Note on a Problem of Elimination connected with Glisettes of an Ellipse or Hyperbola. By Thomas Muir, LL.D.

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The problem occurs in a paper of Professor Tait's ("Glisettes of an Ellipse and of a Hyperbola," *Proc. Roy. Soc. Edin.*, xvii. pp. 2-4). An ellipse whose semi-axes are a and b is considered as moving so as always to be in contact with both axes of coordinates, and the glissette in question is the curve traced out during this motion by a point whose coordinates with respect to the axes of the ellipse are p and q . Professor Tait states, and it is readily seen, that if the current coordinates of the point be x and y , and θ be the variable angle made by one of the axes of the ellipse with the axis of x , we have the equations

$$\begin{aligned} x &= \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + p \cos \theta - q \sin \theta \\ y &= \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} + p \sin \theta + q \cos \theta \end{aligned}$$

or

$$\begin{aligned} (x - p \cos \theta + q \sin \theta)^2 &= a^2 \cos^2 \theta + b^2 \sin^2 \theta \quad \dots \quad (1) \\ (y - p \sin \theta - q \cos \theta)^2 &= a^2 \sin^2 \theta + b^2 \cos^2 \theta \quad \dots \quad (2) \end{aligned}$$

and that to obtain the x -and- y equation of the glissette it remains to eliminate θ between these two equations.

Before attempting the elimination I first satisfied myself that the resultant must be, as affirmed by Professor Cayley, of the 8th degree.

From (1) and (2) by addition there is obtained

$$2(qx - py) \sin \theta = 2(px + qy) \cos \theta + a^2 - p^2 + b^2 - q^2 - x^2 - y^2 \quad (S),$$

and, on squaring both sides of this, and putting for shortness' sake

$$\begin{aligned} a^2 - p^2 &= A, \\ b^2 - q^2 &= B, \\ x^2 + y^2 &= r^2, \end{aligned}$$

there results the quadratic in $\cos \theta$

$$4(p^2 + q^2)r^2 \cos^2 \theta + 4(A + B - r^2)(px + qy) \cos \theta + (A + B - r^2)^2 - 4(qx - py)^2 = 0 \quad (\alpha).$$

Again, if in (1) and (2) the terms involving the first power of $\sin \theta$ be alone retained on the left-hand side, $\sin \theta$ may be eliminated between the two equations by division, and thus a second quadratic in $\cos \theta$ obtained, viz.,

$$\begin{aligned} & \{(A - B + 2p^2)qx + (B - A + 2q^2)py\} \cos^2 \theta \\ & + \{(A + B)pq - pqr^2 - 2(p^2 + q^2)xy\} \cos \theta \\ & + \{px^2y + qxy^2 - Aqx - Bpy\} = 0 \quad (\beta). \end{aligned}$$

This, it may be noted, is the equation also arrived at by using (S) to eliminate $\sin \theta$ from either (1) or (2).

Writing now the equations (α) and (β), for shortness' sake, in the forms

$$\left. \begin{aligned} L_2 \cos^2 \theta + M_3 \cos \theta + N_4 &= 0 \\ \lambda_1 \cos^2 \theta + \mu_2 \cos \theta + \nu_3 &= 0 \end{aligned} \right\},$$

where the suffixes indicate the degrees of the coefficients as regards x and y , the resultant desired is known to be

$$\begin{vmatrix} . & L_2 & M_3 & N_4 \\ L_2 & M_3 & N_4 & . \\ . & \lambda_1 & \mu_2 & \nu_3 \\ \lambda_1 & \mu_2 & \nu_3 & . \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} L_2\mu_2 - \lambda_1 M_3 & L_2\nu_3 - \lambda_1 N_4 \\ L_2\nu_3 - \lambda_1 N_4 & M_3\nu_3 - \mu_2 N_4 \end{vmatrix} = 0.$$

This equation is seen to be of the 10th degree, and the difficulty—not by any means a small one—has to be attacked of finding the extraneous quadratic factor. After many trials a fortunate observation led to the discovery that the factor is

$$(qx - py)^2,$$

and then a lengthy and inelegant transformation brought about its ejection and the final equation

	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	x^0
y^0	Ω		-4Θ		2χ		$-4\sigma^2\Theta$		$\sigma^4\Omega$
y^1		$-16pq(a^2-b^2)$		$32pq(a^2-b^2)\rho$		$-16pq(a^2-b^2)\Lambda$			
y^2			4Γ		-4Ψ		4Δ		$-4\sigma^2\Theta$
y^3				$-16pq(a^2-b^2)$				$16pq(a^2-b^2)\Lambda$	
y^4					2Φ		-4Ψ		2χ
y^5						$16pq(a^2-b^2)$		$-32pq(a^2-b^2)\rho$	
y^6							4Γ		-4Θ
y^7								$16pq(a^2-b^2)$	
y^8									Ω

where

$$\begin{aligned}\Omega &= a^4 + b^4 + p^4 + q^4 - 2a^2b^2 - 2a^2p^2 + 2a^2q^2 + 2b^2p^2 - 2b^2q^2 + 2p^2q^2, \\ &= (p^2 + q^2)^2 - 2(p^2 - q^2)(a^2 - b^2) + (a^2 - b^2)^2, \\ &= (a^2 - b^2 - p^2 + q^2)^2 + (2pq)^2;\end{aligned}$$

$$\begin{aligned}\Gamma &= a^4 + b^4 + p^4 + q^4 - 2a^2b^2 + 2a^2p^2 - 2a^2q^2 - 2b^2p^2 + 2b^2q^2 + 2p^2q^2, \\ &= (p^2 + q^2)^2 + 2(p^2 - q^2)(a^2 - b^2) + (a^2 - b^2)^2, \\ &= (a^2 - b^2 + p^2 - q^2) + (2pq)^2, \\ &= \Omega + 4(p^2 - q^2)(a^2 - b^2);\end{aligned}$$

$$\begin{aligned}\Phi &= 3a^4 + 3b^4 + 3p^4 + 3q^4 - 6a^2b^2 + 10a^2p^2 - 10a^2q^2 - 10b^2p^2 + 10b^2q^2 + 6p^2q^2, \\ &= 3(p^2 + q^2)^2 + 10(p^2 - q^2)(a^2 - b^2) + 3(a^2 - b^2)^2, \\ &= 3\Gamma + 4(p^2 - q^2)(a^2 - b^2); \text{ and } \therefore \Phi + \Omega = 4\Gamma;\end{aligned}$$

$$\begin{aligned}\Theta &= a^6 + b^6 + p^6 + q^6 - a^4b^2 - a^4p^2 + a^4q^2 - b^4a^2 + b^4p^2 - b^4q^2 \\ &\quad - p^4a^2 + p^4b^2 + 3p^4q^2 + q^4a^2 - q^4b^2 + 3q^4p^2 \}, \\ &= (p^2 + q^2)^3 - (p^4 - q^4)(a^2 - b^2) - (a^4 - b^4)(p^2 - q^2) + (a^4 - b^4)(a^2 - b^2);\end{aligned}$$

$$\begin{aligned}\Psi &= 3a^6 + 3b^6 + 3p^6 + 3q^6 - 3a^4b^2 + 5a^4p^2 - 5a^4q^2 - 3b^4a^2 - 5b^4p^2 + 5b^4q^2 \\ &\quad + 5p^4a^2 - 5p^4b^2 + 9p^4q^2 - 5q^4a^2 + 5q^4b^2 + 9q^4p^2 \}, \\ &= 3(p^2 + q^2)^3 + 5(p^4 - q^4)(a^2 - b^2) + 5(a^4 - b^4)(p^2 - q^2) + 3(a^4 - b^4)(a^2 - b^2), \\ &= 3\Theta + 8(p^2 - q^2)(a^2 - b^2)\rho;\end{aligned}$$

$$\rho = a^2 + b^2 + p^2 + q^2;$$

$$\begin{aligned}\chi &= 3a^8 + 3b^8 + 3p^8 + 3q^8 - 4a^6p^2 - 4b^6q^2 - 4p^6a^2 - 4q^6b^2 + 12p^6q^2 + 12q^6p^2 \\ &\quad - 6a^4b^4 + 2a^4p^4 + 2a^4q^4 + 2b^4p^4 + 2b^4q^4 + 18p^4q^4 \\ &\quad + 4a^4b^2q^2 - 4a^4p^2q^2 + 4b^4a^2p^2 - 4b^4p^2q - 4p^4b^2q^2 \\ &\quad - 8p^4a^2q^2 - 4q^4a^2p^2 - 8q^4b^2p^2 \\ &\quad + 16a^2b^2p^2q^2;\end{aligned}$$

$$\begin{aligned}\Lambda &= a^4 + b^4 + p^4 + q^4 + 2a^2b^2 - 2a^2p^2 + 2a^2q^2 + 2b^2p^2 - 2b^2q^2 + 2p^2q^2, \\ &= (p^2 + q^2)^2 - 2(p^2 - q^2)(a^2 - b^2) + (a^2 + b^2)^2, \\ &= \Omega + (2ab)^2;\end{aligned}$$

$$\Delta = \left. \begin{aligned} &3a^8 + 3b^8 + 3p^8 + 3q^8 - 4a^6q^2 - 4b^6p^2 - 4p^6b^2 - 4q^6a^2 + 12p^6q^2 + 12q^6p^2 \\ &- 6a^4b^4 - 6a^4p^4 - 6a^4q^4 - 6b^4p^4 - 6b^4q^4 + 18p^4q^4 \\ &+ 4a^4b^2p^2 + 12a^4p^2q^2 + 4b^4a^2q^2 + 12b^4p^2q^2 - 4p^4a^2q^2 \\ &- 8p^4b^2q^2 - 4q^4b^2p^2 - 8q^4a^2p^2 \\ &- 48a^2b^2p^2q^2; \end{aligned} \right\}$$

$$\sigma = a^2 + b^2 - p^2 - q^2.$$

A complicated result like this, dependent upon lengthy calculations, is more of a hindrance than a help, unless it can be perfectly relied upon.

One test as to its accuracy is suggested by the constitution of the original equations (α) and (β), which on examination will be seen to be symmetrical with respect to the interchange

$$\left\{ \begin{array}{cc} a & b \\ p & q \\ x & y \end{array} \right\}.$$

Now a glance at Ω , Γ , Φ , Θ , Ψ , ρ , χ , Λ , Δ , σ will be sufficient to show that they are all symmetrical with respect to the interchange

$$\left\{ \begin{array}{cc} a & b \\ p & q \end{array} \right\},$$

and this being ascertained it is at once manifest that the resultant is, as it ought to be, symmetrical with respect to the same interchange as the original equations.

Next, there is the test of its agreement with the results otherwise obtainable for special cases,—a test which if satisfied is additionally instructive in that it makes evident the mode in which the degenerate forms of the equations arise.

Taking in the first place the case where $p = q = 0$. We have

$$\begin{aligned}\Omega &= (a^2 - b^2)^2 = \Gamma = \frac{1}{3}\Phi, & \chi &= 3(a^4 - b^4)^2 = \Delta, \\ \Theta &= (a^4 - b^4)(a^2 - b^2) = \frac{1}{3}\Psi, & \Lambda &= (a^2 + b^2)^2, \\ \rho &= a^2 + b^2 = \sigma,\end{aligned}$$

and the equation becomes

$$(a^2 - b^2)^2 \{ (x^2 + y^2) - (a^2 + b^2) \}^4 = 0,$$

$$\text{i.e.,} \quad x^2 + y^2 = a^2 + b^2,$$

as it should be.

Secondly, for the case where the tracing point is a *focus* of the ellipse, that is, where $p^2 = a^2 - b^2$ and $q = 0$, we find

$$\begin{aligned} \Omega &= 0, & \chi &= 8b^4(a^2 - b^2)^2 \\ \Gamma &= 4(a^2 - b^2)^2 = \frac{1}{4}\Phi, & \Lambda &= 4a^2b^2, \\ \Theta &= 0, & \sigma &= 2b^2; \\ \Psi &= 16a^2(a^2 - b^2)^2, \\ \rho &= 2a^2, \end{aligned}$$

and the equation takes the form

$$16(a^2 - b^2)^2(x^2 + y^2) \{ (x^2 + y^2)x^2y^2 - 4a^2x^2y^2 + b^4(x^2 + y^2) \} = 0,$$

$$\text{i.e.,} \quad (x^2 + y^2)(x^2y^2 + b^4) = 4a^2x^2y^2;$$

which is readily seen to be correct, because the original equations then become

$$\begin{cases} 2px \cos \theta + b^2 - x^2 = 0 \\ 2py \sin \theta + b^2 - y^2 = 0 \end{cases}$$

and thus give

$$\left(\frac{x^2 - b^2}{2px} \right)^2 + \left(\frac{y^2 - b^2}{2py} \right)^2 = 1,$$

$$\text{i.e.,} \quad x^2 - 2b^2 + \frac{b^4}{x^2} + y^2 - 2b^2 + \frac{b^4}{y^2} = 4p^2 = 4a^2 - 4b^2,$$

$$\text{i.e.,} \quad (x^2 + y^2) \left(1 + \frac{b^4}{x^2y^2} \right) = 4a^2,$$

$$\text{i.e.,} \quad (x^2 + y^2)(x^2y^2 + b^4) = 4a^2x^2y^2.$$

Thirdly, there is the very interesting case where the tracing point is an extremity of an axis; say, where $p = 0$, $q = b$. In this case we find

$$\begin{aligned} \Omega &= a^4, & \rho &= a^2 + 2b^2, \\ \Gamma &= (a^2 - 2b^2)^2, & \sigma &= a^2, \\ \Phi &= 3a^4 - 16a^2b^2 + 16b^4, & \chi &= 3a^8, \\ \Theta &= a^6, & \Lambda &= a^4 + 4a^2b^2, \\ \Psi &= 3a^6 - 8a^4b^2 - 8a^2b^4 + 16b^6, & \Delta &= 3a^8 - 4a^6b^2 - 12a^4b^4; \end{aligned}$$

and the equation takes at the outset the rather awkward form

$$\begin{aligned}
& \alpha^4 x^8 + 4(\alpha^2 - 2b^2)^2 x^6 y^2 + 2(3\alpha^4 - 16a^2 b^2 + 16b^4) x^4 y^4 + 4(\alpha^2 - 2b^2)^2 x^2 y^6 + \alpha^4 y^8 \\
& - 4a^6 x^6 - 4(3a^6 - 8a^4 b^2 - 8a^2 b^4 + 16b^6) x^4 y^2 - 4(3a^6 - 8a^4 b^2 - 8a^2 b^4 + 16b^6) x^2 y^4 - 4a^6 y^6 \\
& + 6a^8 x^4 + 4(3a^8 - 4a^6 b^2 - 12a^4 b^4) x^2 y^2 + 6a^8 y^4 \\
& - 4a^{10} x^2 - 4a^{10} y^2 \\
& + \alpha^{12} = 0.
\end{aligned}$$

A suspicion attaches to the coefficient of $x^2 y^2$, which, judging from the forms of the other coefficients, we should consider to be incorrect in not having a term in $a^2 b^6$. This suspicion is intensified on proceeding with the simplification of the equation. In the first line we observe that the terms of the expansion of $\alpha^4(x^2 + y^2)^4$ occur, and following up this observation we transform the equation into

$$\begin{aligned}
& \alpha^4(x^2 + y^2)^4 + 16b^2(b^2 - \alpha^2)x^6 y^2 + 32b^2(b^2 - \alpha^2)x^4 y^4 + 16b^2(b^2 - \alpha^2)x^2 y^6 \\
& - 4a^6(x^2 + y^2)^3 + 32b^2(\alpha^4 + a^2 b^2 - 2b^4)x^4 y^2 + 32b^2(\alpha^4 + a^2 b^2 - 2b^4)x^2 y^4 \\
& + 6a^8(x^2 + y^2)^2 - 16a^2 b^2(\alpha^4 + 3a^2 b^2)x^2 y^2 \\
& - 4a^{10}(x^2 + y^2) \\
& + \alpha^{12} = 0.
\end{aligned}$$

But the sum of the first terms of all the lines here is evidently $\alpha^4(x^2 + y^2 - a^2)^4$, consequently we make the further transformation

$$\begin{aligned}
& \alpha^4(x^2 + y^2 - a^2)^4 + 16b^2(b^2 - \alpha^2)x^2 y^2(x^2 + y^2)^2 \\
& + 32b^2(\alpha^2 + 2b^2)(\alpha^2 - b^2)x^2 y^2(x^2 + y^2) \\
& - 16a^2 b^2(\alpha^4 + 3a^2 b^2)x^2 y^2 = 0.
\end{aligned}$$

Now, were it not for the absence of the term above referred to, a most important simplification could have been effected. This is best seen by actually supplying such a term, viz., $-4b^4$ in the bracketed portion of the coefficient of $x^2 y^2$. We then obtain

$$\alpha^4(x^2 + y^2 - a^2)^4 - 16b^2(\alpha^2 - b^2)x^2 y^2 \left(\begin{aligned} & (x^2 + y^2)^2 \\ & - 2(\alpha^2 + 2b^2)(x^2 + y^2) \\ & + a^2(\alpha^2 + 4b^2) \end{aligned} \right) = 64a^2 b^6 x^2 y^2;$$

and since the quadratic expression here in $x^2 + y^2$ resolves into two factors $(x^2 + y^2) - \alpha^2$ and $(x^2 + y^2) - (\alpha^2 + 4b^2)$, the former factor, $x^2 + y^2 - \alpha^2$, becomes a factor of the whole of the left-hand side, and would with a more complete-looking coefficient of $x^2 y^2$, have disappeared altogether and left for this particular case of the glissette an equation of the 6th degree. As a matter of fact the equation takes the form

$$a^4(x^2 + y^2 - a^2)^4 - 16b^2(a^2 - b^2)x^2y^2(x^2 + y^2 - a^2)(x^2 + y^2 - a^2 - 4b^2) = 64a^2b^6x^2y^2.$$

The question therefore is whether a separate investigation of the equation for the case $p=0$, $q=b$ would confirm or remove the suspicion raised in regard to the coefficient of x^2y^2 . When $p=0$ and $q=b$ the original equations become

$$\left. \begin{aligned} x^2 - 2bx \sin \theta &= a^2 \cos^2 \theta \\ y^2 - 2by \cos \theta &= a^2 \sin^2 \theta \end{aligned} \right\}$$

From these we can derive two simple equations in $\sin \theta$, $\cos \theta$, viz. :

$$\left. \begin{aligned} 2bx \sin \theta - 2by \cos \theta &= -P \\ (a^2xP - 4b^2xy^2) \sin \theta + (a^2yP - 4b^2x^2y) \cos \theta &= 0 \end{aligned} \right\}$$

where P stands for $x^2 + y^2 - a^2$. Solving for $\sin \theta$, $\cos \theta$ we obtain

$$\left\{ \frac{P(a^2P - 4b^2y^2)}{4by(a^2P - 2b^2r^2)} \right\}^2 + \left\{ \frac{P(4b^2x^2 - a^2P)}{4bx(a^2P - 2b^2r^2)} \right\}^2 = 1,$$

which readily leads to

$$a^4P^4r^2 - 16a^2b^2x^2y^2P^2(P + a^2) + 16b^4P^2x^2y^2r^2 = -64a^2b^4x^2y^2Pr^2 + 64b^6x^2y^2r^4,$$

so that, on striking out the factor $x^2 + y^2$, we have the equation in the form

$$a^4P^4 - 16b^2x^2y^2(a^2 - b^2)P^2 = -64b^4x^2y^2\{a^2P - b^2(P + a^2)\},$$

or

$$= -64b^4x^2y^2(a^2 - b^2)P + 64a^2b^6x^2y^2,$$

or

$$a^4P^4 - 16b^2x^2y^2(a^2 - b^2)P(P - 4b^2) = 64a^2b^6x^2y^2;$$

and this agrees to the letter with the suspected result obtained by specializing from the general equation.

There thus, I think, can be little doubt that the calculation of the coefficients in the general equation has been accurately performed.