375	8313.4625	611.275	286.925
25	12.1375	8936.875	8625.	311.875	299.4
25.5	4.0625	8940.9375	8940.9375	0.0	311.875

TABLE B.

Initial Velocity $636.1385\sqrt{2}=899.6357$.

Time of Flight.	Elevation.	Range.		Logm. Sine of Elevation.
		Fect.	Yards.	
1	1° 2' 48	837.3	296	8.2614035
2	2 6.20	1749	583	5659336
3	3 11.30	2581	860	7453417
4	4 18	3334.5	1123	8747723
5	5 25.50	4156	1385	9759105
6	6 35	4895	1632	9.0595654
7	7 46.30	5598.5	1866	1312426
8	9	6264.7	2083	1942343
9	10 15.30	6839	2296	2506685
10	11 34	7478	2493	3021649
11	12 55	8021	2674	3492624
12	14 19.12	8518	2839	3932924
13	15 47	8967	2989	4346188
14	17 24	9366.5	3122	4757400
15	18 55	9711.7	3237	5110150
16	20 37.20	10000.6	3333.5	5467733
17	22 24	10229.6	3410	5812694
18	24 19.30	10391	3463	6148063
19	26 21.30	10492	3497	6473401
20	28 32.40	10515.6	3505.2	6792848
21	30 54	10459.1	3486.7	7107053
22	33 29	10314	3438	7417624
23	36 19.30	10072.9	3357.6	7725857
24	39 23.40	9720	3240	8033079
25	43 2	9238.2	3079.4	8340584
25.5	45	8941	2980.3	8494850
	50	8127.6	2709.2	
	55	7252.5	2417.5	
	60	6322.2	2107.4	
	65	5343.7	1781.2	
	70	4324.6	1441.5	
	75	3272.6	1091	
	80	2105.6	732	
	85	1102	367	
	90	0	0	

TABLE C.

Time of Flight 22.75 Seconds. Initial Velocity 586.2865 feet per Second.

	Vertical Ascent (feet).		Fall by Gravity (feet).
	Gradual.	Cumulative.	
0.5	290.809375	290.809375	4.0625
1	285.996875	576.80625	16.2
1.5	281.109375	857.915625	36.3375
2	276.146875	1134.0625	64.4
2.5	271.109375	1405.171875	100.3125
3	265.996875	1671.16875	144
3.5	260.809375	1931.978125	195.3875
4	255.546875	2187.525	254.4
4.5	250.209375	2437.731375	320.9625
5	244.796875	2682.53125	395
5.5	239.309375	2921.840625	476.4375
6	233.746875	3155.5875	565.2
6.5	228.109375	3383.696875	661.2125
7	222.396875	3606.09375	764.4
7.5	216.609375	3822.703125	874.6875
8	210.746875	4033.45	992
8.5	204.809375	4238.259375	1116.2625
9	198.796875	4437.05625	1247.4
9.5	192.709375	4629.765625	1385.3375
10	186.546875	4816.3125	1530
10.5	180.309375	4996.621875	1681.3125
11	173.996875	5170.61875	1839.2
11.5	167.609375	5338.228125	2003.5875
12	161.146875	5499.375	2174.4
12.5	154.609375	5653.984375	2351.5625
13	147.096875	5801.98125	2535
13.5	141.309375	5943.290625	2724.6375
14	134.546875	6077.8375	2920.4
14.5	127.709375	6205.546875	3122.2125
15	120.796875	6326.34375	3330
15.5	113.809375	6440.153125	3543.6875
16	106.746875	6546.9	3763.2
16.5	99.609375	6646.509375	3988.4625
17	92.396875	6738.90625	4219.4
17.5	85.109375	6821.016625	4455.9375
18	77.746875	6901.7625	4693
18.5	70.309375	6972.071875	4945.5125
19	62.796875	7034.86875	5198.4
19.5	55.209375	7090.078125	5456.5875
20	47.546875	7137.625	5720
20.5	39.809375	7177.431375	5983.5625
21	31.996875	7209.43125	6262.2
21.5	24.109375	7233.540625	6540.8375
22	16.146875	7249.6875	6824.4
22.5	8.109375	7259.796875	7112.8125
22.75	1.0171875	7258.8140625	7258.8140625