On the Foundations of Dynamics.

By Dr PEDDIE.

Note on a Theorem in connection with the Hessian of a Binary Quantic.

By CHARLES TWEEDIE, M.A., B.Sc.

Extension of the "Medial Section" problem (Euclid II:11, VI:30, etc.) and derivation of a Hyperbolic Graph.

By R. E. ANDERSON, M.A.

To divide the straight line AB (containing a units) at C so that
 $AB \cdot BC = p \cdot AC^2$.

§ I.

By algebra, taking the positive root,

$$AC = \frac{AB}{2p} (\sqrt{4p+1} - 1), \quad (1.)$$

The number p may therefore have any positive value, integral or fractional, and when negative cannot exceed $\frac{1}{4}$. Secondly, AC and AB are incommensurable except when $4p+1$ is a square:—*e.g.*, if $4p = (q-1)(q+1)$ or if $p = q(q+1)$, q being any positive integer or fraction.

To find the surd-line $\sqrt{4p+1}$ geometrically is the heart of the problem. *Euclid* solves it (II:11) when $p=1$ by I:47, which is also used in Ex. i., ii., iii. following; but II:14 will sometimes be easier. Since equation (1.) becomes $AC = \sqrt{4p+1} - 1$, if $AB = 2p$, *i.e.*, if the unit line is $\frac{AB}{2p}$ or $\frac{AM}{p}$ *, we construct thus:—

(Figure 1.)

Ex. i. Solve $AB \cdot BC = 3AC^2$. Here $p = 3$, surd = $\sqrt{13}$.Take $AK = \frac{1}{3}AM$, $AR = 2AK$, and $KS = RM$.

C the point required is determined by a square described on AS.

(Figure 2.)

Ex. ii. Solve $AB \cdot BC = 4AC^2$. Here $p = 4$, surd = $\sqrt{17}$.Take $AK = \frac{1}{4}AM$ and $KS = KM$.

C the point required is determined by square on AS.

(Figure 3.)

Ex. iii. Solve $AB \cdot BC = 7AC^2$. Here $p = 7$, surd = $\sqrt{29}$.Take $AK = \frac{1}{7}AM$, $AR = 2AK$, $AT = 5AK$, and $KS = RT$.

C is determined by square on AS.

(Figure 4.)

Ex. iv. Solve $AB \cdot BC = AC^2$ by II : 14. Here $p = 1$, surd = $\sqrt{5}$.
 Produce AB both ways till $AK = AM$ the unit, and $AH = 5AM$.
 $\therefore AR = \sqrt{5}$ if $\perp AB$ and limited by $\frac{1}{2} \odot$ on KH : $\therefore$ if $KC = AR$,
 C gives the medial section of AB; and if $KC' = KC$, C' is point of
 external section, corresponding to the negative root of equation (1.).

(Figure 5.)

Ex. v. Solve $AB \cdot BC = \frac{2}{3}AC^2$. Here $p = \frac{2}{3}$, surd = $\sqrt{\frac{11}{3}} = \frac{1}{3}\sqrt{33}$.
 Produce AB both ways. In AB produced take $AK = \frac{3}{2}AM$,
 $AW = 3AK$, and $AY = 11AK$. $\therefore AR = \sqrt{3 \times 11}$ if $\perp AB$ and
 limited by $\frac{1}{2} \odot$ on WY. Hence if $KC = \frac{1}{3}AR$, C is the point
 required; and if $KC' = KC$, C' is the point of external section, as
 in Ex. iv. above.

Thus for a given value of p the surd number $\sqrt{4p+1}$ though
 intractable to Arithmetic can always be found by Geometry. In
 certain cases lower surds should be subsidised: thus for $\sqrt{33}$
 (just found by II : 14) we may say $33 = 5^2 + (2\sqrt{2})^2$.

Similarly $21 = 4^2 + (\sqrt{5})^2$; $181 = 13^2 + (2\sqrt{3})^2$.

* In the diagrams M is the mid-point of AB and in Figs. 1, 2, 3,
 SAK is $\perp AB$.

§ II.

(Figure 6.)

For the general case of equation (1.), the surd line $\sqrt{4p+1}$ can be expressed either by I:47, since $4p+1 = (2\sqrt{p})^2 + 1^2$, or by II:14. Thus, by the latter, produce AB both ways till

$AK = \text{the unit} = \frac{AB}{2p}$ and $BH = BK$, so that $AH = 4p+1$ units.
 $\therefore AR = \sqrt{4p+1}$ if $\perp AB$ and limited by $\frac{1}{2} \odot$ on KH. Hence if $KC = KC' = AR$, C and C' are the required points of section.

(Figure 7.)

Mr G. Duthie suggests a third general construction for eq. (1.).
 "Produce AB till $BK = MB = MA$, and take $MH = \frac{AM}{p}$: with centre K and radius KM describe $\odot MQW$. Finally take $HR = HQ$ the tangent from H. Then if $AC = MR$, C is the point required."

As shown above, C and C' are two points in AB or its production which determine the roots of the original quadratic.

Thus, if $AR = \frac{AB}{2p}$, then

(a.), when p is positive, with limits ∞ and 0,

$$AC = AR(\sqrt{1+4p} - 1), \text{ and } AC' = -AR(\sqrt{1+4p} + 1),$$

$\therefore$ as AR grows continuously from 0 to ∞ , so C and C' move further apart from A in opposite directions;

(β .), when p is negative, with limits $-\frac{1}{4}$ and $-\frac{1}{\infty}$,

$$AC = AR(1 - \sqrt{1+4p}), \text{ and } AC' = AR(1 + \sqrt{1+4p}),$$

$\therefore$ as AR grows continuously from $-2a$ to $-\infty$, so C and C' move in opposite directions further and further apart from Z which is a point in AB produced positively so that $AZ = 2a$.

(γ .). At any instant both for (α .) and (β .), the distance $CC' =$ twice the surd-line; and, numerically, $AC' - AC = 2AR$.

We may note also from equation (1.) that generally $\frac{AO}{AM} = \frac{\sqrt{4p+1}-1}{p}$; $\therefore AC \leq AM$ according as $\sqrt{4p+1} \leq 1+p$, or as $2 \leq p$: [Thus $AC = \frac{1}{2}AB$ when $p=2$; $AC > \frac{1}{2}AB$ when $p < 2$ as in *Euclid* II:11 and Ex. v. above; $AC < \frac{1}{2}AB$ when $p > 2$ as Ex. i., ii., iii.], (2.)

§ III.

(Figure 8.)

To show graphically the variation of the segment AC, as obtained by equation (1.), I have placed P_1R_1 P_2R_2 P_3R_3 perpendicular to the fixed line AB, so that

$P_1R_1 = AR_1(\sqrt{5}-1)$, $P_2R_2 = AR_2(\sqrt{9}-1)$, $P_3R_3 = \text{etc., etc.,}$
and generally $PR = AR(\sqrt{4p+1}-1)$, where $PR = AC$, $AR = \frac{AB}{2p}$

$$\begin{aligned} \therefore PR + AR &= PS = AR \sqrt{4p+1} = \frac{AS}{\sqrt{2}} \sqrt{4p+1} \\ \therefore \left. \begin{aligned} 2PS^2 &= AS^2(4p+1) \\ &= AS(2a\sqrt{2}+AS) \\ &= AS.A'S, \end{aligned} \right\} \text{since } 4p = \frac{2a}{AR} = \frac{2a\sqrt{2}}{AS}, \end{aligned} \quad (3.)$$

Hence for P, any point in the locus, the square of PS has a constant ratio to the rectangle AS . A'S, and that is the geometrical property of a Hyperbola having A'OASE as a diameter and PS an ordinate to it. Thus $PS = SQ$, FAD is a tangent at A, F'A' another at A', and O is the centre of the curve.

With reference to the original problem, PS (or SQ) is the surd-line $\sqrt{4p+1}$, AR (or RS) the unit-line, and the two roots of equation (1.) are PR and RQ. If, for example,

$$\left. \begin{aligned} p=1, SP &= SQ = \sqrt{5} \\ PR &= \sqrt{5}-1 = AC, \text{ internal segment} \\ RQ &= -\sqrt{5}-1 = AC', \text{ external segment} \end{aligned} \right\} \begin{array}{l} \text{cf. Ex. iv. of § I.} \\ \text{and } Euclid \text{ II:11.} \end{array}$$

The ordinate at P' if produced upward to meet the branch P'V'A' in Q', and downward to meet AB in R', gives P'R' = AC and Q'R' = AC', for p negative. The Euclidian solution therefore of this case must place C and C' in AB produced through B, as already shown, § II.

§ IV.

The Cartesian equation to this Hyperbola is at once derived from equation (3.) thus, choosing OFG as the axis of x ,

$$\begin{aligned} 2PS^2 &= AS \cdot A'S = (OS - a\sqrt{2})(OS + a\sqrt{2}) \\ &= OS^2 - 2a^2 \end{aligned} \left. \begin{array}{l} \\ \end{array} \right\} \text{putting } x+y \text{ for PS and } x\sqrt{2} \text{ for OS.}$$

$$\therefore 2(x+y)^2 = 2x^2 - 2a^2 \quad \therefore y^2 + 2xy + a^2 = 0. \quad (4.)$$

The form of this equation shows that one asymptote is parallel to the axis of x $\therefore$ OFG is that asymptote. But $y^2 + 2xy = y(y + 2x)$, $\therefore y + 2x = 0$ is the other asymptote, viz., the line OD. Thus V'OV bisecting the angle FOD is the transverse axis of the Hyperbola, V and V' are the vertices.

Finally, referring the curve to its own axes, equation (4.) becomes

$$\frac{x^2}{\sqrt{5}+1} - \frac{y^2}{\sqrt{5}-1} = \frac{a^2}{2}, \quad \text{or} \quad \left(\frac{x}{h}\right)^2 - \left(\frac{y}{k}\right)^2 = 1, \quad (5.)$$

where $h^2 = OV^2 = \frac{a^2}{2}(\sqrt{5}+1) = 2a^2 \cos \frac{\pi}{5}$

$$k^2 = OW^2 = \frac{a^2}{2}(\sqrt{5}-1) = 2a^2 \cos \frac{2\pi}{5}$$

Thus $hk = a^2 = \frac{1}{4}LL'$, where L is the lat. rectum of the curve, and L' that of its conjugate. Hence

$$L = 4k \cos 72^\circ.$$

A curious property therefore of our Hyperbola is that

$$\frac{L}{WW'} = \frac{WW'}{VV'} = \frac{VV'}{L'} = 2 \cos 72^\circ, \quad (6.)$$

In other words, an isosceles Δ satisfying *Euclid* IV : 10 is found by taking the two terms of any one of those three fractions, and making the numerator the base.

Another singular property, easily deduced, is that

$$r^{\frac{2}{3}} + a^{\frac{2}{3}} = (4L)^{\frac{2}{3}}, \quad (7.)$$

where r = radius of curvature at the extremity of L the latus rectum.