General Theory of Irrational Number; Integration of dy/y

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discontinuity of a Fourier Series must be of this character. The case which is usually discussed in this connection is that of the series

$$\sin \theta + \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} + \dots,$$

which is not so simple as the one I have chosen, as the function is not constant between its discontinuities. The number of examples which can be

FIG. 3.

constructed is of course legion, and, although the diagrams take rather a long while to draw, I believe it will be admitted that the insight into the theory of infinite series is quite worth the trouble.

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GENERAL THEORY OF IRRATIONAL NUMBER; INTEGRATION OF dy/y .

THERE is a perfectly natural way of arriving at the differential coefficient of a^x , and the corresponding integration of dy/y , provided that the ordinary inadequate bookwork of indices is properly supplemented to include irrational indices. (This is really necessary for logarithms to be treated rigorously.) The graph of $y = a^x$, x any real number and a positive,

is then a continuous curve for which it is easily proved that $\frac{dy}{dx} = \frac{y}{x}$, where k is a constant (the subtangent of x, y) which depends on the value of a , and may be defined as the *modulus* of the system of exponentials to *base* a . In this case e is the base whose modulus is unity, and e^a is readily found by Taylor's theorem, since

$$\frac{de^x}{dx} = e^x = \frac{d^2e^x}{dx^2} = \dots$$

I. Something like the following chain of theorems is necessary ; the base a is always considered as positive.

(i) 'If $a > 1$, a^x can be made $>$ any assigned quantity, by making x a great enough integer ; and if $a < 1$, $a^x <$ any assigned quantity, etc.'

We can prove successively

$$(1+x)^2 > 1+2x ; (1+x)^3 > (1+x)(1+2x) > 1+3x ;$$

and so on ;

$$\therefore (1+x)^n > 1+nx > \text{any assigned quantity, } n \text{ great enough.}$$

This proves the theorem when $a > 1$, and by using the reciprocal of a , the corresponding theorem is proved when $a < 1$.

(ii) 'If $a > 1$, n a given integer, a number $b > 1$ can be found such that $b^n < a$; and if $a < 1$, $c < 1$ can be found so that $c^n > a$.'

Without the binomial theorem we can show that the sum of coefficients of $(1+x)^n = 2^n$; hence if $x < 1$

$$\begin{aligned} (1+x)^n &< 1 + c_1x + c_2x^2 + c_3x^3 + \dots + c_nx^n \\ &< 1 - x + 2^n x < 1 + x(2^n - 1). \end{aligned}$$

By making x small enough, this can be made $< a$.

This proves when $a > 1$; by taking reciprocals, we prove when $a < 1$.

(iii) 'Every real positive number a has one only real positive n^{th} root' ; this root we shall call the *principal n^{th} root* of a .

If $a > 1$, numbers b exist > 1 , such that $b^n < a$.

Choose b the greatest number, such that $b^n \nless a$;

$$\therefore \text{either } a > b^n \text{ or } a = b^n.$$

But if $a > b^n$, and c is any number $> b$, $c^n > a$; and

$$\begin{aligned} c^n - b^n &= (c-b)(c^{n-1} + c^{n-2}b + \dots + b^{n-1}), \\ &< \text{any assigned quantity, if } c-b \text{ is small enough,} \\ &< a - b^n, \quad \text{if } c-b \text{ is small enough.} \end{aligned}$$

But this requires $c^n < a$, which is not true ;

$$\therefore a \nless b^n ; \therefore a = b^n.$$

Also, the value of b^n increases ($b > 1$) as b increases, hence no other real positive quantity than b can satisfy the condition $b^n = a$.

This proves the theorem when $a > 1$, and by reciprocals we can prove when $a < 1$.

We can now define $a^{\frac{p}{q}}$ as the *principal q^{th} root* of a^p , proving the surd laws first and deducing from them the index laws for fractional indices.

The simplest method for the next stage is the decimal method of representing number, of which I have given a brief outline, sufficient for geometry, in my *Elementary Pure Geometry*.

It is easily shown that every real number a (defined as the measure of a magnitude X in terms of another Y as unit) can be *represented* by a decimal $d \cdot d_1 d_2 d_3 \dots d_n \dots$ *ad inf.*

This decimal may be regarded *pro tem.* as a symbol or mark and not as itself a number.

If $a_n = d_1 d_2 d_3 \dots d_n$, terminating in the n^{th} place, and $a'_n = a_n + \frac{1}{10^n}$, obtained by adding unity to the last digit of a_n , then we may call a_n, a'_n the n^{th} decimal approximations of a . Also,

(iv) Every $a_n < \text{any } a'_n$,

and $a > a_n$ but $\overline{\overline{a}} \leq a'_n$, whatever integer n is.

If the decimal $d_1 d_2 d_3 \dots d_n \dots$ is given, we easily show that there is one only number $a > a_n$ but $\overline{\overline{a}} \leq a'_n$, so that each decimal represents one only number; and the ordinary laws of operation are easily proved.

We may now define for an irrational index x , ' a^x is intermediate between a^{x_n} and $a^{x'_n}$ for all values of n , where x_n , and x'_n are the n^{th} decimal approximations of x .'

We prove first

(v) 'There is one only number, viz. a^x between a^{x_n} and $a^{x'_n}$ for all values of n .'

If $a > 1$, and b is the least quantity $>$ all a^{x_n} , and c is any quantity $> b$;

then
$$\frac{a^{x'_n}}{a^{x_n}} = a^{\frac{1}{10^n}} \left(\because x'_n = x_n + \frac{1}{10^n} \right).$$

But $\left(\frac{c}{b} \right)^{10^n} > a$, if n is great enough, by (i) above;

$$\therefore \frac{a^{x'_n}}{a^{x_n}} = a^{\frac{1}{10^n}} < \frac{c}{b}, \quad n \text{ great enough};$$

$$\therefore c > a^{x'_n} \text{ since } b < a^{x_n};$$

$\therefore$ there is one only number $b > a^{x_n}$ but $\overline{\overline{a}} \leq a^{x'_n}$ for all values of n .

Similarly for $a < 1$.

A similar method enables us to prove the index laws for irrational indices.

Negative indices and a^0 are defined as usual.

(vi) Every real number b can be expressed as some power of a given real number a .

Some powers of a can be found $> b$ and $< b$; and we have only to take the greatest power of a , not greater than b , or some equivalent according to the values of a and b relative to unity, to have a power $a^x = b$.

If now a is any positive number > 1 , a^x is a quantity which increases continuously from zero to 1 to $+\infty$ as x increases from $-\infty$ to zero to $+\infty$, and it has one real value for each value of x . The graph of a^x is therefore a continuous curve consisting of one branch.

II. Graph of $y = a^x$. $\int \frac{dy}{y}$.

If $PQ, P'Q'$ are chords of $y = a^x$, whose projections on the axis of x are equal ($MN = M'N'$), and if $PQ, P'Q'$ meet the x -axis in T, T' , we may call $TM, T'M'$ the subchords.

Let the coordinates of P, Q, P', Q' be $x, y; v, z; x', y'; v', z'$; then

$$\frac{z}{y} = \frac{a^v}{a^x} = a^{v-x} = a^{v'-x'} = \frac{a^{v'}}{a^{x'}} = \frac{z'}{y'};$$

$$\therefore \frac{TM}{PL} = \frac{PM}{QL} = \frac{y}{z-y} = \frac{y'}{z'-y'} = \frac{P'M'}{Q'L'} = \frac{T'M'}{P'L'};$$

$$\therefore TM = T'M';$$

i.e. the subchord is constant when the x projection of the chord is constant, and also the ratio of ordinates is constant.

Move Q, Q' to coincidence with P, P' , keeping $MN = M'N'$; then, in the limit,

subtangent $TM = T'M' = \text{constant} = k$ say.

$$\text{Also } \frac{dy}{dx} = \text{lt. of } \frac{QL}{PL} = \frac{PM}{\text{subtangent}} = \frac{y}{k} = \frac{\alpha^x}{k}.$$

It is easy to see that for different bases, this quantity k will have different values. k is the *modulus*.

We may now define e as that base whose subtangent or modulus is unity; hence, since y can be expressed as a power of e , we may write

$$y = e^x, \text{ modulus unity;}$$

$$\therefore \frac{dy}{dx} = \frac{y}{1} = e^x,$$

and

$$\int \frac{dy}{y} = \int dx = x = \log_e y.$$

Again, any real quantity y can be expressed as some power, say α^x , of α , and α can be expressed as some power, say e^c , of e ;

$$\therefore y = \alpha^x = e^{cx}, \text{ and } c = \log_e \alpha;$$

$$\therefore \frac{dy}{dx} = \frac{\alpha^x}{k} = ce^{cx} = c\alpha^x;$$

$$\therefore k = \frac{1}{c} = \frac{1}{\log_e \alpha} = \log_a e.$$

We thus have the modulus of any base α , in terms of e .

Also, if $y = e^x$,

$$\frac{dy}{dx} = e^x, \quad \frac{d^2y}{dx^2} = e^x, \text{ etc.; } e^0 = 1;$$

hence, by Taylor's theorem,

$$e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \dots,$$

and

$$e = 1 + 1 + \frac{1}{1 \cdot 2} + \dots$$

We thus arrive quite naturally at this fundamental theorem of the Theory of Functions.

III. If n ordinates to the curve are drawn at successive intervals along the x -axis equal to $\frac{1}{n}$, the ratio of any two consecutive ordinates is constant $= 1 + \frac{m}{n}$ say; hence, if a is the base,

$$\left(1 + \frac{m}{n}\right)^n = a \quad \left(\text{the ordinate at dist. } n \times \frac{1}{n}\right),$$

and

$$a^x = \left(1 + \frac{m}{n}\right)^{nx}.$$

If we now consider the part of the graph of $y = e^x$ from 0, 1 to 1, e , and if OI is divided into n equal parts, and the ratio of successive ordinates is $1 + \frac{m}{n}$;

then the subtangent TO at J =unity;

$\therefore$ the tangent of slope at J is 1, and $m > 1$;

also, if n is indefinitely increased, m becomes unity in the limit ($QL = PL$ in previous figure if P, Q coincide with J);

$$\therefore e = \text{lt. of } \left(1 + \frac{1}{n}\right)^n, \quad n \text{ infinite,}$$

and

$$e^x = \text{lt. of } \left(1 + \frac{1}{n}\right)^{nx}, \quad n \text{ infinite.}$$

We can thus connect the ordinary method of developing e with a characteristic property of the graph of $y = e^x$.

We easily derive the most satisfactory general method of differentiating x^n , when n is any real number whatever.

$$\text{Let } y = x^n, \quad x = e^z; \quad \therefore y = e^{zn};$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = ne^{zn} \times \frac{1}{e^z} = \frac{nx^n}{x} = nx^{n-1}.$$

The great advantage of the complete method, as I have given it, is that the continuity of a^x when x varies is fully established at the commencement; it therefore serves as an admirable example of a continuously varying function such as is implied in the fundamental processes of differentiation and integration.

E. BUDDEN.