The Complete Solution of the Differential Equation of $J_{[n]}$. By the **REV. F. H. JACKSON**, H.M.S. "Irresistible."
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In connection with the function $J_{[n]}$, it may be of interest to give briefly the complete solution of the differential equation satisfied by the function $J_{[n]}$. The method of Frobenius will be employed. Consider the differential equation

$$px^{[2]}f^{(2)} \left\{ 1 - [n] - [-n] \right\} x^{[1]}f^{(1)}(x) \\ + x^{[2]}f(x^{p^2}) + [n][-n] f(x) = 0 \quad \dots (a)$$

in which

$$[n] = \frac{p^n - 1}{p - 1} \\ f^{(2)}(x) = \frac{d}{d(x^p)} \left\{ \frac{df}{dx} \right\}$$

If $p = 1$, the equation reduces to

$$x^2 f'' + x f' + (x^2 - n^2) f = 0,$$

which is Bessel's equation for functions of order n .

Substituting an expression

$$f(x) = \sum c_s x^{[a+s]}$$

in the equation (a), we have an indicial equation

$$[a+n][a-n] = 0,$$

and an indicial function

$$f = c_0 \left[x^{[a]} - \frac{x^{[a+2]}}{[a-n+2][a+n+2]} + \frac{x^{[a+4]}}{[a-n+2][a+n+2][a-n+4][a+n+4]} - \dots \right]$$

The principal roots of the indicial equation are

$$a = +n, \quad a = -n.$$

If n be not an integer, the corresponding integrals are $J_{[n]}$ and $J_{[-n]}$.

$$J_{[n]}(x) = \frac{1}{[2]^{-n} \Gamma_{p^2}([n+1])} \left\{ x^{[n]} - \frac{x^{[n+2]}}{[2][2n+2]} + \dots \right\}$$

$$J_{[-n]}(x) = \frac{1}{[2]^{-n} \Gamma_{p^2}([1-n])} \left\{ x^{[-n]} - \frac{x^{[-n+2]}}{[2][2-2n]} + \dots \right\}$$

If $n=0$ these integrals are identical, while if n be an integer, one or other becomes ineffective according as n is positive or negative. In these cases, then, it is necessary to form a second distinct and effective integral corresponding to Hankel's solution of Bessel's equation.

When a is integral, we write¹

$$f(x) = C \left\{ [a+2n] - [n] \right\} \left\{ [a+2n] - [-n] \right\}$$

$$\left[x^{[a]} - \frac{x^{[a+2]}}{[a-n+2][a+n+2]} + \dots \right.$$

$$\left. \dots - (-1)^n \frac{x^{[a+2n-2]}}{\prod_{r=1}^{n-1} \left\{ [a+2r] - [n] \right\} \left\{ [a+2r] - [-n] \right\}} \right]$$

$$+ E \left[x^{[a+2n]} - \left\{ [a+2n+2] - [n] \right\} \left\{ [a+2n+2] - [-n] \right\} \right.$$

$$\left. + \dots \right]$$

$$= \omega_1 + \omega_2.$$

¹ Cf. Forsyth, *Theory of Differential Equations*, vol. iii. pp. 101, 102.

From

$$(\omega_1 + \omega_2)_{a=-n}, \left(\frac{\partial \omega_1}{\partial a} + \frac{\partial \omega_2}{\partial a} \right)_{a=-n}$$

we obtain

$$(\omega_1)_{a=-n} = 0 \quad \dots \quad (1)$$

$$(\omega_2)_{a=-n} = [2]^n \Gamma_p([n+1]) E J_{[n]}(x) \quad \dots \quad (2)$$

$$\left(\frac{\partial \omega_1}{\partial a} \right)_{a=-n} = C \frac{\log p}{p-1} \sum_{r=1}^{n-1} (-1)^r \frac{[2n]}{[2][4] \dots [2r] \cdot [2-2n] \dots [2r-2n]} x^{[2r-n]} \quad (3)$$

$$\begin{aligned} \left(\frac{\partial \omega_2}{\partial a} \right)_{a=-n} &= E \frac{\log p}{p-1} \log x \sum_{r=1}^{r=\infty} (-1)^r \frac{p^{n+2r} x^{[n+2r]}}{[r]![n+r]!(2)_r(2)_{n+r}} \\ &\quad - E \frac{\log p}{p-1} \sum_{r=1}^{r=\infty} (-1)^r \left\{ \frac{p^2}{[2]} + \frac{p^4}{[4]} + \dots + \frac{p^{2r}}{[2r]} + \frac{p^{2n+2}}{[2n+2]} + \dots \right. \\ &\quad \left. \dots + \frac{p^{2n+2r}}{[2n+2r]} \right\} \frac{x^{[n+2r]}}{[r]![n+r]!(2)_r(2)_{n+r}} \quad \dots \quad (4) \end{aligned}$$

If we give C the value $-\frac{(2)_{n-1} \Gamma_p([n])}{[n]}$, so that $E = \frac{1}{(2)_{n-1} \Gamma_p([n+1])}$

or what is equivalent

$$E = \frac{1}{[2]^{n-1} \Gamma_p([n+1])}$$

we obtain an integral from (3) and (4) which may be termed $W_1 + W_2$. If $p=1$, this integral reduces to that given on p. 102, vol. iii., Forsyth's *Theory of Differential Equations*.

In the case when $n=0$, the integral is

$$\begin{aligned} f_{=c} J_{[0]}(p, x) \log x - c \sum_{s=1}^{s=\infty} \left\{ \frac{2p^2}{[2]} + \frac{2p^4}{[4]} + \dots \right. \\ \left. \dots + \frac{2p^{2s}}{[2s]} \right\} \frac{x^{[2s]}}{[2]^2 [4]^2 \dots [2s]^2} \quad \dots \quad (5) \end{aligned}$$

These functions satisfy the same recurrence equations as the

function $J_{[n]}$ given in the *Transactions* of the Society, vol. xli., part i., Nos. (1) and (6). The expression for $\Gamma_p([x])$ given on p. 105, No. 6, vol. xli., should be

$$\frac{p^{\frac{1}{2}x(x-1)}}{\Gamma_p([x])} = [x]e^{px} \prod_{s=1}^{s=\infty} \left\{ \left(1 + p^{-x} \frac{[x]}{[s]} \right) e^{\frac{x}{[s]}} \right\}.$$

The class of differential equations integrable by Bessel's functions, and discussed by Lommel in vol. xiv., *Mathematische Annalen*, may without difficulty be formally extended in the same way that Bessel's equation and its solutions have been extended in the above work.

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