envelope of circles which cut S orthogonally and have their centres on the parabola $\alpha^2 - \beta = 0$ (see Casey's "Bicircular Quartics").

This latter result was proposed as a question in the *Educational Times*, July, 1866, by Prof. Sylvester. (See *Educational Times*, August, 1866, for Prof. Cayley's solution of the above; also October, 1866, for a solution by Prof. Crofton.)

Note on certain Symbolic Operators, and their application to the Solution of certain Partial Differential Equations. By J. W. L. GLAISHER, M.A., F.R.S.

[Read June 9th, 1881.]

1. Poisson's well-known theorem .

$$e^{a^2 D^2} \phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} \phi(x + 2au) du \dots\dots\dots(1),$$

where D denotes $\frac{d}{dx}$, may be proved very simply as follows.

We have
$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-u^2} du \dots\dots\dots(2),$$

whence, putting $u - a$ for u , a being a constant,

$$\begin{aligned} \sqrt{\pi} &= \int_{-\infty}^{\infty} e^{-(u-a)^2} du \\ &= e^{-a^2} \int_{-\infty}^{\infty} e^{-u^2 + 2au} du, \end{aligned}$$

and therefore
$$\sqrt{\pi} e^{a^2} = \int_{-\infty}^{\infty} e^{-u^2 + 2au} du \dots\dots\dots(3).$$

Writing aD in place of a , and taking $\phi(x)$ as the subject of operation,

$$\begin{aligned} \sqrt{\pi} e^{a^2 D^2} \phi(x) &= \int_{-\infty}^{\infty} e^{-u^2 + 2auD} du \cdot \phi(x) \dots\dots\dots(4) \\ &= \int_{-\infty}^{\infty} e^{-u^2} \phi(x + 2au) du. \end{aligned}$$

2. This proof is rigorous, for (3), regarded as an equation involving a , is true identically; that is to say, if both members of the equation are expanded in powers of a , we have

$$\sqrt{\pi} \left(1 + a^2 + \frac{a^4}{2!} + \&c. \right) = \int_{-\infty}^{\infty} e^{-u^2} \left(1 + 2au + \frac{(2au)^2}{2!} + \&c. \right) du \dots\dots(5),$$

and it is easily seen that the coefficients of the same powers of a on each side of the equation are equal to one another, the terms involving

uneven powers of a on the right-hand side being zero. When therefore a is replaced by aD , the operations indicated by each side of the equation are identical; and thus (4), and therefore also (1), must be true.

3. It may be remarked that (1) may be readily verified by expanding $\phi(x + 2au)$ by Taylor's theorem in ascending powers of a , and showing that the coefficient of a^{2n} on the right-hand side of the equation is $\frac{\phi^{(2n)}(x)}{n!}$. This method of proving (1), which is that which most naturally presents itself, is, it will be observed, equivalent to actually verifying the equality, term by term, of the series in (5). In the proof given in § 1, the equality of these two series is deduced at once from the formula (2).

4. Professor Crofton's formula,*

$$e^{aD^2} e^{kx^2} = (1 - 4ak)^{-\frac{1}{2}} e^{\frac{kx^2}{1 - 4ak}},$$

may, as he has remarked himself, be readily deduced from (1); for, putting $\phi(x) = e^{kx^2}$, we find

$$\begin{aligned} e^{aD^2} e^{kx^2} &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} e^{k(x+2\sqrt{a}u)^2} du \\ &= \frac{1}{\sqrt{\pi}} e^{kx^2} \int_{-\infty}^{\infty} e^{-u^2 + 4akxu + 4\sqrt{a}kxu} du \\ &= \frac{1}{\sqrt{\pi}} e^{kx^2} \frac{\sqrt{\pi}}{\sqrt{1 - 4ak}} e^{\frac{4ak^2 x^2}{1 - 4ak}} = (1 - 4ak)^{-\frac{1}{2}} e^{\frac{kx^2}{1 - 4ak}}. \end{aligned}$$

5. The formula may, however, be proved somewhat differently as follows. Writing kx^2 for a^2 in (3), we have

$$\sqrt{\pi} e^{kx^2} = \int_{-\infty}^{\infty} e^{-u^2 + 2\sqrt{k}xu} du.$$

Now

$$\phi(D) e^{mx^2} = \phi(m) e^{mx^2},$$

and therefore

$$\begin{aligned} \sqrt{\pi} e^{aD^2} e^{kx^2} &= \int_{-\infty}^{\infty} e^{-u^2} a^2 D^2 e^{\frac{1}{2}\sqrt{k}xu} du \\ &= \int_{-\infty}^{\infty} e^{-u^2} e^{a^2 \cdot 4kx^2} e^{\frac{1}{2}\sqrt{k}xu} du \\ &= \int_{-\infty}^{\infty} e^{-(1 - 4ak)u^2 + 2\sqrt{k}xu} du \\ &= \frac{\sqrt{\pi}}{\sqrt{1 - 4ak}} e^{\frac{kx^2}{1 - 4ak}}. \end{aligned}$$

* *Philosophical Transactions*, 1870, pp. 186, 187.

6. The method of proof employed in the last section shows that in general,

$$\phi(D) e^{kx} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2 + 2x\sqrt{k}u} \phi(2\sqrt{k}u) du;$$

or, transforming the integral, that

$$\phi(D) e^{kx} = \frac{1}{2\sqrt{(k\pi)}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{4k} + xv} \phi(v) dv.$$

7. Poisson's theorem enables us to express $e^{aD}\phi(x)$ as a single definite integral, and it seems worthy of notice that by the repeated application of this theorem we may express $e^{aD}\phi(x)$ as a double definite integral, $e^{aD}\phi(x)$ as a triple definite integral, and so on.

For

$$e^a = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2 + 2\sqrt{a}v} dv,$$

whence

$$\begin{aligned} e^{aD}\phi(x) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2 + 2aDv} dv \cdot \phi(x) \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} dv \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2 + 2\sqrt{a}v\sqrt{a}u} du \cdot \phi(x) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2 - v^2} \phi(x + 2^{1+\frac{1}{2}} auv^{\frac{1}{2}}) du dv; \end{aligned}$$

and, similarly, we find

$$e^{aD}\phi(x) = \frac{1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2 - v^2 - w^2} \phi(x + 2^{1+\frac{1}{2}} auv^{\frac{1}{2}}w^{\frac{1}{2}}) du dv dw \dots (6),$$

$$e^{aD}\phi(x)$$

$$= \frac{1}{\pi^{\frac{3}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2 - v^2 - w^2 - t^2} \phi(x + 2^{1+\frac{1}{2}} auv^{\frac{1}{2}}wt^{\frac{1}{2}}) du dv dw dt,$$

and so on, the general law being evident.

These results may be verified by expanding the function subject to the sign ϕ , by Taylor's theorem, in ascending powers of a , and integrating term by term. Taking for example (6), it is easy to see that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2 - v^2 - w^2} (uv^{\frac{1}{2}}w^{\frac{1}{2}})^n du dv dw,$$

viz.,

$$\int_{-\infty}^{\infty} e^{-u^2} u^n du \times \int_{-\infty}^{\infty} e^{-v^2} v^{\frac{1}{2}n} dv \times \int_{-\infty}^{\infty} e^{-w^2} w^{\frac{1}{2}n} dw,$$

is equal to zero unless n is a multiple of 8, and to verify that, when n is a multiple of 8, the coefficient of a^n on the right-hand side of the equation (6) is equal to $\frac{\phi^{(n)}(x)}{(\frac{1}{8}n)!}$.

8. If, now, in the known formula

$$e^{-a} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2 - \frac{a^2}{4u^2}} du \dots\dots\dots (7),$$

we substitute $a\sqrt{D}$ for a , and operate upon $\phi(x)$, we find

$$e^{-a\sqrt{D}} \phi(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2 - \frac{a^2 D}{4u^2}} du \cdot \phi(x) \dots\dots\dots (8),$$

and therefore
$$e^{-a\sqrt{D}} \phi(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} \phi\left(x - \frac{a^2}{4u^2}\right) du \dots\dots\dots (9).$$

In connexion with this result, it is especially to be noticed that the formula (7), unlike (3) in § 1, cannot be proved by expanding both sides of the equation in powers of a , and verifying the equality of the coefficients, for the integral involves a only through a^2 , and if it is expanded in powers of a^2 the coefficients are infinite.

The substitution of $a\sqrt{D}$ for a has therefore not been shown to be justifiable, for $e^{a\sqrt{D}}$ can be interpreted only as indicating the performance of the operations

$$1 - aD^{\frac{1}{2}} + \frac{a^2 D}{2!} - \frac{a^3 D^{\frac{3}{2}}}{3!} + \&c.,$$

and it does not appear that the operations indicated by the integral in (8) are identical with these.

It will be noticed that, in (9), if the sign of a is changed, the left-hand member becomes $e^{a\sqrt{D}} \phi(x)$, but the right-hand side remains unaffected.

9. By the repeated application of (8), it can be shown that

$$e^{-aD^{\frac{1}{2}}} \phi(x) = \left(\frac{2}{\sqrt{\pi}}\right)^2 \int_0^{\infty} \int_0^{\infty} e^{-u^2 - v^2} \phi\left(x - \frac{a^2}{2u^2v^2}\right) du dv,$$

$$e^{-aD^{\frac{1}{2}}} \phi(x) = \left(\frac{2}{\sqrt{\pi}}\right)^3 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-u^2 - v^2 - w^2} \phi\left(x - \frac{a^2}{2u^2v^2w^2}\right) du dv dw,$$

and so on, the general law being evident.

If, in the first of these formulæ, we replace a by $-a$, ai or $-ai$, the exponent on the left-hand side becomes $aD^{\frac{1}{2}}$, $aiD^{\frac{1}{2}}$, or $-aiD^{\frac{1}{2}}$, but the right-hand side remains unaltered; similarly, in the second formula, there are eight values of the exponent on the left-hand side which correspond to the same value of the right-hand member, and so on.

The remainder of this note relates to the application of the symbolic formulæ, which have been considered in this and the preceding sections, to the expression of the solutions of certain partial differential equations.

10. The linear partial differential equation

$$\frac{dz}{dx} = a \frac{dz}{dy}$$

may, as is well known, be solved as follows.

Let a denote $a \frac{d}{dy}$, then the differential equation is

$$\frac{dz}{dx} = az,$$

and therefore

$$z = Ce^{ax} = e^{ax \frac{d}{dy}} \phi(y) \\ = \phi(y + ax).$$

The solution in a form involving an arbitrary function is thus obtained by means of the symbolic expression for the series in Taylor's theorem.

In the same manner we find that the solution of the differential equation

$$\frac{dz}{dx} = a^2 \frac{d^2 z}{dy^2} \dots \dots \dots (10)$$

is

$$z = e^{a^2 x \frac{d^2}{dy^2}} \phi(y) \dots \dots \dots (11),$$

and therefore, by (1),

$$z = \int_{-\infty}^{\infty} e^{-u^2} \phi(y + 2a \sqrt{x} u) du \dots \dots \dots (12);$$

so that, in this case, Poisson's theorem enables us to express the solution in a form involving an arbitrary function.

Similarly, the solution of the differential equation

$$\frac{dz}{dx} = a^4 \frac{d^4 z}{dy^4}$$

may, by § 7, be expressed in the form

$$z = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2 - v^2} \phi(y + 2^{\frac{1}{2}} a x^{\frac{1}{2}} uv) du dv;$$

and we obtain also solutions, in which z is equated to an expression involving an arbitrary function, of the differential equations

$$\frac{dz}{dx} = a^3 \frac{d^3 z}{dy^3}, \text{ \&c.}$$

11. Returning to the differential equation (10), if we denote $\frac{1}{a^2} \frac{d}{dx}$ by β , the equation becomes

$$\frac{d^2 z}{dy^2} = \beta z,$$

and therefore

$$z = Ae^{\sqrt{\beta} y} + Be^{-\sqrt{\beta} y} \\ = e^{\frac{\gamma}{a} \sqrt{\frac{d}{dx}}} f(x) + e^{-\frac{\gamma}{a} \sqrt{\frac{d}{dx}}} F(x) \dots \dots \dots (13),$$

and either of these two terms leads, by § 8, to the formula

$$z = \int_0^{\infty} e^{-u^2} f\left(x - \frac{y^2}{4a^2u^2}\right) du \dots\dots\dots (14).$$

From this expression for z , we find

$$\frac{d^2z}{dy^2} = \int_0^{\infty} e^{-u^2} f''\left(x - \frac{y^2}{4a^2u^2}\right) \frac{y^2}{4a^4u^4} du - \int_0^{\infty} e^{-u^2} f'\left(x - \frac{y^2}{4a^2u^2}\right) \frac{1}{2a^2u^2} du,$$

$$\frac{dz}{dx} = \int_0^{\infty} e^{-u^2} f'\left(x - \frac{y^2}{4a^2u^2}\right) du,$$

and, substituting these expressions in the differential equation,

$$\begin{aligned} & \int_0^{\infty} e^{-u^2} f''\left(x - \frac{y^2}{4a^2u^2}\right) \frac{y^2}{4a^4u^4} du \\ &= \int_0^{\infty} e^{-u^2} f'\left(x - \frac{y^2}{4a^2u^2}\right) \frac{1}{2a^2u^2} du + \frac{1}{a^2} \int_0^{\infty} e^{-u^2} f'\left(x - \frac{y^2}{4a^2u^2}\right) du. \end{aligned}$$

Integrating by parts, the left-hand member

$$= \frac{1}{a^2} \left[\frac{e^{-u^2}}{2u} f'\left(x - \frac{y^2}{4a^2u^2}\right) \right]_0^{\infty} + \frac{1}{a^2} \int_0^{\infty} f'\left(x - \frac{y^2}{4a^2u^2}\right) \left(e^{-u^2} + \frac{1}{2u^2} e^{-u^2} \right) du,$$

so that the differential equation is satisfied if $\frac{e^{-u^2}}{2u} f'\left(x - \frac{y^2}{4a^2u^2}\right)$ vanishes when $u = 0$. We observe also that f must be such that both the integrals in the expression for $\frac{d^2z}{dy^2}$ are finite.

Subject to these conditions, the formula (14) satisfies the differential equation; and it may be remarked, in reference to these conditions, that it seems pretty clear that they will generally be satisfied whenever the form of the function f is such that the integral in (14) is finite.

12. The formula (12) satisfies the differential equation unconditionally, for we have

$$\frac{d^2z}{dy^2} = \int_{-\infty}^{\infty} e^{-u^2} \phi''(y + 2a\sqrt{x}u) du,$$

$$\frac{dz}{dx} = \int_{-\infty}^{\infty} e^{-u^2} \phi'(y + 2a\sqrt{x}u) \frac{au}{\sqrt{x}} du,$$

and, integrating the expression for $\frac{d^2z}{dy^2}$ by parts, we find that it

$$= \left[\frac{e^{-u^2} \phi'(y + 2a\sqrt{x}u)}{2a\sqrt{x}} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-u^2} \phi'(y + 2a\sqrt{x}u) \frac{u}{a\sqrt{x}} du,$$

of which the first term is zero owing to the presence of the factor e^{-u^2} .

13. In his "Examples of the Processes in the Differential and Integral Calculus" (Cambridge, 1841, p. 350), Gregory gives the solution of the differential equation (10) in the two symbolic forms (11) and (13); from the former he deduces that

$$z = \phi(y) + a^2 x \phi''(y) + \frac{a^4 x^2}{2!} \phi^{iv}(y) + \&c.,$$

and from the latter, by putting

$$f(x) + F(x) = \psi(x), \quad \left(\frac{d}{dx}\right)^4 \{f(x) - F(x)\} = \chi(x),$$

he deduces that

$$\begin{aligned} z = & \psi(x) + \frac{1}{2!} \frac{y^2}{a^2} \psi'(x) + \frac{1}{4!} \frac{y^4}{a^4} \psi''(x) + \&c. \\ & + \frac{y}{a} \chi(x) + \frac{1}{3!} \frac{y^3}{a^3} \chi'(x) + \frac{1}{5!} \frac{y^5}{a^5} \chi''(x) + \&c. \end{aligned}$$

In reference to these two solutions in series, he remarks that "it seems anomalous that the same equation should admit of two solutions differing so essentially in character that the one contains two arbitrary functions, and the other only one," and he mentions certain considerations "which may serve to explain the difficulty."

It is partly on account of the attention thus directed to the two forms (11) and (13) of the solution of the differential equation, that I have thought it worth while to notice the formula (14), which is derived from (13), and involves only one arbitrary function, as it is curious that both terms of (13) lead by means of (9) to the same expression. It also seemed of interest, having regard to the completeness and rigour with which Poisson's theorem admits of being established, and to the unsatisfactory character of the process by which (9) was obtained, to point out that the former leads to an exact solution of the differential equation, and the latter to a solution which is true only under certain conditions.

14. If the differential equation

$$\frac{dz}{dx} = a^4 \frac{d^4 z}{dy^4},$$

is integrated by treating $\frac{d}{dx}$ as a constant, the four terms in the solution all lead to the formula

$$z = \int_0^\infty \int_0^\infty e^{-u^2 - v^2} \phi\left(x - \frac{y^4}{16a^4 u^4 v^4}\right) du dv,$$

and so on; but the solutions of these differential equations of higher orders present no additional point of interest.

The following presents were made to the Society's Library during the Recess :—

"Mittheilungen der Naturforschenden Gesellschaft in Bern in dem Jahre 1880," Nos. 979—1003; Bern, 1881.

"On Binomial Congruences, comprising an Extension of Fermat's and Wilson's Theorems, and a Theorem of which both are Special Cases," by O. H. Mitchell (from "American Journal of Mathematics," Vol. iii., No. 4): from the Author.

"Observations of Double Stars made at the United States Naval Observatory," by Asaph Hall; Washington, 1881.

"Account of the Operations of the Great Trigonometrical Survey of India," Vol. vi.—The principal Triangulation of the South-east Quadrilateral: from Her Majesty's Secretary of State for India, in Council.

"Educational Times," August, September, October, November, 1881; and "Reprint from the Educational Times," Vol. xxxv.: from the Publishers.

"Crelle," 91^o Band, 2^{te}, 3^{te}, 4^{te} Heft; Berlin, 1881.

"Proceedings of the Royal Society," Vol. xxxii., Nos. 214, 215.

"Physical Society. Proceedings," Vol. iv., Pt. iii., May to June, 1881; Vol. iv., Pt. iv., July to October, 1881.

"Bulletin des Sciences Mathématiques et Astronomiques," 2^{me} Série, Tome v., Févr., Mars, Avril, Mai, Juin, 1881.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang 24, 1^{te}, 2^{te}, 3^{te}, 4^{te} Heft, 1879; Jahrgang 25, 1^{te}, 2^{te}, 3^{te}, 4^{te} Heft, 1880.

"Proceedings of Royal Irish Academy—Science," Vol. iii., Series ii., No. 5, December, 1880; No. 6, April, 1881.

"Proceedings of Royal Irish Academy—Polite Literature and Antiquities," Vol. ii., Series 2, No. 2, December, 1880.

"Transactions of the Royal Irish Academy—Science," Vol. xxviii.; i. (December, 1880), ii. (February, 1881), iii., iv. (February, 1881), v. (March, 1881).

"Transactions of the Royal Irish Academy—Polite Literature and Antiquities," Vol. xxvii., iv. (June, 1881).

"Monatsbericht," April, Mai, Juni, 1881.

"Beiblätter zu den Annalen der Physik und Chemie," Band v., Stück 7, No. 7; St. 8, No. 8; St. 9, No. 9; St. 10, No. 10.

"Jahrbuch über die Fortschritte der Mathematik," elfter Band, Jahrgang 1879, Heft 1, 2; Berlin, 1881.

"Reale Istituto Lombardo—Rendiconti," Serie ii., Vol. xii., 1879.

"Jornal do Sciencias Mathematicas e Astronomicas," publicado pelo Dr. Fr. Gomes Teixeira, Vol. i., 1878; Vol. ii., 1880; Vol. iii., Nos. 1, 2, 3, 1881; Coimbra.

"Verhandlungen der Schweizerischen Naturforschenden Gesellschaft in Brieg," den 13, 14, 15 Sept., 1880 (63 Jahresversammlung).

"Jahresbericht," 1879, 1880; Lausanne, 1881.

"A Determination of the Solar Parallax from Observations of Mars made at the Island of Ascension in 1877," by David Gill, LL.D., 1881 ("Memoirs of Royal Astronomical Society," Vol. xlv.): from the Author.

"Catalogue of the Literary, Historical, and Scientific Library of the celebrated Mathematician Mr. Michel Chasles, of the Institute of France," to be sold by auction at Paris on the 27th June, 1881, and eighteen following days (Sundays excepted), (with corresponding French title), 3 Fasciculi in 390 8vo. pp.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," No. 1, Mai, 1881.

"The Mathematical Visitor," edited and published by Artemas Martin, M.A., Vol. i., 1877—81, Erie, Pa., 1881: from the Editor.

"American Journal of Mathematics," Vol. iii., No. 4.

"Measures, Weights, and Moneys of all Nations, and an Analysis of the Christian, Hebrew, and Mahometan Calendars," by W. S. B. Woolhouse, F.R.A.S., 6th edition, 1881.

"On the Adjustment of Mortality Tables, a sequel to former papers on the same subject," by W. S. B. Woolhouse, F.R.A.S.; from the Author.

"Annali di Matematica," Tomo x°, Fasc. 3° (Ott. 1881).

"In Memoriam Dominici Chelini, Collectanea Mathematica," nunc primum edita cura et studio L. Cremona et E. Beltrami; Hoepli, Mediolani, 1881: from the Editors.

The following is a list of the volumes from the late Mr. Henry Warburton's Library, which were presented to the Society by Mr. Howard Elphinstone (see p. 36):—

"Du Calcul des Dérivations," par L. F. A. Arbogast; Strasbourg, 1890.

"Essai sur la Théorie des Nombres," par A. M. Legendre, 2^e édition; Paris, 1808.

"Lilawati, or a Treatise on Arithmetic and Geometry," by Bhascara Acharya, translated by J. Taylor, M.D.; Bombay, 1816 (presented to John Pond, Astronomer Royal, by the Translator).

"Histoire de l'Astronomie Ancienne," par M. Delambre, Vols. i., ii.; Paris, 1817.

"Astronomie Théorique et Pratique," par M. Delambre, Vols. i., ii., iii.; Paris, 1814.

"Histoire de l'Astronomie du Moyen Age," par M. Delambre; Paris, 1819.

"Mécanique Analytique," par J. L. Lagrange, nouvelle édition, Tome i., 1811; Tome ii., 1815; Paris.

"Meditationes Analyticae," ab Edvardo Waring, M.D.; Cantab. 1776.

"The History and Present State of Discoveries relating to Vision, Light, and Colours," by Joseph Priestley, F.R.S.; London, 1772.

J. A. Borelli, "Atrium Physico-mathematicum apertum . . . (De Vi Percussionis . . . et de Motu Animalium . . .);" Lugd. Bat., 1686. (Editio prima Belgica accurante J. Broen).

"The Anti-logarithmic Canon, being a Table of Numbers . . . to which is prefixed an Introduction by James Dodson;" London, 1742.

"Miscellanea Philosophico-Mathematica Societatis Privatorum Taurinensis," Toin. i., 1769; Tom. ii., 1760—61; Tom. iii., 1762—65; Tom. iv., 1766—69; Tom. v., 1770—73; Turin.

"Christiani Hugonii Opera reliqua," Vols. i., ii., 1728; Amstelodami.

"Christiani Hugonii Opera varia," Vols. i., ii., 1724; Lugduni Batavorum.

"Leonhardi Euleri Opuscula Analytica," Tom. i., 1783; Tom. ii., 1785; and

"Tentamen Novæ Theoriæ Musicæ," 1739 (in one volume); Petropoli.

"Leonhardi Euleri Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes" . . . ; Lausannæ et Genevæ, 1744.

"Leonhardi Euleri Mechanica, sive Motus Scientia analytico exposita," Tom. i., ii.; Petropoli, 1736.

"Leonhardi Euleri Opuscula varii argumenti" (6 papers); Berolini, 1746.

"G. G. Leibnitii et Joh. Bernoullii commercium Philosophicum et Mathematicum," Tom. i., ii. (in one Vol.); Lausannæ et Genevæ, 1745.

"Conjectura physica circa propagationem Soni et Luminis, una cum aliis dissertationibus analyticis de numeris amicabilibus, de natura æquationum ac de rectifica-

tionem ellipsis" (1750); opusculum Tomus iii., continens novam Theoriam Magnetis" (1751); (all in one volume).

"Geometria Organica, sive descriptio linearum curvarum universalis," auctore C. MacLaurin; Londini, 1720.

"Leon. Euleri Theoria motus corporum solidorum seu rigidorum;" Rostochii, 1765.

"A Treatise of Mechanics, Theoretical, Practical, and Descriptive," by Olinthus Gregory, Vol. i., ii., and Volume of plates; London, 1806.

"Archimedis Opera; Apollonii Pergæi Conicorum Libri iv.; Theodosii Sphærica, Methodo novâ illustrata et succincte demonstrata per Isaacum Barrow;" Londini, 1675.

"Elements of Mechanical Philosophy," by J. Robison, LL.D.; Vol. i., Dynamics and Astronomy; Edinburgh, 1804.

"A Treatise on Algebra, with Appendix concerning the General Properties of Geometrical Lines," by Colin Maclaurin (4th edition); London, 1779.

"Jacobi Bernoulli ars Conjectandi, Tractatus de Seriebus Infinitis, et Epistola Gallice scripta de ludo Pilæ reticularis;" Basileæ, 1713. "Joh. Bernoulli de motu Musculorum," and "Nic. Bernoulli de usu Artis Conjectandi in Jure." (These five in one volume.)

"Leçons sur le Calcul des Fonctions," nouvelle édition, J. L. Lagrange; Paris, 1806.

"Leon. Euleri Introductio in Analysin Infinitorum," Tom. i., ii.; Lugduni, 1797. (In one volume.)

"Leon. Euleri Elémens d'Algèbre" (Nouvelle édition, par J. G. Garnier), Tome i., ii., 1807; Paris.

"Opticks, or a Treatise of the Reflections, Refractions, Inflections, and Colours of Light," by Sir Isaac Newton, Second edition; London, 1718.

"Cours de Mathématiques," par Charles Bossut, Tome second (Géométrie et application de l'Algèbre à la Géométrie); Paris, 1800 (An. ix.)

"Bija Ganita, or the Algebra of the Hindus," by Edward Strachey; London, 1813.

"Essay on the Resolution of Algebraic Equations," by Giffin Wilson (Phil. Trans., read June, 1799).

"On Self-repeating Series," by H. Warburton (Camb. Phil. Soc. Trans., Vol. ix., Pt. iv., duplicate).

"On Simultaneous Differential Equations of the First Order in which the Number of the Variables exceeds by more than one the Number of the Equation," by G. Boole, F.R.S. (Phil. Trans., read June 19, 1862).

"The Diffusion of Liquids," by T. Graham, F.R.S. (Phil. Trans., Pt. 1 for 1850).

"On the Partition of Numbers, and on Combinations and Permutations" (Camb. Phil. Soc. Trans., Vol. viii., Pt. iv.), by H. Warburton, F.R.S. (interleaved and bound copy).

In a case—"Animal Mechanics," "Hydrostatics," "Elements of Trigonometry" (W. Hopkins); "Elements of Spherical Trigonometry" (De Morgan); "On Probability," "On the Study of Mathematics," and other numbers of Mathematical Works in the Library of Useful Knowledge series.

"On the Application of Liquids formed by the Condensation of Gases as Mechanical Agents," by Sir H. Davy, P.R.S. (Phil. Trans., 1823).

"Origine, trasporto in Italia, primi progressi in essa dell' Algebra: Storia critica di nuove disquisizioni analitiche o metafisiche," abricchita di D. Pietro Cossali, C.R., Vol. i., 1797; Vol. ii., 1799; Parma.

"*Isaaci Newtoni Opera quæ exstant omnia*," commentariis illustrabat Sam. Horsley, LL.D., R.S.S., Londini; Vols. i., ii., 1779; Vols. iii., iv., 1782; Vol. v., 1786.

"*Scriptores Logarithmici . . .*" (F. Maseres), Vols. i., ii., 1791; Vol. iii., 1796; Vol. iv., 1801; Vol. v., 1804; Vol. vi., 1807.

"*Bibliographie Astronomique, avec l'Histoire de l'Astronomie depuis 1781 jusqu'à 1802*," par Jérôme de la Lande; à Paris, 1803.

"*Traité de l'Astronomie Indienne et Orientale*," par M. Bailly; Paris, 1787.

"*Introduction à l'Analyse des Lignes Courbes Algébriques*," par G. Cramer; Genève, 1760.

"*The Differential and Integral Calculus . . .*," by Augustus De Morgan (Library of Useful Knowledge); London, 1842.

"*Traité élémentaire du Calcul des Probabilités*," par S. F. Lacroix; Paris, 1822.

"*Jacobi Bernoulli Basileensis Opera*," Tom. i., ii.; Genève, 1744.

"*Johannis Bernoulli Opera omnia*," Tome i., ii., iii., iv., Lausannæ et Genève, 1742.

"*A Treatise on the Rectilinear Motion and Rotation of Bodies, with a description of original experiments relative to the subject*," by G. Atwood, F.R.S.; Cambridge, 1784.

"*Christiani Hugonii Κοσμοθεωρῆσις*," Hagæ-Comitum, 1699.

"*Treatise of Algebra in two books*," by W. Emerson; London, 1764.

"*Miscellanies, or a Miscellaneous Treatise, containing several Mathematical subjects*," by W. Emerson; London, 1776.

"*The Theory of the Moon, and on the Perturbations of the Planets*," and other Tracts, by J. W. Lubbock, F.R.S.; London, 1833 (in duplicate).

"*Miscellanea Mathematica*," by C. Hutton, F.R.S.; London, 1775.

"*The Diarian Miscellany*," by Chas. Hutton (3 Vols.); London, 1775.

"*Traité de Mécanique élémentaire*," par L. B. Francœur; Paris (An. ix.)

"*Cours complet de Mathématiques Pures*," par L. B. Francœur; Paris, 1819. Seconde édition, Tomes i., ii.

"*Uranographie, ou Traité élémentaire d'Astronomie*," par L. B. Francœur, Seconde édition; Paris, 1818.

"*Recueil de diverses Propositions de Géométrie*," par L. Puissant, Seconde édition; Paris, 1809.

"*Réflexions sur la Métaphysique du Calcul Infinitésimal*," par M. Carnot; Paris, 1813.

"*Essais de Géométrie sur les Plans et les Surfaces Courbes*," par S. F. Lacroix, 3^{me} édition; Paris, 1808. And "*Traité élémentaire de Trigonométrie rectiligne et sphérique*, 1810." And "*Elémens de Géométrie à l'usage de l'Ecole Centrale des Quatre-nations*," Neuvième édition, 1811. (These three in one volume.)

"*A Geometrical Treatise of Conic Sections*" (in 4 books), by Rev. Abram Robertson, F.R.S.; Oxford, 1802.

"*The Phenomena and Order of the Solar System*," by J. P. Nichol, LL.D.; Edinburgh, 1838.

"*Complément des Elémens d'Algèbre*," par S. F. Lacroix, 4^e édition; Paris, 1817.

"*Catoptricæ et Dioptricæ Sphæricæ Elementa*," auctore D. Gregorio, M.D.; Oxonii, 1695.

"*Histoire générale des Mathématiques*," par Charles Bossut, Tome i., ii., 1810; Paris.

"*Elements of the Conic Sections*," by Dr. Rob. Simson; Edinburgh, 1775.

"*The First Book of Euclid's Elements, with alterations and familiar notes*," by a Member of the University of Cambridge (P. Col. Peyronnet Thompson), in usum serenissimæ filioliæ; London, 1830.

"Elementary Principles of the Theories of Electricity and Heat, and Molecular Actions," by Rev. R. Murphy, Pt. i., on Electricity; Cambridge, 1833.

"A Syllabus of the Elementary System of Astronomy" (interleaved).

"A Treatise on Isoperimetrical Problems and the Calculus of Variations," by R. Woodhouse, F.R.S.; Cambridge, 1810.

"A Treatise on Plane and Spherical Trigonometry," by R. Woodhouse, F.R.S.; Cambridge, 1819.

"Geometrical Problems . . . , and an Appendix containing the Elements of Plane Trigonometry," by Miles Bland, D.D., F.R.S.; London, 1842.

"A Familiar Introduction to Crystallography, with an Appendix," by Henry James Brooke, F.R.S.; London, 1823.

"The Heat of Vapours, and on Astronomical Refractions," by J. W. Lubbock, F.R.S.; London, 1840.

"When is Venus brightest?" by A. S. Herschel ("Quarterly Journal of Pure and Applied Mathematics," February, 1861).

"On an Approximate and Graphical Rectification of the Circle," by A. S. Herschel (from same, October, 1860).

"Factorial Notation," by H. W. Elphinstone (from same, May, 1867).

"Memoir on a New and Certain Method of ascertaining the Figure of the Earth by means of Occultations of the Fixed Stars," by A. Cagnoli, with Notes and an Appendix by Francis Baily; London, 1819.

"Trigonometry and Double Algebra," by Aug. de Morgan; London, 1849.

APPENDIX.

THE remarks on Reversion (p. 3) were merely an abstract of what now appears in Mr. Taylor's *Geometry of Conics*, pp. lxxxvi. and 321—328; compare also the *Quarterly Journal of Mathematics*, Vol. xiv., pp. 25—31.

Prof. Teixeira's Note (p. 14) was a trifling one, though the theorem it contained is interesting, "a kind of Leibnitz's Theorem for Determinants."

Prof. Cayley put it in the following compact form:—

$$\delta_x^n D = \Sigma \frac{\Pi n}{\Pi \alpha \cdot \Pi \beta \cdot \Pi \gamma \cdot \Pi \delta} \cdot \begin{vmatrix} \delta_x^n & \phi_1 & \psi_1 & \theta_1 & \omega_1 \\ \delta_x^p & \phi_2 & \psi_2 & \theta_2 & \omega_2 \\ \delta_x^q & \phi_3 & \psi_3 & \theta_3 & \omega_3 \\ \delta_x^r & \phi_4 & \psi_4 & \theta_4 & \omega_4 \end{vmatrix},$$

$$\alpha + \beta + \gamma + \delta = n;$$

where the δ_s , &c. apply to the several lines of the determinant. But this is nothing else than $\delta_s'' D = (\delta_1 + \delta_2 + \delta_3 + \delta_4)'' D$,

the $\delta_1, \delta_2, \delta_3, \delta_4$ applying to the several lines of the determinant.

Mr. H. M. Jeffery's paper, referred to, p. 26, is given in *Quarterly Journal of Mathematics*, No. 69, pp. 1—40; there is also a paper by him in No. 70, under the title "On the Stappete-points of Class-Quartics with Quadruple Foci," pp. 158—166.

In the *Revue Maritime et Coloniale*, for November, 1880, is published a paper by Lieut. A. Corrad, in which the author prefaced a discussion on the collision of ships by some considerations of elementary geometry on the uniform rectilinear motion of two vessels. Mr. Merrifield, in his short communication (p. 36), showed that, though the author preferred to use demonstrations dependent upon the straight line and circle only, the question really turned upon the consideration that the line joining their centres envelopes a parabola.

In § 19 (p. 134) of Mr. Crofton's paper, read

$$r^{x(\log x)D} F(x) \equiv F(x^r) r^{x(\log x)D},$$

and

$$F(D^r) \equiv r^{-D(\log D)^2} F(D) r^{D(\log D)^2}.$$

Mr. Emory McClintock, of Milwaukee, the author of a paper entitled "An Essay on the Calculus of Enlargement" (*American Journal of Mathematics*, Vol. ii.), calls attention to the fact that a theorem of some importance, equation (69) of his paper,

$$S^h = 1 + hR + \frac{h^2 R^2}{1 \cdot 2} + \dots,$$

where

$$R = \log S,$$

[later on, he shows that $Rv = \frac{Dv}{D\psi x}$, equation (90),] has been substantially rediscovered by Prof. Crofton (pp. 127—129). "The plan of my work required the postponement of the discussion of differentiation . . . I am glad that he (Prof. C.) has so well illustrated the practical utility of the theorem, from a point of view other than that in which the plan of my essay led me to exhibit it." Mr. Crofton admits that the theorem is immediately deducible, and writes, "of course I would have stated this in my paper, had I been aware of it." Speaking very favourably of what appears to him to be "a very original and remarkable essay," Prof. Crofton concludes, — "I do not think the theorem, in the form I have given, can be said to have been given by Mr. McClintock in his essay, while I quite admit that, by putting together his series (69) and one of the values (90), given later on for R , it may be at once deduced."

Take any point P on a curve, O the source of light, OP , PA the incident and reflected rays respectively, PG , PT the normal and tangent to the curve at P ; the path of the ray $= OP + PA \equiv v$. In order to discriminate between the maximum and minimum paths, we must ascertain the sign of $\frac{d^2v}{dx^2}$,

$$v = \sqrt{x^2 + y^2} + \sqrt{(a-x)^2 + y^2}, \quad \frac{OG}{OP} = \frac{GA}{PA},$$

$$\frac{d^2v}{dx^2} = \left(\frac{1}{OP} + \frac{1}{PA} \right) \left\{ \frac{(y-px)^2}{x^2 + y^2} + qy \right\}.$$

The object of Mr. H. M. Taylor's question (p. 109) was to get a geometrical value for the second factor.

The following is an abstract of Mr. Merrifield's Note on Ptolemy's Theorem (Euc. vi., D), p. 142:—

If A, B, C, D be four points on a circle, Ptolemy's theorem tells us that the lengths of the lines joining them are subject to the relation

$$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0.$$

But, since the four points lie in a plane, the lines joining them must also be subject to the further relation $V = 0$, where V is the volume of the tetrahedron $ABCD$. The question may be asked, what becomes of the lines which satisfy Ptolemy's relation, when the restriction as to their lying in a plane is removed?

This question is very easily answered with the help of Carnot's expression, which gives the radius of the circumscribed sphere; for, calling this R , we have

$$576 R^3 V^2 \equiv (+ AB \cdot CD + AC \cdot DB + AD \cdot BC) \times \\ \begin{array}{cccc} (- & + & + &) \times \\ (+ & - & + &) \times \\ (+ & + & - &) \end{array}.$$

If the second side of this equation vanishes, as it does when Ptolemy's equation holds, either R or V must vanish. If R vanishes, the four points of the tetrahedron coalesce in a single point. If V vanishes, the four points lie in a plane, and R takes the value $\frac{0}{0}$, as it ought to do,

being only restricted by the condition that it should not be less than the radius of the circle through the four points in plane.

The existence of Ptolemy's relation therefore implies, that the four points must either lie in a plane (and, if so, in a circle), or else be coincident.

In connexion with Mr. Carpmael's paper, we may refer to *Proceedings*,

Vol. vii., p. 235, adding thereto a reference to the *Quarterly Journal of Mathematics*, Vol. xi. (1871), pp. 26—37, where is given a “General Solution and Extension of the Problem of the 15 School Girls,” by Andrew Frost.

Mr. McColl has printed in the *Philosophical Magazine* (Jan. 1881, p. 40) a paper entitled “Implicational and Equational Logic,” after the method of his communications to the *Proceedings*.

The following two articles on Linkages have been noted :—“Linkages for X^n ,” by F. T. Freeland (*American Journal of Mathematics*, Vol. iii., No. 4); and “Note sur le système articulé du Colonel Peaucellier,” par M. D’Ocagne, in the *Nouvelles Annales* (Oct. 1881, pp. 456, &c.).

In reference to a letter from Signor Brioschi, mentioned in the last volume of the *Proceedings* (xi., p. 157), Sir J. Cockle cites three papers by Signor Brioschi (*Annali de Matematica*, Tom. ix., p. 11, Tom. x., p. 1 and p. 4), and a paper by M. Hermite (*ib.*, Tom. ix., p. 21); and further remarks that references to all, or nearly all, the papers bearing on the subject will be found in Prof. Cayley’s recent memoir “On the Schwarzian Derivative,” &c. (*Cambridge Philosophical Society Transactions*, Vol. xiii., p. 5).

Herr W. Schlötel, of Strassburg, in a letter to the Secretaries, dated January 21, 1881, “in order to secure priority” as against Messrs. C. S. Peirce and McColl, refers to papers communicated by him to the *Augsburger Allgemeine Zeitung*, in 1868, 1871, and 1876.

Mr. T. Muir has contributed to the *Quarterly Journal of Mathematics*, Vol. xviii., No. 70, a very complete “List of Writings on Determinants” (*cf. Proceedings London Mathematical Society*, Vol. xi., p. 157).

The Subscription List of the De Morgan “Memorial Medal” Fund having been closed, it may be stated here that the money obtained has been invested by the Treasurer as £104. 19s. 3d. Reduced 3 per cent. Consols.

A brief allusion is due to the losses the Society has sustained in the past session by the deaths of two of its members, each of whom surpassed the threescore years and ten.

In M. Chasles, we lost our first Foreign member. No one could value this title more highly than M. Chasles did.

For notices of his life and works, we refer to the funeral speeches (extracted from the *Comptes Rendus*), published in the *Bulletin des Sciences Mathématiques et Astronomiques*, Tome iv., December 1880, by M. Bertrand (pp. 433—435), and M. Bouquet (pp. 435, 436).

In the same number of the *Bulletin* is an appreciative notice by M. Darboux (pp. 436—442); in *Nature* (Vol. xxiii., No. 584), January 6, 1881, there is also a brief memoir, from which we make a few extracts.

Michel Chasles was born at Épernon (Eure-et-Loire), November 15th, 1793, and entered the Ecole Polytechnique in 1812. He was present at the defence of Paris, in 1814, and reentered the above-named school in 1815. In 1839 Chasles was elected a corresponding member of the Academy, was "decorated" the same year, and in 1841 was made "Professeur de Machines et de Géodésie" at the Ecole Polytechnique. This chair he held for ten years. Meanwhile he was elected to the newly created chair of Modern Geometry in 1846. The following are a few among the many honours Chasles obtained:—in 1851 he was elected a member of the Academy; in 1854 he became a Foreign Member of the Royal Society; in 1865 he was awarded the Copley Medal, and in April 1867 he was elected our first Foreign Member, a distinction which he for some time enjoyed alone. "M. Chasles a poursuivi son œuvre sans interruption depuis sa sortie du Lycée jusqu' à l'âge de quatre-vingt-sept ans. Soixante-huit années séparent la première note de l'élève Chasles, insérée dans la *Correspondance sur l'Ecole Polytechnique*, du dernier mémoire présenté à l'Académie des Sciences. Tous les géomètres, sans distinction de nationalité ni d'école, se sont inclinés devant ce vénérable vieillard; tous ont admiré sa puissance d'invention, sa fécondité, que l'âge semblait rajeunir, son ardeur, et son zèle, continués jusqu' aux derniers jours." M. Chasles died December 18th, 1880. "La vie de M. Chasles a été heureuse et simple; il a trouvé dans la science, avec les plus grandes joies, une gloire qui sera immortelle, et dans la vive affection de ses amis, dans leur assiduité empressée aux réunions où il les conviait avec une grâce si aimable, dans leur respectueuse déférence en toute circonstance, la consolation de sa vieillesse."

We give here the dates and titles of M. Chasles's larger works:—

1. "Aperçu historique sur l'origine et le développement des méthodes en Géométrie . . . suivi d'un mémoire . . . sur deux principes généraux . . . et l'Homographie," 1837, reprinted verbatim in 1875.
2. "Traité de Géométrie supérieure," 1852, reprinted in 1880.
3. "Les Trois Livres de Porismes d'Euclide, rétablis pour la première fois, d'après la Notice et les Lemmes de Pappus, et conformément au sentiment de R. Simson sur la Forme des Enoncés de ces Propositions," 1863.
4. "Traité des Sections Coniques, faisant suite au Traité de Géométrie supérieure," Vol. i., 1835.
5. "Rapport sur les progrès de la Géométrie," 1870.

The number of the smaller papers printed in Chasles's lifetime, we think, exceeded 200; and the number of his scientific MSS., presented to the Academy of Sciences by M. Favre, is stated to be 113 (*Nature*, Jan. 5, 1882, No. 636).

M. Chasles was personally known to a few only of our members, but no one who attended our meetings from November 1865 to June 1878,

with any regularity could have failed to have seen Mr. T. Cotterill, who during that time was rarely absent and was a most interested auditor of all papers bearing on plane curves. Of him the late Prof. Clifford wrote (*Mathematical Papers*, p. 42),—"Mr. Cotterill, is, I believe, the first person that ever saw a curve of the third class."

The Rev. J. E. B. Mayor, in a notice of "The Cotterells, Cotterills, and Cottrells, of Cambridge" (*Notes and Queries*, November 12, 1881), states that Mr. T. Cotterill was the son of the Rev. Thomas Cotterill,* formerly Fellow of St. John's College, and was born at Lane End, county Stafford, of which place his father was then the minister. Subsequently his father was Perpetual Curate of St. Paul's, Sheffield, and young Cotterill was educated at the Grammar School of that town. He was entered as a pensioner at St. John's College, Cambridge, January 26, 1828, "aged 18"; he graduated B.A. 1832, M.A. 1835, and was admitted Foundation Fellow of his College, March 18th, 1834. He died February 16th, 1881, aged (according to the notice in the papers of the time) "73 years."† For twenty-one years he was a mathematical master under the Rev. T. Day, of Brixton, having after a brief trial relinquished the idea of studying for the Bar. He often mentioned that he himself had prepared for the Army fourteen of the officers who fell in the Crimean War.

Mr. Cotterill's connexion with the Society dates from October 16, 1865, when he was elected a member thereof; he became a member of the Council, March 19th, 1866, a position he held until November 14th, 1878, at which date he ceased to attend our meetings.

Mr. Cotterill was an excellent geometer, but his mathematical contributions were not numerous. The causes which operated to bring this result about also account for his (apparently) never having completed to his satisfaction one or two very interesting papers which, we fear, are now irrecoverably lost. The following is as complete a list as we have been able to make of his published and unpublished papers:—

1. A Geometrical Property of Curves of the Third Order; *Cambridge and Dublin Mathematical Journal*, vii., pp. 14—16; 1852.
2. On the Calculus of Distances, Areas, and Volumes, and its relations to the other Forms of Space; *Cambridge and Dublin Mathematical Journal*, vii., pp. 275—279; 1852.
3. Certain Properties of Plane Polygons of an even number of sides; *Proceedings London Mathematical Society*, Vol. i., No. v., pp. 13—16; 1866.
4. Property of a certain Curve described on a Sphere.
5. Some Properties of Cubic Curves.

* For other particulars of this gentleman than those given by Mr. Mayor, we are referred by Mr. Leader, of Sheffield, to Gatty's edition of Hunter's "History of Hallamshire" (p. 276)

† Application was made to the present incumbent of Lane End for information as to exact date of birth from Church books, but no answer came to hand.

6. On an Involution-System of Circular Cubics, and description of the Curve by Points, when the Double Focus is on the Curve; *Proceedings London Mathematical Society*, Vol. i., No. viii. p. 14; 1866.

7. A Goniometrical Problem: to be solved Analytically in one move, or more simply Synthetically in two moves; *Quarterly Journal of Pure and Applied Mathematics*, No. 27; 1866.

8. Property of Six Points on a Plane or Sphere; *Proceedings London Mathematical Society*, Vol. ii., p. 49; 1867.

9. The Eight Points of Intersection of three Quadric Surfaces.

10. Proofs of the Fundamental Formulæ of the Higher Geometry in Space of two and three Dimensions; *Messenger of Mathematics*, Vol. iv., pp. 98—106; 1868.

11. On a Correspondence of Points, such that a curve of the n^{th} order in one plane corresponds to a curve of the $4n^{\text{th}}$ in another plane, with three multiple points of the order n on the line of intersection of the planes, and three other multiple points of the order $2n$; *Proceedings London Mathematical Society*, Vol. ii., pp. 119—126; 1868.

12. An Envelope in the Cubic Correspondence of Points.

13. Opposite Points on a Curve.

14. Propositions connected with Residuals.

15. On the Intersection of Plane Curves.

16. On an Algebraical Form, and the Geometry of its Dual connexion with a Polygon, Plane or Spherical; *Proceedings London Mathematical Society*, Vol. iv., pp. 139—143; 1872.

17. On the Correspondence of Points Collinear with a Fixed Origin; *Proceedings London Mathematical Society*, Vol. vi., p. 200 (abstract); 1876.

18. On a New View of Pascal's Hexagram; *Proceedings London Mathematical Society*, Vol. viii., p. 311 (abstract); 1877.

Mr. Cotterill was a frequent contributor to the mathematical pages of the *Educational Times*.

R. T.