SUBCONSCIOUS PANGEOMETRY.

FROM the press of Teubner in Leipsic has just appeared a work which perhaps can best be described as a book on "The Non-Euclidean Geometry Inevitable." This book, *The Theory of Parallels*,[1] by Paul Staeckel, in conjunction with Friedrich Engel, is a marvel of German accuracy, depth, and withal enterprise.[2]

It confers an inestimable boon on thinkers by giving them the actual documents which are the slow, groping awakening of the world-mind at the gradual dawning of what has now become the full day of self-conscious non-Euclidean geometry.

To one who appreciates the judicial weight of German scholarship, it must be highly gratifying to recognise its sanction of the position first put forth in *The Monist*, beginning, *loc. cit.* p. 486: "Euclid did not try to hide the non-Euclidean geometry. That was done by the superstitious night of the fanatic dark ages, from which night we have finally emerged, to find again what Euclid knew," etc.

Says Staeckel, p. 3: "Es ist kein Zufall, dass die ersten achtundzwanzig Sätze von der fünften Forderung, dem sogenannten Parallelenaxiom, durchaus unabhängig sind, und dass dieses erst beim Beweise des neunundzwanzigsten Satzes eintritt; es ist kein Zufall, dass der Aussenwinkel des Dreiecks an zwei Stellen behandelt wird: zuerst, in Satz 16, wird nur gezeigt, dass er grösser

[1] See *The Monist*, July, 1894, pp. 483–493.

[2] The full title of the book runs: *Die Theorie der Parallellinien von Euklid bis auf Gauss, Eine Urkundensammlung zur Vorgeschichte der nichteuklidischen Geometrie.* In Gemeinschaft mit Friedrich Engel herausgegeben von Paul Stäckel. Mit 145 Figuren im Text und der Nachbildung eines Briefes von Gauss. Leipsic: B. G. Teubner. 1895. Pages, 325. Price, 9 Mks.

ist als jeder der beiden ihm gegenüberliegenden inneren Winkel, und erst später, in Satz 32, stellt sich heraus, dass der Aussenwinkel der Summe jener beiden inneren Winkel genau gleich ist.

"Diese Anordnung berechtigt zu dem Schlusse, dass Euklid die in der Parallelentheorie verborgene Schwierigkeit sehr wohl durchschaut hat."

The very pretty point made[1] against all the modern English translations and editions in reference to the different and more elegant form given by Euclid in Proposition 29 to his celebrated Parallel-postulate is confirmed by Staeckel's re-translation of the original Greek, "wie er in Heiberg's neuer ausgezeichneter Ausgabe vorliegt."

Saccheri discussed the contribution made by Wallis to the theory of parallels, and Staeckel, after his re-translation of Euclid's Book I., through Prop. 32, gives this passage from Wallis, and then proceeds to Saccheri himself.

In *The Monist*, p. 489, a sentence was quoted from Dr. Emory McClintock[2] in regard to Saccheri, with grave doubts. It reads: "He confessed to a distracting heretical tendency on his part in favor of the *hypothesis anguli acuti*, a tendency against which, however, he kept up a perpetual struggle (*diuturnum proelium*)."

Translating Saccheri's book into English strengthened these doubts into the conviction that the whole was an error based on a mistranslation of the passage pointed out by the two Latin words retained in parenthesis. A letter embodying this conviction was written to Dr. McClintock, who thereupon made a special trip to the Astor Library to read again Beltrami's article on Saccheri, entitled: *Un precursore italiano di Legendre et di Lobatschewsky*. He thereupon answered:

"I have just read Beltrami in the Astor Library, also my own paper. Saccheri was always fighting against the heretical results of his own logic on behalf of what he obviously considered God's truth.

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[1] *The Monist*. p. 488.

[2] *Bulletin of the New York Mathematical Society*, Vol. II., p. 145.

"I did not speak of him as yielding; but one who is battling manfully against the productions of his mind may fairly be described, I think, in the words you dispute, though Saccheri's 'confession' is implicit and not explicit.

"I should have done better to use the words 'suffered from' for 'confessed to,' though there is sufficient confession in the 'proelium.'

"Beltrami is disgusted by the unexpected triumph of faith over logic.

"'Or qui crederebbe che subito dopo la proposizione testé citata il lettore dovesse vedersi comparire innonzi quest' altra. [Prop. 33.] Eppure è proprio cosi. L'Autore fa un lunghissimo discorso per conestare piuttosto che dimostrare cotesto suo asserto. . . . *Si direbbe quasi che l'Autore, più che a convincere altrui, si adoperi a persuadere sè stesso. . . .*'"

But still the conviction remained that there was no adequate ground in Saccheri for this interpretation of the "diuturnum proelium" passage.

A transcript of a considerable portion of the only copy of Saccheri's book then on this continent was made and sent to Dr. McClintock. He at once replied:

"I thank you for the manuscript, which I shall take care of and return. Now I need to consult Beltrami's article again.

"The original context of the 'diuturnum proelium' gives me a wholly novel view of it, instantly. It was a reference to a 'running fight' on paper, part of a mere summary of the book.

"I had supposed it to be a bit of mental autobiography.

"I do not doubt that Beltrami's mention of it is not inconsistent with the meaning Saccheri intended,—yet it failed, even the other day after your question, to suggest to me the true meaning. I will write again after I can get to the Library.

"You can blame me and the lack of context, not Beltrami, unless his suggestion that *Saccheri was trying to persuade himself*, may have helped."

The article in *The Monist* continues as follows: "The Inquisitor-general and the Archbishop of Milan saw Saccheri's book on

July 13, 1733; the Provincial of the Company of Jesus on August 16, 1733. Within less than two months Saccheri was dead and buried. Not so his book. It was reviewed in the *Acta Eruditorum* in 1736. It was probably in the library at Göttingen about 1790–1800, for it is marked with an asterisk in the *Bibliotheca Mathematica* of Murhard. In this work it is signalised (I. II., p. 43) among the writings consecrated to the explication, to the criticism, or to the defence of Euclid (*Einleitungs- und Erläuterungsschriften, auch Angriffe und Vertheidigungen des Euklides*). It therefore attained a certain notoriety. Did it escape the notice of Gauss?"

This suggestion has now been verified by Engel and Staeckel (p. 38) with truly German minuteness. "Der *Euclides ab omni naevo vindicatus* scheint ein ziemlich verbreitetes Buch gewesen zu sein. In Deutschland haben wir sein Vorhandensein auf den Königlichen Bibliotheken zu Berlin und Dresden und auf den Universitätsbibliotheken *in Göttingen* (*seit 1770*), Halle, Rostock und Tübingen festgestellt."

In the very brief sketch of Lambert by F. W. Cornish of Eton College, inserted in the *Encyclopædia Britannica* in 1882, how did it happen that from the mass of Lambert's papers one of the few mentioned should be that on *parallel lines*? If any hint of its known or possible interest was meant, it bore fruit; for only in 1893 and by accident did Staeckel discover in Lambert a precursor of Bolyai and Lobachewski. In the present book seventy-two pages are devoted to this treatise of Lambert. It is a developed consistent non-Euclidean geometry.

In some points it falls short of Saccheri; for instance, in not reaching Lobachewski's highly interesting "boundary-lines."

But in other respects it goes beyond Saccheri. Its examination, as compared to the writings on which the claims for Gauss are made, shows some startling coincidences.

That it was familiar to Gauss is clear from the letter of Bessel to Gauss, Feb. 10, 1829, where it is referred to as something well-known in the following paragraph:

"Durch das, was Lambert gesagt hat und was Schweikardt mündlich äusserte, ist mir klar geworden, dass unsere Geometrie unvollständig ist und eine Korrektion erhalten sollte, welche hypothetisch ist, und wenn die Summe der Winkel des ebenen Dreiecks $= 180°$ ist, verschwindet.

"Das wäre die wahre Geometrie, die Euklidische aber die praktische, wenigstens für die Figuren auf der Erde."

Says Lambert, § 79: "Ich habe aber vornehmlich bey der dritten Hypothese [angle-sum $< 180°$] solche Folgsätze aufgesucht, um zu sehen, ob sich nicht Widersprüche äussern würden. Aus Allem sah ich, dass sich diese Hypothese gar nicht leicht umstossen lässt.

"Die erheblichste von solchen Folgen ist, dass, *wenn die dritte Hypothese statt hätte*, wir *absolutes Maass der Länge* haben würden."

Says Gauss in his letter to Taurinus, 1824: "Die Annahme, dass die Summe der 3 Winkel kleiner sei als 180°, führt auf eine eigne von der unsrigen (Euklidischen) ganz verschiedene Geometrie. . . . Alle meine Bemühungen, einen Widerspruch, eine Inconsequenz in dieser Nicht-Euklidischen Geometrie zu finden, sind fruchtlos gewesen, und das Einzige was unserm Verstande darin widersteht, ist, dass es, wäre sie wahr, im Raum eine an sich bestimmte (obwohl uns unbekannte) Lineargrösse geben müsste."

Says Lambert, p. 200: "Diese Folge hat etwas Reizendes, welches leicht den Wunsch abdringt, die dritte Hypothese möchte doch wahr seyn!"

Says Gauss, p. 250: "Ich habe daher wohl zuweilen im Scherz den Wunsch geäussert, dass die Euklidische Geometrie nicht die Wahre wäre, weil wir dann ein absolutes Maass *a priori* haben würden."

Again Lambert shows that the formulas of this non-Euclidean geometry are simply those of spherics on an imaginary sphere. Now what Dr. McClintock (*Bulletin*, Vol. II., p. 146), calls "the important formula for the circumference of a circle published later by the younger Bolyai," given in 1831 by Gauss in a letter to Schumacher, is nothing but the elementary expression for the circumference of a circle on a sphere where the radius r has been replaced by $r\sqrt{-1}$. Moreover it is now known that Bolyai János discovered his system of Pangeometry in 1823.

In a letter of May 17, 1831, Gauss says: "Von meinen eignen Meditationen, . . . wovon ich aber nie etwas aufgeschrieben habe, . . . habe ich vor einigen Wochen doch einiges aufzuschreiben angefangen. Ich wünschte doch, dass es nicht mit mir unterginge."

It is mentioned in *The Monist* that in a letter to Schumacher, Gauss tells him that "a certain Schweikardt has given to this geometry the name of *Astralgeometrie*," and Gauss added in regard to him the brief note: "Früher in Marburg, jetzt Professor der Jurisprudenz in Königsberg." On p. 9, of the English translation of Vasiliev's Address on Lobachewski is the sentence: Taurinus in his *Theorie der Parallellinien* (1825) says: "The idea of a geometry in which the sum of the angles of a triangle is less than two right angles was already communicated to me four years ago (by my uncle, Prof. S., in K., then still in M.)."

Ferdinand Karl Schweikart (1780–1857) studied from 1796 to 1798 in Marburg, attending there the mathematical lectures of J. K. F. Hauff, who since 1793 had published different writings on the question of parallels. From 1812 he was professor in Charkov; from 1816 in Marburg; from 1820 in Königsberg. Entirely by himself, without the slightest suggestion from any man, he developed and taught a non-Euclidean geometry.

Engel and Staeckel seem to delight in the perfect proof of his independence from even the remotest connexion with Gauss.

Gerling (1788–1864) from 1817 professor of astronomy at Marburg, wrote to Bolyai Farkas: "We had here about this time [1819] a law professor, Schweikart, who had previously been in Charkov, and had attained similar ideas, since, without aid of the Euclidean axiom he developed in its elements a geometry, which he called *astralgeometry*. What he communicated to me in regard to it, I sent Gauss, who then communicated how much farther had already been advanced on this way [wie viel weiter man schon auf diesem Wege gekommen]." Can this refer to Saccheri or Lambert? Our authors say, p. 252: "Schweikart's achievement consists in this, that independently he clearly recognised and declared the possibility and the justification of a non-Euclidean geometry."

It is satisfactory to give every one the place justly due in what will perhaps be eventually looked upon as the profoundest achievement of modern thought, but it is really comforting to have reaffirmed as the mature outcome of this splendid work what has already long been the world's judgment, that Bolyai and Lobachewski must be looked upon as the real founders of the non-Euclidean geometry.

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