XVII.—*On Superposition.* By the Rev. PHILIP KELLAND, M.A., Professor of Mathematics in the University of Edinburgh. (With a Plate.)

(Read 19th February 1855.)

The subject which I propose to discuss in this paper is the value of the method of demonstration by superposition. I am satisfied that it has been much underrated, and in some cases misunderstood. It may be stated, that the essential characteristic of this method of demonstration, is the mental comparison of two magnitudes, by placing one of them upon the other. EUCLID's axiom of equality (which, perhaps, is rather a definition) is this: "Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal." Accordingly, in his first four books, EUCLID never regards two magnitudes as equal, except under circumstances wherein it can be shown that this condition is satisfied. Only in one proposition has he avoided the labour which a strict attention to this requirement necessarily imposes; and perhaps, even in that case, it is hypercritical to object to what he has done. All that he assumes is this: it being admitted that when A fills the same space as B, A is equal to B; it must therefore be admitted, that when A and C together fill the same space as B and C together, A is also equal to B.

In the 6th book, as depending on the 5th, EUCLID makes another step in the assumptions necessary, and, in the 2d proposition, he admits the test of equality to be this: When two magnitudes can be multiplied equally to any extent, and when it can be shown, that on every occasion in which one exceeds or falls short of a magnitude, the other does the same exactly; then the two cannot fill other than the same space.

For the next step in this line of argument, we must advance to NEWTON's Principia. In the 7th Lemma will be found a beautiful process, by which it is shown that two inconceivably small magnitudes are, as they get smaller, approaching to a ratio of equality. The process is simply that of applying a minute magnifying power, so that one of the things to be compared shall always be magnified up to a fixed standard.

This is the *method* of demonstration in geometry. It is not too much to say that geometry owes nine-tenths of its value, as an educational agent, to its being a consistent and systematic exemplification of this method. The treatises of LEGENDRE, in French, and Sir JOHN LESLIE, in English, in which *method* is set at defiance, however valuable as introductory to other branches of science, are com-

paratively useless as elements of training. The loose way in which many of these treatises are compiled, causes us to be astonished at the celebrity they have attained. In the treatise of LEGENDRE, for instance, the author attempts, in his first edition, to place the doctrine of parallels on a more simple basis,—at that time, it may be presumed he was tolerably ignorant of the real state of the question,—but when his book became pretty generally adopted, he gave his close attention to the subject, and became convinced that his emendation required amendment. New editions were called for, and new amendments succeeded the old ones, until, in the tenth edition, we are presented with the following significant words as an advertisement: “By the advice of several distinguished professors, I have determined to re-establish, in this tenth edition, the theory of parallels nearly on the same basis as EUCLID’S.”

The author has thus sailed round nearly all the points of the compass, and agreeably to his own confession arrives nearly at the point at which he started. *Nearly*, he says; and, truth to tell, not so very nearly at all. There is certainly an abandonment of all novelty in the exposition of the doctrine; but any one may see how very far the author is behind EUCLID, even in his tenth edition. If we could examine the last edition, we should probably find EUCLID’S method completely restored. At any rate, LEGENDRE tells us, in his note, that he is not satisfied with the theory as it stands, and he attributes its imperfection to the definition of a straight line; but whether he means his own definition, which is imperfect enough, no doubt, or EUCLID’S, does not appear.

But it is time we should leave LEGENDRE, and offer one positive argument in favour of the method of demonstration by superposition. It is admitted, I think, that a chain of reasoning upon an abstract definition is the most healthful exercise of the mental powers, at least in the days of youth. Viewed as such, EUCLID’S Elements stand above all other writings; but a class of objectors of a totally different order from LEGENDRE has arisen, who, with considerable show of reason, urge against geometry, as based on superposition, that it excludes all exercise of ingenuity, inasmuch as it only demands a uniform and unvaried march, to deviate from which is to wander into error. There is some truth in this,—but it is not altogether true; and I have here exhibited, in reply to it, some of the different solutions of a single problem where we are bound down by a specific requirement; and it will be seen that the solutions present themselves in tolerable variety, and that their discovery must have brought out some ingenuity.

The problem was proposed to me by the late lamented Secretary of this Society, Sir JOHN ROBISON. I gave him the first solution. The others have arisen out of it, partly from my own suggestions, partly from my students’ exercises. It is certainly a very remarkable problem. I have never met with one which presents such a variety of altogether independent solutions. There can be no doubt that the problem has appeared in the Ladies’ Diary, or elsewhere, but I

have never met with it, and I do not suppose, at any rate, that half the solutions have been dreamt of before. About these solutions I have not many words to say. It may suffice if I describe the first, and indicate the others by a letter or two placed in the figures. (See Plate V.)

PROBLEM. *From a given square one quarter is cut off, to divide the remaining gnomon into four such parts that they shall be capable of forming a square.*

Let ABE be the gnomon. Let AX be a side of the square which is equal to this gnomon (Euc., ii, 14): call it x , and call AB a . In Figure I. draw HK parallel to AB and equal to $x - a$.

Then (1), (2), and (4) will unite as in Figure II. Also (3) will fit in between (1), (2), and (4). Now, if the four do not make up a square, either (3) will reach below P, or (2) will not reach so low as P, or *vice versâ*, seeing that the area is equal to the square of MP. But neither of these circumstances can happen, because then $HK + GE$ would be unequal to MR or x , which it is not. It follows that the four parts exactly make up a square.

The *second* method of making the sections differs from the first only in the different method of producing the portions (2) and (4).

The *third* differs from the first in the way of producing (2) and (3). This method may be modified *ad libitum*. It is only necessary that the portion which (3) takes from (2) shall be symmetrical with respect to C and K.

The *fourth*, like the second, is a slight modification of (2) and (4) in the first. An isosceles triangle is taken out of (4) into (2).

The *fifth* is a modification of the second. A figure equal to (4) is cut out of (2) and remains in (3), whilst (4) itself replaces that figure in (2).

The *sixth* is a new form: (3) and (4) instead of being, as in the first method, cut from the right side, are cut from the left and reversed.

The *seventh* is again a new form.

The *eighth* has two pieces in common with the first method. In other respects it is new.

The *ninth* is a slight modification of the eighth, the piece (4) being cut from the top instead of from the bottom.

The *tenth* is a modification of the seventh.

The *eleventh* is another modification of the eighth; the piece is now cut out of the top of (3).

The *twelfth* is new; and the demonstration is effected by showing that the figures form a rectangular parallelogram, of which one side is x .

I must not omit to add, that the major part of these solutions are due to the ingenuity of my students.