in which $\zeta(n)$ denotes the excess of the sum of the uneven divisors of n above the sum of the even divisors; and $\Delta'(n)$ denotes the sum of those divisors of n whose conjugates are uneven.

The general theorems relating to the actual divisors are derived from the other Zeta functions by a method similar to that employed in §§ 25-27.

On a certain Riemann's Surface. By W. BURNSIDE.

[Read May 14th, 1891.]

The present note was suggested by the paper of Mr. R. A. Roberts, pp. 28-34 of the current volume of the Society's *Proceedings*, dealing with a particular case of Abel's theorem. The special interest of the case that Mr. Roberts treats lies in the degradation of the apparently Abelian integrals to ordinary elliptic integrals; and though, as Mr. Greenhill has pointed out to me, this property, for the form of integral in question, is sufficiently well known, being originally due to Legendre, still its consideration from quite another point of view, namely, that of Riemann's theory, is perhaps not altogether superfluous.

The equation
$$s^3 = \frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}$$

defines a three-sheeted Riemann's surface. The only branch points are α , β , γ , δ , and at each of these the three sheets are connected together. The points at infinity on each sheet are ordinary points. The "Verzweigungs-schnitte," or lines of passage, will consist of a line from α to γ , connecting sheets I. and II., a second from α to γ , connecting II. and III., and two similar ones from β to δ . The lines may be taken straight, so that on crossing $\alpha\gamma$ or $\beta\delta$ from the left to the right a point will pass from I. to II., or from II. to III., or from III. to I.; the separate sheets being denoted by I., II., and III. The number of sheets being three, while there are four lines of passage, Riemann's number " p " for the surface will be 2, and there are, therefore, according to Riemann's theory, two independent integrals of the first kind upon it.

The normal series of sections, by which the surface is reduced to a simply connected one, will consist of two a -curves, two b -curves, and a line joining any point of the system a_1b_1 to any point of a_2b_2 . A figure will indicate the system of curves chosen more readily than a verbal description, and in the figure the line joining a_1b_1 to a_2b_2 , being of no importance for the integrals, is omitted.

Lines drawn in the first sheet are full, in the second broken, and in the third dotted.

The arrowheads on the a and b curves indicate the directions round them in which the integrals giving the periods will be taken.

The two simplest integrals of the first species on the surface are

$$\int \frac{dz}{s(z-\gamma)(z-\delta)} \quad \text{and} \quad \int \frac{dz}{s^2(z-\gamma)(z-\delta)},$$

or

$$\int \frac{dz}{\{(z-\alpha)(z-\beta)(z-\gamma)^2(z-\delta)^2\}^{\frac{1}{2}}},$$

and

$$\int \frac{dz}{\{(z-\alpha)^2(z-\beta)^2(z-\gamma)(z-\delta)\}^{\frac{1}{2}}},$$

and they will be denoted by

$$\int dw \quad \text{and} \quad \int dw'.$$

The periods of w and w' , that is, the values of

$$\int dw \quad \text{and} \quad \int dw',$$

taken round the sections, may be expressed in terms of rectilinear integrals in the following forms:—

$$\begin{aligned} \int_{b_1} dw &= (1-T) \int_{\cdot} dw, & \int_{a_1} dw &= (1-T) \int_{\cdot} dw, \\ \int_{b_1} dw &= -T^2(1-T) \int_{\cdot} dw, & \int_{a_1} dw &= T(1-T) \int_{\cdot} dw, \\ \int_{b_1} dw' &= (1-T^2) \int_{\cdot} dw', & \int_{a_1} dw' &= (1-T^2) \int_{\cdot} dw', \\ \int_{b_1} dw' &= -T(1-T^2) \int_{\cdot} dw', & \int_{a_1} dw' &= T^2(1-T^2) \int_{\cdot} dw', \end{aligned}$$

where T is one of the imaginary cube roots of unity, $\int_{\cdot}$ denotes integration in sheet I. along the right-hand side of $\alpha\gamma$, and $\int_{\cdot}$ denotes integration in sheet I.

Hence the constant differences of w and w' , when the variable is restricted to the simply connected surface given by the sections, at corresponding points on opposite sides of the sections, are given by the table:—

	a_1	b_1	a_2	b_2
w	A	B	$-T^2A$	TB
w'	A'	B'	$-TA'$	T^2B'

or, in other words, the table gives the relations between the periods of the simple integrals.

The fundamental relation, connecting the periods of any two independent integrals of the first kind, gives, when applied to the above table,

$$AB' - A'B - T^4AB' + T^2A'B = 0,$$

or

$$TAB' + A'B = 0.$$

The normal integrals of the first kind, namely those whose constant

differences at the a -curves are $(\pi i, 0)$ and $(0, \pi i)$ respectively, are given by

$$W_0 = \frac{\pi}{\sqrt{3}} \left(\frac{Tw}{A} - \frac{T^2 w'}{A'} \right),$$

and

$$W' = \frac{\pi}{\sqrt{3}} \left(\frac{w}{A} - \frac{w'}{A'} \right);$$

and, on substituting, it will be found that the periods of these normal integrals are given by the following table:—

	a_1	b_1	a_2	b_2
W	πi	$\frac{2\pi}{\sqrt{3}} \frac{B}{A} T$	0	$-\frac{\pi}{\sqrt{3}} \frac{B}{A} T$
W'	0	$-\frac{\pi}{\sqrt{3}} \frac{B}{A} T$	πi	$\frac{2\pi}{\sqrt{3}} \frac{B}{A} T$

the simplification arising from the previously proved relation

$$TAB' + A'B = 0.$$

An inspection of this table shows at once that W and W' are elliptic integrals, for the most general value of W is

$$W + m\pi i + (2n_1 - n_2) \frac{\pi}{\sqrt{3}} \frac{B}{A} T,$$

and $2n_1 - n_2$ being an integer, the integral has therefore only two periods.

The ratio A/A' , involved in the passage from w and w' to W and W' , may be shown to be algebraical. For this purpose let b'_1 be a closed curve on I , enclosing $\beta\delta$ and not cutting b_1 ; then, in the space outside b_1 and b'_1 , the integrand of w is everywhere finite, and hence

$$\int_{b_1} dw + \int_{b'_1} dw = 0.$$

Now the linear substitution, given by

$$\frac{z-a}{z-\beta} \cdot \frac{z-\delta}{z-\gamma} = \frac{z'-a}{z'-\beta} \cdot \frac{z'-\delta}{z'-\gamma},$$

obviously changes $\alpha, \beta, \gamma, \delta$ into $\delta, \gamma, \beta, \alpha$; also it must be of the form

$$\frac{z-z_1}{z-z_2} = -\frac{z'-z_1}{z'-z_2},$$

for it is its own reciprocal.

$$\text{Hence} \quad z-\alpha = \frac{\frac{1}{2}(z_1+z_2)-\alpha}{D} (z'-\delta),$$

$$z-\beta = \frac{\frac{1}{2}(z_1+z_2)-\beta}{D} (z'-\gamma),$$

$$z-\gamma = \frac{\frac{1}{2}(z_1+z_2)-\gamma}{D} (z'-\beta),$$

$$z-\delta = \frac{\frac{1}{2}(z_1+z_2)-\delta}{D} (z'-\alpha),$$

where

$$D = z' - \frac{1}{2}(z_1+z_2);$$

therefore

$$dw = \frac{-\left(\frac{z_1-z_2}{2}\right)^3}{\left[\left(\frac{z_1+z_2}{2}-\alpha\right)\left(\frac{z_1+z_2}{2}-\beta\right)\left(\frac{z_1+z_2}{2}-\gamma\right)\left(\frac{z_1+z_2}{2}-\delta\right)\right]^{\frac{1}{2}}} \\ \times \frac{dz'}{\left[(z'-\alpha)^3(z'-\beta)^3(z'-\gamma)(z'-\delta)\right]^{\frac{1}{2}}}.$$

Also

$$\frac{z_1+z_2}{2} = \frac{\alpha\delta-\beta\gamma}{\alpha+\delta-\beta-\gamma},$$

$$\frac{z_1-z_2}{2} = \frac{\sqrt{(\alpha-\beta)(\alpha-\gamma)(\delta-\beta)(\delta-\gamma)}}{\alpha+\delta-\beta-\gamma};$$

therefore

$$dw = \left(\frac{\alpha-\beta}{\delta-\gamma}\right)^{\frac{1}{2}} dw',$$

and

$$\int_{\rho} dw = -\left(\frac{\alpha-\beta}{\delta-\gamma}\right)^{\frac{1}{2}} \int_{\rho} dw',$$

where the determination of the cube root will depend on which side of $\alpha\gamma$ the circular arc lies, into which $\beta\delta$ is changed by the substitution.

$$\text{Now} \quad A' = (1-T^2) \int_{\rho} dw' = (T^2-1) \left(\frac{\delta-\gamma}{\alpha-\beta}\right)^{\frac{1}{2}} \int_{\rho} dw,$$

and

$$A = (1-T) \int_{\rho} dw = (T-1) \int_{\rho} dw;$$

therefore $A' = - \left(\frac{\delta - \gamma}{\alpha - \beta} \right)^{\frac{1}{2}} A,$

with a proper determination of the root.

[Addition. October 30th, 1891.

The equation $s^3 = \frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)} \dots \dots \dots (A),$

defining the Riemann's surface considered in this note, can, by a rational and reversible transformation, be changed into the form

$$s^2 = z(1-z)(1-\kappa z)(1-\lambda z)(1-\kappa\lambda z) \dots \dots \dots (B),$$

where κ, λ are determinate in terms of $\alpha, \beta, \gamma, \delta$. But without performing this transformation the equations (A) and (B) have clearly this property in common: that each can be changed into itself by a transformation, of period two, which at the same time transforms into themselves two independent integrals of the first species.

Thus, for (A), if

$$\frac{(z-\alpha)(z-\gamma)}{(z-\beta)(z-\delta)} = \frac{(z'-\alpha)(z'-\gamma)}{(z'-\beta)(z'-\delta)}, \text{ and } s = \frac{1}{s'},$$

then

$$s'^3 = \frac{(z'-\alpha)(z'-\beta)}{(z'-\gamma)(z'-\delta)},$$

and the two normal integrals considered in this paper are each unchanged; while for (B), if

$$z = \frac{1}{\kappa\lambda z'}, \text{ and } s = \frac{s'}{(\kappa\lambda)^{\frac{1}{2}} z'^2},$$

then

$$s'^2 = z'(1-z')(1-\kappa z')(1-\lambda z')(1-\kappa\lambda z'),$$

and $\frac{1 + \sqrt{\kappa\lambda} \cdot z}{s} dz, \frac{1 - \sqrt{\kappa\lambda} \cdot z}{s} dz$ are unchanged except as regards sign.

If in either case the point (s, z) describes a period-path on the Riemann's surface, the point (s', z') must do the same, and a very little consideration shows that the corresponding paths are not reconcilable. Hence, at once, each integral can only have two independent periods. This reasoning may be applied to any equation

$$s^3 = f_s(z) \dots \dots \dots (C),$$

[where $f_6(z)$ is a sextic function,] which is such that it can be transformed into itself by a substitution, of period two, of the form

$$z = \frac{az' + b}{cz' + d}, \quad s = A \frac{s'}{(cz' + d)^3},$$

and therefore the two integrals of the first species belonging to equation (C) can be expressed by means of elliptic integrals, if the equation

$$f_6(z) = 0$$

can be transformed into itself by such a linear substitution. The roots of the sextic are in this case in involution, and hence the condition is that the skew invariant should vanish. This is the result originally proved (in a different manner) by Prof. Gordan.]

Quartic Equations interpreted by the Parabola.

By G. HEPPLE, M.A.

[Read February 12th, 1891.]

The geometrical solution of quartic equations by means of the parabola forms the subject of a paper by Hoppe in the *Archiv der Mathematik*, Vol. LXIX., 1883, Leipzig; and also of a paper by Hofmann in the *Nouvelles Annales*, March, 1888. The object of the present paper is to attempt an extension of the method adopted in the former of these, so as to be in fact supplementary to Hoppe's investigation.

In Descartes' solution of a quartic, we derive from the quartic a certain cubic which must have at least one positive root, and if that root is found, all the roots, real or imaginary, of the quartic are at once easily deduced.

Hoppe shows how this preliminary solution may be effected by measuring the ordinates of the points of intersection of a variable circle with a fixed parabola. The following is a brief abstract of the method he employs.

Let the cubic be $y^3 + \alpha y^2 + \beta y + \gamma = 0$;