XXXVIII.

An Analytical Discussion of Dr MATTHEW STEWART'S General Theorems. By THOMAS STEPHENS DAVIES, Esq., F.R.S.L. & Ed., F.A.S., Royal Military Academy, Woolwich.

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PART FIRST.

DURING the century which has nearly elapsed since Dr MATTHEW STEWART published his *General Theorems*, many eminent geometers, both English and Foreign, have attempted to discover their solutions. Those attempts have, however, been rewarded with but limited success, and by far the most general and the most difficult of them remain still without a single published remark in the way of discussion or solution. Dr STEWART did not, as far as I know, make allusion to them himself in any of his subsequent writings, though he describes them as “of considerable use in the higher parts of mathematics ;” and we learn from the preface to Mr GLENIE’S *Demonstration of the 42d Proposition (Tract, 1813)*, that in conversation, Professor DUGALD STEWART, in 1805, stated that, “he had not been able to find amongst his father’s posthumous papers one word respecting them ; that he had, oftener than once, observed mention made of them, in terms of admiration and respect, by some of the first mathematicians on the continent of Europe ; but that as both they and the geometers in this country had tried their strength on them without success, and they had so long remained without demonstrations, he never expected to see them demonstrated.” This circumstance, of neither any demonstrations nor even memoranda, on the subject being found amongst Dr STEWART’S papers, is readily accounted for by Professor PLAYFAIR, in his biography of that distinguished geometer (*Edin. Trans.* vol. i. p. 74), in the description which he gives of the habits of study of Dr STEWART. “He rarely wrote down any of his investigations till it became necessary to do so for the purpose of publication. When he discovered any proposition, he would put down the enunciation with great accuracy, and on the same piece of paper would construct very neatly the figure to which it referred. To these he trusted for recalling to his mind, at any future period, the demonstration or the analysis, however complicated it might be.”

It thus appears that no ground exists for our hoping to discover the means by which Dr STEWART originally investigated these theorems ; whilst we cannot but be surprised at the powers of attention and invention of that mind which could carry on, without the aid of writing, such extended and complicated inquiries as are implied in the discovery of them. Neither can we be surprised if some oversights should occur in their investigations, conducted in such a manner. That oversights do exist, will, however, presently appear.

It is curious enough that, among the geometers who have written concerning these theorems, very few have seen their true character. In fact, Professor PLAYFAIR is the only one who has distinctly stated that “they are, for the most part, *porisms*,” but he nowhere enters into any discussion of them, either under this or any other aspect. In one place, however, (*Ed. Rev.* vol. xvii. p. 129,) he recommends to the attention of geometers these propositions, as fitting subjects for the employment of the *trigonometrical analysis*, and speaks of “the difficulties which they will present even to those who come armed with that powerful instrument.” Amongst the other authors who have spoken of these propositions as to logical character, it may be sufficient to quote two merely; but the scientific rank and high acquirements of these two will prove that very precise views are not entertained by mathematicians, even in this country, respecting these propositions. Mr BABBAGE says, that “many of them are *capable of forming, with a slight alteration in their enunciations*, the most beautiful porisms,” (*Quarterly Journal of Science*, vol. i.; and Mr ELLIS affirms that, “whether they are in reality porismatic, is a question on which it would not be worth while to enter.” (*Cambr. Journal*, May 1841.) Adopting SIMSON’s definition, however, of the porism, it will be quite clear that a considerable number of them—that is, all which are really porismatic—have the *strictly porismatic form of enunciation*. Of the remaining ones, a very small number are local theorems; and the rest are given in the ordinary form of indeterminate theorems.

In all the attempts at solution of the porismatic part of these propositions that I have met with, they have invariably been treated as indeterminate theorems, the porismatic constructions being first supplied; and in supplying these, the authors, having no mode of analysis adapted to their object (except from conjecture), had to encounter difficulties which would inevitably render their success impossible. In fact, the skill and address manifested by Dr SMALL (*Ed. Trans.* vol. ii.), and Messrs LOWRY and SWALE (*Leyb. Repos.*, O.S., vols. i. ii.), manifest the most profound geometrical sagacity, and will reflect a lasting honour on their names: but, at the same time, it must always be regretted that their degree of success was not proportioned to the labour and ability employed in their researches. (Note A.)

At a very early period of my own studies, the porisms engaged much of my attention, and excited a deep interest in the inquiry. This interest, in the outset, was created by the paper of Professor PLAYFAIR, in the *Edinburgh Transactions*,—one of the most luminous and philosophical discussions of a mathematical subject it has ever been my good fortune to read. His suggestion of an algebraical analysis of the porism, which unfortunately he never published, led me to attempt such an application myself; and it could not long escape notice, under these circumstances, that the method of treatment must be identical with that employed in the “method of indeterminate co-efficients;” in fact, that this latter method always occurs in the shape of a porism, and all the propositions in

which it can be applied are strictly, in form and essence, porisms. At the same time, I saw that, in most geometrical porisms, the co-ordinate method would give considerable facility in conducting the actual solution; and having applied this method to a considerable number of porisms which had been treated geometrically, its application to Dr STEWART'S general propositions became natural. In this way, by the use of *rectangular* co-ordinates, nearly the whole of the propositions which had been discussed by Dr SMALL, and Messrs LOWRY and SWALE, were readily established, together with the last five porisms of Dr STEWART respecting the circle. A few of these were sent to a periodical work; but some circumstances connected with that paper, induced me to lay aside the subject altogether, till a recent period. The views to which I was at that time led, have been since explained in the "MATHEMATICIAN" (Nos. 1 and 2), to which I must refer for details which would be unsuitable to the present paper. I had intentionally omitted from this latter paper all reference to Dr STEWART'S theorems, for two reasons:—*first*, That I had found the insufficiency of the *rectangular* co-ordinate system to meet the object of the more general propositions, from its always giving a redundancy of conditional equations, arising out of a peculiarity in the expressions; and, *secondly*, that I had found the method of *polar* co-ordinates free from this embarrassing objection in all the cases I had tried, and hoped to find it so in all cases whatever. Having now found that such is the case, and having likewise discovered a method of solving the equations to which Dr STEWART'S porisms give rise, I am desirous of laying the results before the Royal Society of Edinburgh.

Many reasons induce this wish. Dr STEWART'S position in the University of Edinburgh, and his being one of the most distinguished of the original Fellows of the Royal Society, are reasons, however, paramount to all others; and I am led to believe that an interest will be felt (even in a subject purely relative to speculative mathematics) by that Society, to which I ought to pay respect. Another is, that the *polar equation of the straight line*, of which so much use is made in this discussion, was first given, incidentally, in the Edinburgh Transactions (vol. xii.); and the present is the first application made of that system of equations, except to comparatively elementary inquiries. The subject of these equations has, however, been more amply developed in my recent edition of Dr HUTTON'S *Mathematics* (12th edit.), to which reference may be made in any case where the first sketch, already referred to, may be considered incomplete.

Adopting, as I do, without modification, Dr SIMSON'S definition of the porismatic proposition, and taking into account that the point from which lines are drawn (either to points or perpendicular to lines), is arbitrary, the following statement of the process which I employ will appear both simple and obvious. It will, however, be necessary to remark, that the points, lines, or other entities, which the proposition affirms to be determinable, are called, for precision, *poris-*

matic points, porismatic lines, etc.; and that these points, lines, *etc.*, are said to be *porismatised*, instead of *given*, as usually expressed.

The arbitrary point is invariably denoted by the polar co-ordinates $r \theta$; the porismatic are denoted by the *unknown co-ordinates* of the point, if a point be porismatised; and by the equation of a locus with *unknown co-efficients*, if a line or any other locus. The equation of the porism is then formed by means of these co-ordinates of points, or equations of lines, the several data of the proposition, and the arbitrary point $r \theta$.* Then, since $r \theta$ is perfectly arbitrary, the general equation of the porism can only be fulfilled by the co-efficients of the several combinations of r and θ which appear in it, being separately and simultaneously equal to zero. This equating to zero of those several co-efficients, gives a number of conditional equations, involving the several porismatic unknowns; and we must have as many equations, independent of each other, as there are unknowns porismatised in the statement of the proposition. Should the number of these conditional equations be in excess or defect of the number of porismatised unknowns, the porism is incorrectly stated. However, it is always easy to correct the enunciated porism so as to fulfil these conditions, either by abstraction from the number of porismatised entities, or by addition to them, as the case may require.

The number of conditional equations may, however, be correct, and yet the porism not true: for if there be not corresponding *real* values for each of the unknowns deducible from these equations, it will follow that the conditions of the porism are inconsistent with each other. The complete algebraical solution of a porism requires, therefore, that the conditional equations shall be either actually resolved, or at least that it shall be shewn that the roots of the final equations in each of them, from which all the others have been eliminated, are real.

In the first part of this discussion, I have, in the main, attended to the formation of the conditional equations of the porisms, and the correct determination of the number of porismatic points and lines: but still I have resolved the equations themselves in a great number of cases, including those belonging to several porisms that have not been before established by any method. The equations, however, that result are of a peculiar class and admit of easy discussion by one general method. The preliminary discussions which force themselves upon us in the solution of these, would occupy so much space, that I have thought it better to defer them to the second, or concluding part of the paper. I have, however, judged it proper to give, in one case, a separate proof of the erroneous number of porismatic entities, enunciated by Dr STEWART, in order to remove any latent suspicion that certain of the equations were vir-

* When the point is not entirely arbitrary (as in most porisms is the case), r and θ will be connected by an equation which defines the locus of the partially arbitrary point. Any detail upon this head would, however, be altogether irrelevant in this place. See "The Mathematician," as above.

tually contained in the others. (*Note E.*) A formal proof of the correctness of my own determination will be contained in the actual solution of the general conditional equations. It will there appear, not only that this determination is correct, but also that, under these modified conditions, the values of all the porismatic entities are *essentially real*.

The *indeterminate theorems* except those established by Dr STEWART, are also proved in the present part of the discussion. As these have been established already in different ways, it was not necessary to dwell upon them at any considerable length, the mere indication of the application of our lemmas to this purpose, being deemed sufficient for the object in view.

Of the few propositions regarding *loci*, it is sufficient to remark, that this branch of the subject is too well understood at the present day, to need discussion here; and, at the same time, that some of them are true, *only* under special relations amongst the data, instead of universally true, as Dr STEWART has enunciated them.

The five properties of the circle which form the concluding ones of the *General Theorems*, are deferred to the next part of this paper, not from any peculiarity in the manner of treating them, but to equalise, as much as possible, the extent of the two parts into which the discussion is divided.

In the enunciations, the forms are altered to suit the view under which the method of solution here employed would give them in the most convenient shape for use. An abbreviated notation, too, is employed in the expression of the theorems: but the principle of that notation is so simple, and, indeed, the notation itself is so commonly in use for analogous purposes, as to render it almost needless to distinctly specify it. It is

$$a_1 + a + \dots + a_m = S a_m,$$

$$a_1 r_1^n + a_2 r_2^n + \dots + a_m r_m^n = S (a_m r_m^n); \text{ etc.}$$

The classification, also, of the *General Theorems* is here added.

1. *Indeterminate Theorems.* 1, 2, 3, 4, 5, 6, 7, 8, 22, 23, 26, 27, 28, 29, 34, 39, 40, 41, 42, 45.

2. *Porismatic, relating to points and lines.* 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 24, 25, 30, 31, 32, 33, 35, 36, 37, 38, 43, 44, 46, 47, 48, 49, 50, 51, 52, 53.

3. *Loci.* 54, 55, 56, 57, 58, 59.

4. *Porismatic, relating to the circle.* 60, 61, 62, 63, 64.

A few notes are added on some articles in the paper, as it appeared more convenient to place these remarks in such a form than in *scholia* or foot-notes.

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SECTION I.—LEMMAS TO BE USED IN THE DISCUSSION.

LEMMA I.

To expand $(r^2 - 2 r r_1 \cos \omega + r_1^2)^n$ in multiple cosines, n being a positive integer.

Put $2 \cos \omega = u + \frac{1}{u}$: then,

$$(r^2 - 2 r r_1 \cos \omega + r_1^2)^n = (r - r_1 u)^n \left(r - \frac{r_1}{u}\right)^n.$$

Expand both these by the binomial theorem, writing $t_0, t_1, t_2, \dots, t_{n-1}, t_n$ for the co-efficients of the first, second, third, . . . , n th, $(n+1)$ th terms of the expansion $(1+1)^n$. We thus have the two series.

$$\begin{aligned} & t_0 r^n - t_1 r^{n-1} r_1 u + t_2 r^{n-2} r_1^2 u^2 - t_3 r^{n-3} r_1^3 u^3 + t_4 r^{n-4} r_1^4 u^4 - \dots \\ & t_0 r^n - t_1 r^{n-1} \frac{r_1}{u} + t_2 r^{n-2} \frac{r_1^2}{u^2} - t_3 r^{n-3} \frac{r_1^3}{u^3} + t_4 r^{n-4} \frac{r_1^4}{u^4} - \dots \end{aligned}$$

The following simple arrangement of the order of multiplication will enable us to obtain $(r_1 u)^k$ and $\left(\frac{r_1}{u}\right)^k$ simultaneously, and thence the value of $\cos(k\omega)$ for the successive values of k from 0 to n .

1. Multiply each term of the first series by that one of the second which stands immediately beneath it: the sum of these products is that term of the expansion which is clear of cosines, or that in which $k=0$.

2. Multiply each term of the first series by that one of the second which stands immediately to the right of it, which will be the co-efficient of $\frac{1}{u}$: then multiply each term of the second series by that one of the first which stands immediately to the right of it, which gives the co-efficients of u . These co-efficients will evidently be identical. Whence we shall have the compound co-efficient of $u + \frac{1}{u}$ or of $2 \cos \omega$.

3. Multiply cross-ways as before, the multipliers being in this case two steps to the right, instead of one, as in the preceding case: the result will be the compound co-efficient of $u^2 + \frac{1}{u^2}$ or of $2 \cos 2\omega$.

4. Taking, in like manner, the multipliers three steps to the right of the terms multiplied, we shall obtain the compound co-efficient of $u^3 + \frac{1}{u^3}$ or of $2 \cos 3\omega$.

Proceeding thus, we shall have every term of the first series multiplied by every term of the second, without any repetition of the same factors: and the required expansion will be found to be (R^2 denoting the vinculated trinomial),

$$\begin{aligned} R^{2n} = & \{ t_0^2 r^{2n} + t_1^2 r^{2(n-1)} r_1^2 + t_2^2 r^{2(n-2)} r_1^4 + \dots + t_n^2 r_1^{2n} \} \\ & - 2 r r_1 \cos \omega \{ t_0 t_1 r^{2(n-1)} + t_1 t_2 r^{2(n-2)} r_1^2 - t_2 t_3 r^{2(n-3)} r_1^4 + \dots + t_{n-1} t_n r_1^{2(n-1)} \} \\ & + 2 r^2 r_1^2 \cos 2\omega \{ t_0 t_2 r^{2(n-2)} + t_1 t_3 r^{2(n-3)} r_1^2 + t_2 t_4 r^{2(n-4)} r_1^4 + \dots + t_{n-2} t_n r_1^{2(n-n)} \} \\ & - 2 r^3 r_1^3 \cos 3\omega \{ t_0 t_3 r^{2(n-2)} + t_1 t_4 r^{2(n-4)} r_1^2 + t_2 t_5 r^{2(n-5)} r_1^4 + \dots + t_{n-3} t_n r_1^{2(n-3)} \} \\ & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ = & \pm 2 r^{n-1} r_1^{n-1} \cos (n-1) \omega \{ t_0 t_{n-1} r^2 + t_1 t_n r_1^2 \} \\ = & \pm 2 r^n r_1^n \cos n \omega \{ t_0 t_n \} \end{aligned}$$

The upper or lower signs of the concluding terms applying as n is even or odd.

It will be obvious that a slight simplification of this result may be obtained from the consideration that n being integer, $t_0=t_n$, $t_1=t_{n-1}$, *etc.* Our present object will not, however, require any special attention to this circumstance.

For example, let $n=2$, then

$$(r^2 - 2 r r_1 \cos \omega + r_1^2)^2 = r^4 + 4 r^2 r_1^2 + r_1^4 - 2 r r_1 \cos \omega \{2 r^2 + 2 r_1^2\} + 2 r^2 r_1^2 \cos 2 \omega.$$

Let $n=3$: then

$$\begin{aligned} (r^2 - 2 r r_1 \cos \omega + r_1^2)^3 = & r^6 + 9 r^4 r_1^2 + 9 r^2 r_1^4 + r_1^6 \\ & - 2 r r_1 \cos \omega \{3 r^4 + 9 r^2 r_1^2 + 3 r_1^4\} \\ & + 2 r^2 r_1^2 \cos 2 \omega \{3 r^2 + 3 r_1^2\} \\ & - 2 r^3 r_1^3 \cos 3 \omega. \end{aligned}$$

LEMMA II.

To expand $(p-r \cos \omega)^n$ in multiple cosines, n being a positive integer.

By the binomial theorem we have

$$(p-r \cos \omega)^n = t_0 p^n - t_1 p^{n-1} r \cos \omega + t_2 p^{n-2} r^2 \cos^2 \omega - t_3 p^{n-3} r^2 \cos^3 \omega + \dots$$

For the powers of $\cos \omega$ put their values in multiple cosines: then the expansion becomes,

$$\begin{aligned} & \left\{ t_0 p^n + \frac{t_2 p^{n-2} r^2}{2^1} + \frac{3 t_4 p^{n-4} r^4}{2^3} + \frac{10 t_6 p^{n-6} r^6}{2^5} + \dots \right\} \\ & - \left\{ \frac{t_1 p^{n-1} r}{2^0} + \frac{3 t_3 p^{n-3} r^3}{2^2} + \frac{10 t_5 p^{n-5} r^5}{2^4} + \frac{35 t_7 p^{n-7} r^7}{2^6} + \dots \right\} \cos \omega \\ & + \left\{ \frac{t_2 p^{n-2} r^2}{2^1} + \frac{4 t_4 p^{n-4} r^4}{2^3} + \frac{15 t_6 p^{n-6} r^6}{2^5} + \dots \right\} \cos 2 \omega \\ & - \left\{ \frac{t_3 p^{n-3} r^3}{2^2} + \frac{5 t_5 p^{n-5} r^5}{2^4} + \frac{21 t_7 p^{n-7} r^7}{2^6} + \dots \right\} \cos 3 \omega \\ & \qquad \qquad \qquad \&c. \qquad \qquad \&c. \qquad \qquad \&c. \end{aligned}$$

The law of the numerical co-efficients of the several powers of p and r are

already known : but as (except those of the first line, which is clear of the cosines) we shall not require them in these inquiries, it will be unnecessary to discuss them, beyond the extent which our present purpose demands.

It will be observed, that all the terms, after the first, of the first line, arise from the expansion of the *even powers* of $\cos \omega$, and are, in fact, the absolute terms of those expanded even powers. Let these powers be denoted generally by 2μ : then the general form of these terms is

$$\frac{1}{2^{2\mu}} \cdot \frac{2\mu(2\mu-1)(2\mu-2)\dots(\mu+1)}{1 \cdot 2 \cdot 3 \cdot \dots \mu}.$$

This will take, very readily and obviously, the following forms in succession :

$$\begin{aligned} & \frac{1}{2^{2\mu}} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2\mu-1) \times 2 \cdot 4 \cdot 6 \dots 2\mu}{(1 \cdot 2 \cdot 3 \dots \mu)^2} \\ &= \frac{1}{2^{2\mu}} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2\mu-1) \times 2^\mu \cdot 1 \cdot 2 \cdot 3 \dots \mu}{(1 \cdot 2 \cdot 3 \dots \mu)^2} \\ &= \frac{1}{2^{2\mu}} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2\mu-1)}{1 \cdot 2 \cdot 3 \dots \mu} \\ &= \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2\mu-1)}{2 \cdot 4 \cdot 6 \cdot 8 \dots 2\mu}. \end{aligned}$$

Giving to μ the successive values which are applicable to the successive cases viz. 1, 2, $\dots$ n , we have the form of the first line changed to

$$p^n + \frac{1}{2} \cdot t_2 p^{n-2} r^2 + \frac{1 \cdot 3}{2 \cdot 4} t_4 p^{n-4} r^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} t_6 p^{n-6} r^6 + \dots$$

For examples of the general expansion, let $n=2$: then

$$(p - r \cos \omega)^2 = (p^2 + \frac{1}{2} r^2) - 2 p r \cos \omega + \frac{1}{2} r^2 \cos 2 \omega,$$

Let $n=3$: then

$$(p - r \cos \omega)^3 = p^3 + \frac{3 p r^2}{2} - 3 (p^2 r + \frac{1}{4} r^3) \cos \omega + \frac{3}{2} p r^2 \cos 2 \omega - \frac{1}{4} r^3 \cos 3 \omega.$$

Let $n=4$: then

$$\begin{aligned} (p - r \cos \omega)^4 &= p^4 + 3 p^2 r^2 + r^4 - (4 p^3 r + 3 p r^3) \cos \omega \\ &\quad + (3 p^2 r^2 + \frac{1}{2} r^4) \cos 2 \omega - p r^3 \cos 3 \omega + \frac{1}{8} r^4 \cos 4 \omega. \end{aligned}$$

LEMMA III.

To expand $(1 - \cos \omega)^n$ in multiple cosines of ω , n being an integer.

This is, in fact, but a particular case of each of the preceding lemmas, and its expansion might be deduced from either of them by making the requisite modifications in the formulæ.

In the first lemma, by making $r_1=r$, we should get

$$(r^2 - 2 r r_1 \cos \omega + r_1^2)^n = (2 r^2)^n (1 - \cos \omega)^n;$$

and in the second, by making $p=r$, we should get

$$(p - r \cos \omega)^n = r^n (1 - \cos \omega)^n.$$

We should thus obtain two different forms: but the expansion will be better adapted to our present object when obtained as follows:—

Since $(1 - \cos \omega)^n = 2^n \sin^{2n} \frac{\omega}{2}$; the ordinary formula, making the requisite change in the form of the last term, gives

$$\begin{aligned} \sin^{2n} \frac{1}{2} \omega &= \frac{1}{(-1)^n 2^{2n-1}} \cdot \left\{ \cos 2n \frac{1}{2} \omega - \frac{2n}{1} \cos (2n-2) \frac{1}{2} \omega + \dots \right. \\ &\quad \left. + (-1)^n \cdot \frac{1}{2} \cdot \frac{2n(2n-1) \dots (n+1)}{1 \cdot 2 \cdot 3 \dots n} \right\} \\ &= \frac{1}{(-1)^n 2^{2n-1}} \cdot \left\{ \cos n \omega - \frac{2n}{1} \cos (n-1) \omega + \dots \right. \\ &\quad \left. + (-1)^n \cdot \frac{1}{2} \cdot \frac{2n(2n-1) \dots (n+1)}{1 \cdot 2 \cdot 3 \dots n} \right\} \end{aligned}$$

(for 2^n is always even, when n is an integer.)

$$\begin{aligned}
&= \frac{1}{(-1)^n 2^{2n-1}} \left\{ \cos n \omega - \frac{2^n}{1} \cos (n-1) \omega + \dots \right\} \\
&\quad + \frac{1}{2^{2n}} \cdot \frac{2^n (2^n - 1) \dots (n-1)}{1 \cdot 2 \cdot 3 \dots n} . \\
&= \frac{1}{(-1)^n 2^{2n-1}} \left\{ \cos n \omega - \frac{2^n}{1} \cos (n-1) \omega + \dots \right\} \\
&\quad + \frac{1}{2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} .
\end{aligned}$$

LEMMA IV.

If $\theta_1, \theta_2, \theta_3, \dots, \theta_n$, be n angles in arithmetical progression, whose common difference is $\frac{2\pi}{n}$ (or, which is the same thing, if they be the angles formed by lines from the summits of a regular polygon with any arbitrary line, the centre of the polygon being the origin of co-ordinates), we shall have simultaneously,

$$\begin{array}{rcl} \cos \theta_1 & + \cos \theta_2 & + \cos \theta_3 + \dots + \cos \theta_n = 0 \\ \cos 2 \theta_1 & + \cos 2 \theta_2 & + \cos 2 \theta_3 + \dots + \cos 2 \theta_n = 0 \\ \cos 3 \theta_1 & + \cos 3 \theta_2 & + \cos 3 \theta_3 + \dots + \cos 3 \theta_n = 0 \\ & \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \cos (n-1) \theta_1 & + \cos (n-1) \theta_2 & + \dots + \cos (n-1) \theta_n = 0 \end{array}$$

PROPOSITIONS XI., XII. PORISMS.

Let there be given m points and m magnitudes, as in the preceding : then there may be found a circle, and likewise a point, such, that drawing any line through the point found to cut the circle in X and Y , and that Z be any point whatever in the same plane, we shall always have

$$S(A_m Z^2) = \frac{1}{2} S a_m \cdot (X Z^2 + Y Z^2)$$

For, let the given points, referred to the centroid as origin, and any axis whatever be $r_1 \theta_1, r_2 \theta_2, \dots, r_m \theta_m$; and denote the points X, Y by $u_1 \omega_1, u_2 \omega_2$, and the radius of the circle by ρ ; and let Z be $r \theta$.

Then, expressing the lines concerned in terms of these quantities, cancelling $S a_m \cdot r^2$ from both sides, and equating to zero the co-efficients of r ,

$$S(a_m r_m^2) = \frac{1}{2} S a_m (u_1^2 + u_2^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$u_1 \cos(\theta - \omega_1) + u_2 \cos(\theta - \omega_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But since also θ is arbitrary, (2) becomes

$$u_1 \cos \omega_1 + u_2 \cos \omega_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$u_1 \sin \omega_1 + u_2 \sin \omega_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These two equations are satisfied by *any* two points in a line passing through the origin, and equidistant from it, on opposite sides; or $u_1 = u_2$, and $\omega_1 = \pi + \omega_2$.

The point required to be found is hence the centroid, and the circle has that point for its centre.

It also follows, that $u_1 = u_2 = \rho$ the radius of the circle; and hence from (1) we have

$$\rho^2 = \frac{S(a_m r_m^2)}{S a_m}.$$

Whence the circle is entirely determined.

Dr STEWART porismatises, not a circle, but two points, X and Y , to be found. The equations to which in such form the proposition gives rise, are precisely the same as those above: wherefore there would be given only the three equations (1, 2, 3) for the determination of four quantities $u_1, u_2, \omega_1, \omega_2$, which obviously leaves one of the quantities indeterminate. Dr SMALL notices, in another form, this indeterminateness.

[See also note on these.]

PROPOSITIONS XIII., XVIII. PORISMS.

Let there be given m parallel lines and m magnitudes $a_1, a_2, \dots, a_m$: then there can be found another line parallel to these, and likewise a space s^2 , such, that

Insert these in the equation of the porism (putting the expansion in multiple cosines), cancel, and equate to zero the co-efficients of $\cos 2\theta$ and $\sin 2\theta$. Then there results,

$$\cos 2\omega_1 + \cos 2\omega_2 = \frac{2 S(a_m \cos 2\theta_m)}{S a_m},$$

$$\sin 2\omega_1 + \sin 2\omega_2 = \frac{2 S(a_m \sin 2\theta_m)}{S a_m}.$$

The solution of these equations gives

$$\cos 2\omega_1 = \frac{S(a_m \cos 2\theta_m) \pm R \cdot S(a_m \sin 2\theta_m)}{S a_m},$$

$$\cos 2\omega_2 = \frac{S(a_m \cos 2\theta_m) \mp R \cdot S(a_m \sin 2\theta_m)}{S a_m};$$

$$\text{where } R^2 = \frac{(S a_m)^2 - \{S(a_m \cos 2\theta_m)\}^2 - \{S(a_m \sin 2\theta_m)\}^2}{\{S(a_m \cos 2\theta_m)\}^2 + \{S(a_m \sin 2\theta_m)\}^2}.$$

Since the angles are symmetrically involved in the general expression, we see that the double sign does not imply different possible solutions, but merely that the sign of $\cos 2\omega_2$ depends upon that which we select as belonging to $\cos 2\omega_1$.

That it is always real, is at once obvious: for the denominator of the radical is essentially positive; and the numerator is convertible into

$$\begin{aligned} & 2 a_1 a_2 \{1 - \cos 2(\theta_1 - \theta_2)\} + 2 a_1 a_3 \{1 - \cos 2(\theta_1 - \theta_3)\} + \dots \\ & + 2 a_2 a_3 \{1 - \cos 2(\theta_2 - \theta_3)\} + 2 a_2 a_4 \{1 - \cos 2(\theta_2 - \theta_4)\} + \dots \\ & + \dots \dots \dots \\ & + 2 a_{m-1} \cdot a_m \{1 - \cos 2(\theta_{m-1} - \theta_m)\} \end{aligned}$$

which is, also, obviously positive.

Again, the expressions for the single angles ω_1 , ω_2 are found from the preceding (3, 4), by means of the familiar relation,

$$\cos \omega_1 = \pm \sqrt{\frac{1 + \cos 2\omega_1}{2}} \text{ and } \cos \omega_2 = \pm \sqrt{\frac{1 + \cos 2\omega_2}{2}}.$$

But it is easy to see that these double signs, in each case, only refer to the two opposite branches of the same line in respect of the origin: so that, on the whole, the solutions are found to be single, and that there is one, and only one, pair of lines which fulfils the conditions of the porism.

PROPOSITIONS XVI., XX. PORISMS.

Let there be given m lines, which are neither all parallel nor all meet in one point, and m magnitudes, $a_1, a_2, \dots, a_m$: then there can be found two other straight lines and a space s^2 , such, that if perpendiculars $Z P_1, Z P_2, \dots, Z P_m$ be drawn to all the given lines, and others $Z Q_1, Z Q_2$ to those found, we shall always have

$$2 S (a_m Z P_m^2) = S a_m \cdot \{Z Q_1^2 + Z Q_2^2 + s^2\}.$$

For, let the several given lines be

$$p_1 = r \cos (\theta - \theta_1), p_2 = r \cos (\theta - \theta_2), \dots, p_m = r \cos (\theta - \theta_m);$$

and let the porismatic lines be denoted by

$$q_1 = r \cos (\theta - \omega_1), \text{ and } q_2 = r \cos (\theta - \omega_2).$$

Then the perpendiculars upon these from the arbitrary point Z , or $r \theta$, will be (HUTTON, ii. p. 268),

$$\begin{aligned} &\pm \{p_1 - r \cos (\theta - \theta_1)\}, \pm \{p_2 - r \cos (\theta - \theta_2)\}, \text{ etc.}; \text{ and} \\ &\pm \{q_1 - r \cos (\theta - \omega_1)\}, \text{ and } \pm \{q_2 - r \cos (\theta - \omega_2)\}. \end{aligned}$$

Insert these values in the equation of the porism, and arrange the results in terms $r \cos \theta, r \cos 2 \theta, r \sin \theta$, and $r \sin 2 \theta$; then, as r^2 cancels, we have

$$2 S (a_m p_m^2) = S a_m \cdot \{q_1^2 + q_2^2 + s^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$2 S (a_m p_m \cos \theta_m) = S a_m \cdot \{q_1 \cos \omega_1 + q_2 \cos \omega_2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$2 S (a_m p_m \sin \theta_m) = S a_m \cdot \{q_1 \sin \omega_1 + q_2 \sin \omega_2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$2 S (a_m \cos 2 \theta_m) = S a_m \cdot \{\cos 2 \omega_1 + \cos 2 \omega_2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$2 S (a_m \sin 2 \theta_m) = S a_m \cdot \{\sin 2 \omega_1 + \sin 2 \omega_2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (4, 5) will be found, as in the preceding porism, the values of ω_1, ω_2 ; from these results, combined with (2, 3), we shall obtain, by a simple equation, the values of q_1, q_2 ; and from these values inserted in (1) we finally obtain the value of the space s^2 .

PROPOSITIONS XVII., XXI. PORISMS.

Let there be given m lines, which are neither all parallel nor all meet in one point, and m magnitudes, $a_1, a_2, \dots, a_m$: then there can be found three lines, such, drawing from any point, Z , the perpendiculars $Z P_1, Z P_2, \dots, Z P_m$ to the given lines, and $Z Q_1, Z Q_2, Z Q_3$ to those found, we shall always have

$$3 S (a_m Z P_m^2) = S a_m \cdot (Z Q^2).$$

Such is Dr STEWART'S statement; but forming the equations of condition, as in the preceding Propositions, we have

$$3 S(a_m p_m^2) = S a_m \cdot \{q_1^2 + q_2^2 + q_3^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$3 S(a_m p_m \cos \theta_m) = S a_m \cdot \{q_1 \cos \omega_1 + \dots + q_3 \cos \omega_3\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$3 S(a_m p_m \sin \theta_m) = S a_m \cdot \{q_1 \sin \omega_1 + \dots + q_3 \sin \omega_3\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$3 S(a_m \cos 2 \theta_m) = S a_m \cdot \{\cos 2 \omega_1 + \dots + \cos 2 \omega_3\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$3 S(a_m \sin 2 \theta_m) = S a_m \cdot \{\sin 2 \omega_1 + \dots + \sin 2 \omega_3\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now, in the present case we have only five equations for the determination of six quantities, $\omega_1, \omega_2, \omega_3, q_1, q_2$, and q_3 . The condition of the porism cannot, therefore, be fulfilled without another condition.

This indeterminateness, in respect of this proposition, has been noticed by Dr SMALL, *Ed. Trans.*, ii. p. 46. In the discussion of Props. 46-53 of this Series, will be noticed again.

PROPOSITIONS XXIV., XXV. PORISMS.

Let there be given m lines and m magnitudes as before: then p straight lines can be found, such, that if we draw the perpendiculars from any point Z to all the lines given, and to all the lines found, we shall have

$$p \{S(a_m \cdot Z P_m^3)\} = S a_m \cdot S(Z Q_4^3).$$

The general form of the component terms of this equation is, *Lemma ii.*,

$$Z P_m^3 = (p_m^3 + \frac{3}{2} p_m r^2) - 3(p_m^2 r + \frac{1}{4} r^3) \cos(\theta - \theta_m) + \frac{3}{2} p_m r^2 \cos 2(\theta - \theta_m) - \frac{1}{4} r^3 \cos 3(\theta - \theta_m)$$

Whence, forming the equations of condition, we have the following series,

$$\begin{aligned} p S(a_m \cos 3 \theta_m) &= S a_m \cdot S(\cos 3 \omega_4) \\ p S(a_m \sin 3 \theta_m) &= S a_m \cdot S(\sin 3 \omega_4) \\ p S(a_m p_m \cos 2 \theta_m) &= S a_m \cdot S(q_4 \cos 2 \omega_4) \\ p S(a_m p_m \sin 2 \theta_m) &= S a_m \cdot S(q_4 \sin 2 \omega_4) \\ p S(a_m \cos \theta_m) &= S a_m \cdot S(\cos \omega_4) \\ p S(a_m \sin \theta_m) &= S a_m \cdot S(\sin \omega_4) \\ p S(a_m p_m^2 \cos \theta_m) &= S a_m \cdot S(q_4^2 \cos \omega_4) \\ p S(a_m p_m^2 \sin \theta_m) &= S a_m \cdot S(q_4^2 \sin \omega_4) \\ p S(a_m p_m) &= S a_m \cdot S(q_4) \\ p S(a_m p_m^3) &= S a_m \cdot S(q_4^3) \end{aligned}$$

Now, as there are ten equations, all independent of each other, it follows that $p=5$; for there will be five q s and five ω s to be determined from this system, and these are the requisite conditions for finding five lines. Whence the number of lines is incorrectly given by Dr STEWART who porismatises only *four* lines; but the Porism is evidently possible with the condition altered as here proposed.

PROPOSITIONS XXX., XXXI. PORISMS.

Let there be given m points $A_1, A_2, \dots, A_m$, and m magnitudes, $a_1, a_2, \dots, a_m$: then there can be found two lines $O X, O Y$, and a point P , together with two magnitudes a, b ; such, that if from any point whatever, Z , there be drawn lines to all the given points, and to the point found, together with perpendiculars $Z X, Z Y$ to the lines found, we shall always have

$$S(a_m \cdot A_m Z^4) = S a_m \cdot \{P Z^4 + a^2(Z X^2 + Z Y^2 + b^2)\}.$$

Let, as before, the given points be referred to the centroid as origin of polar co-ordinates, and denoted by $r_1 \theta_1, r_2 \theta_2, \dots, r_m \theta_m$; also denote P by $r_0 \theta_0$, Z by $r \theta$, and the lines $O X, O Y$ by

$$\begin{aligned} p_1 &= r \cos(\theta - \omega_1) \\ p_2 &= r \cos(\theta - \omega_2) \end{aligned}$$

Then the perpendiculars on these from $r \theta$ will be expressed as before by

$$\begin{aligned} Z X &= \pm\{p_1 - r \cos(\theta - \omega_1)\} \\ Z Y &= \pm\{p_2 - r \cos(\theta - \omega_2)\} \end{aligned}$$

Also, the lines from Z to the several points will be expressed in power by

$$A_m Z^4 = r^4 + 4 r^2 r_m^2 + r_m^4 - 4 r r_m (r_m^2 + r^2) \cos(\theta - \theta_m) + 2 r_m^2 r^2 \cos 2(\theta - \theta_m)$$

$$P Z^4 = r^4 + 4 r^2 r_0^2 + r_0^4 - 4 r r_0 (r_0^2 + r^2) \cos(\theta - \theta_0) + 2 r_0^2 r^2 \cos 2(\theta - \theta_0)$$

$$Z X^2 = (p_1^2 + \frac{1}{2} r^2) - 2 p_1 r \cos(\theta - \omega_1) + \frac{1}{2} r^2 \cos 2(\theta - \omega_1)$$

$$Z Y^2 = (p_2^2 + \frac{1}{2} r^2) - 2 p_2 r \cos(\theta - \omega_2) + \frac{1}{2} r^2 \cos 2(\theta - \omega_2)$$

With these values form the equation of the porism, cancel common terms, and equate to zero the co-efficients of the arbitrary quantities which remain; then we get the following system of conditional equations:

$$S(a_m r_m^2) = S a_m \cdot \{r_0^2 + \frac{1}{2}a^2\} \dots \dots \dots (1)$$

$$S(a_m r_m^4) = S a_m \cdot \{r_0^4 + a^2(p_1^2 + p_2^2 + b^2)\} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot (2)$$

$$S(a_m r_m \cos \theta_m) = S a_m . r_0 \cos \theta_0 (3)$$

[illegible]

$$S(a_m r_m^3 \cos \theta_m) = S a_m \cdot \{r_0^3 \cos \theta_0 + \frac{1}{2}a^2(p_1 \cos \omega_1 + p_2 \cos \omega_2)\} \quad . \quad (5)$$

$$S(a_m r_m^3 \sin \theta_m) = S a_m \cdot \{r_0^3 \sin \theta_0 + \frac{1}{2} a^2 (p_1 \sin \omega_1 + p_2 \sin \omega_2)\} \quad . \quad (6)$$

$$S(a_m r_m^2 \cos 2 \theta_m) = S a_m \cdot \{r_0^2 \cos 2 \theta_0 + \frac{1}{4} a^2 (\cos 2 \omega_1 + \cos 2 \omega_2)\} \quad . \quad (7)$$

$$S(a_m r_m^2 \sin 2\theta_m) = S a_m \cdot \{r_0^2 \sin 2\theta_0 + \frac{1}{4} a^2 (\sin 2\omega_1 + \sin 2\omega_2)\} \quad . \quad (8)$$

Now, since the left sides of (3, 4) are zero, we find that $r_0=0$, and that θ_0 is indeterminate. The point P is, therefore, the origin of co-ordinates, or the centroid of the given system itself.

Insert this value of r_0 in (1): then we get

$$a^2 = \frac{2 S (a_m r_m^2)}{S a_m} \dots \dots \dots (9)$$

Insert it in (7, 8) : then these become

$$\cos 2\omega_1 + \cos 2\omega_2 = \frac{4 S (a_m \cdot r_m^2 \cos 2\theta_m)}{a^2 \cdot S a_m} \quad \dots \dots \dots (10)$$

$$\sin 2 \omega_1 + \sin 2 \omega_2 = \frac{4 S (a_m r_m^2 \sin 2 \theta_m)}{a^2 S a_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

from which ω_1 and ω_2 may be found as in (xv. xix); and which, as in that place, are real.

Put the value 0 of r_0 and the values of ω_1, ω_2 in (5, 6): then we get the values of p_1, p_2 .

Finally, from the insertion of the values of p_1 , p_2 and r_0 in (2), we obtain the value of b^2 .

PROPOSITIONS XXXII., XXXIII. PORISM.

Let there be given m points $A_1, A_2, \dots, A_m$, and m magnitudes $a_1, a_2, \dots, a_m$: then there may be found four points B_1, B_2, B_3, B_4 , such, that if from any point, Z , we draw lines to all the given points, and likewise lines to all the points found, we shall always have

$$4 \ S(a_m \cdot A_m \mathbb{Z}^4) = S a_m \cdot S(B_4 \mathbb{Z}^4).$$

[Dr STEWART says *three* points B_1, B_2, B_3 , and the multiplier 3 on the left side of the equation.]

Taking the expressions for the fourth powers of the lines concerned, as before, and equating to zero the co-efficients of the arbitrary quantities, we get the following conditional equations:—

$$4 S (a_m r_m \cos \theta_m) = S a_m \cdot S (u_4 \cos \omega_4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$4 S (a_m r_m \sin \theta_m) = S a_m \cdot S (u_4 \sin \omega_4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$4 S (a_m r_m^2 \cos 2 \theta_m) = S a_m \cdot S (u_4^2 \cos 2 \omega_4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$4 S (a_m r_m^2 \sin 2 \theta_m) = S a_m \cdot S (u_4^2 \sin 2 \omega_4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$4 S (a_m r_m^3 \cos \theta_m) = S a_m \cdot S (u_4^3 \cos \omega_4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$4 S (a_m r_m^3 \sin \theta_m) = S a_m \cdot S (u_4^3 \sin \omega_4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$4 S (a_m r_m^2) = S a_m \cdot S (u_4^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$4 S (a_m r_m^4) = S a_m \cdot S (u_4^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which, as *eight* conditional equations inevitably result from the porism, there must be *four* points porismatised to be determined from them.

Since the origin is the centroid, the left sides of (1, 2) become 0, and we see that the porismatic points have the same centroid as the given system.

PROPOSITIONS XXXV TO XXXVIII. PORISMS.

Let there be given m lines and m magnitudes, $a_1, a_2, \dots, a_m$: then there can be found p other right lines, such, that if from any point, Z , there be drawn perpendiculars $Z P_1, Z P_2, \dots, Z P_m$, to the given lines, and $Z Q_1, Z Q_2, \dots, Z Q_p$, to those found, we shall always have

$$p S (a_m Z P_m^4) = S a_m \cdot S (Z Q_p^4)$$

Forming the conditional equations, as in the former cases, we have

$$p \cdot S (a_m p_m^4) = S a_m \cdot S (q_p^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$p \cdot S (a_m p_m^2) = S a_m \cdot S (q_p^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$p \cdot S (a_m p_m^3 \cos \theta_m) = S a_m \cdot S (q_p^3 \cos \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$p \cdot S (a_m p_m^3 \sin \theta_m) = S a_m \cdot S (q_p^3 \sin \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$p \cdot S (a_m p_m \cos \theta_m) = S a_m \cdot S (q_p \cos \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$p \cdot S (a_m p_m \sin \theta_m) = S a_m \cdot S (q_p \sin \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$p \cdot S (a_m p_m^2 \cos 2 \theta_m) = S a_m \cdot S (q_p^2 \cos 2 \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$p \cdot S (a_m p_m^2 \sin 2 \theta_m) = S a_m \cdot S (q_p^2 \sin 2 \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$p \cdot S (a_m \cos 2 \theta_m) = S a_m \cdot S (\cos 2 \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$p \cdot S (a_m \sin 2 \theta_m) = S a_m \cdot S (\sin 2 \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$p \cdot S(a_m p_m \cos 3 \theta_m) = S a_m \cdot S(q_p \cos 3 \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$p \cdot S(a_m p_m \sin 3 \theta_m) = S a_m \cdot S(q_p \sin 3 \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$p \cdot S(a_m \cos 4 \theta_m) = S a_m \cdot S(\cos 4 \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$p \cdot S(a_m \sin 4 \theta_m) = S a_m \cdot S(\sin 4 \omega_p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which giving *fourteen* independent equations furnishes data for finding *seven* lines, instead of *five*, as stated by Dr STEWART: that is $p=7$, instead of $p=5$.

The preceding statement is that of *Prop.* 38, of which the others are particular cases. It becomes, *Prop.* 37, when $a_1=a_2=\dots=a_m$. When all the lines are parallel it becomes the first case of *Prop.* 36. In this instance we have

$$\theta_1 = \theta_2 = \dots = \theta_m = \frac{1}{2}\pi.$$

Substituting these values in the preceding equations, we have, by transposition,

$$S a_m \cdot S(q_p^4) = p \cdot S(a_m p_m^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$S a_m \cdot S(q_p^2) = p \cdot S(a_m p_m^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$S a_m \cdot S(q_p^3 \cos \omega_p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$S a_m \cdot S(q_p^3 \sin \omega_p) = p \cdot S(a_m p_m^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$S a_m \cdot S(q_p \cos \omega_p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$S a_m \cdot S(q_p \sin \omega_p) = p \cdot S(a_m p_m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$S a_m \cdot S(q_p^2 \cos 2 \omega_p) = -p \cdot S(a_m p_m^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$S a_m \cdot S(q_p^2 \sin 2 \omega_p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$S a_m \cdot S(\cos 2 \omega_p) = -p \cdot S a_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$S a_m \cdot S(\sin 2 \omega_p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$S a_m \cdot S(q_p \cos 3 \omega_p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$S a_m \cdot S(q_p \sin 3 \omega_p) = -p \cdot S(a_m p_m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$S a_m \cdot S(\cos 4 \omega_p) = p \cdot S a_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$S a_m \cdot S(\sin 4 \omega_p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Now of these fourteen equations, eight (viz. 17, 19, 22, 23, 24, 25, 27, 28) are fulfilled by $\omega_1=\omega_2=\dots=\omega_p=\frac{1}{2}\pi$, giving the lines to be found, parallel to the given lines. The other six equations are, with this condition, reduced to the four following ones:—

$$\text{from (20, 26)} \dots S a_m \cdot S(q_p) = p \cdot S(a_m p_m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$\text{from (16)} \dots S a_m \cdot S(q_p^2) = p \cdot S(a_m p_m^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\text{from (18, 21)} \dots S a_m \cdot S(q_p^3) = p \cdot S(a_m p_m^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$\text{from (15)} \dots S a_m \cdot S(q_p^4) = p \cdot S(a_m p_m^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

In this case we have $p=4$, instead of $p=3$, as stated by Dr STEWART.

[See also, NOTE E.]

expansion, to zero; and we shall thus obtain the conditional equations of the porism, viz. :

First, from terms clear of θ .

$$\begin{array}{lcl} r^0 \dots p S(a_m r_m^{2n}) & = & S a_m \cdot S(u_p^{2n}) \\ r^2 \dots p S(a_m r_m^{2(n-1)}) & = & S a_m \cdot S(u_p^{2(n-1)}) \\ r^4 \dots p S(a_m r_m^{2(n-2)}) & = & S a_m \cdot S(u_p^{2(n-2)}) \\ \vdots & & \vdots \\ r^{2(n-1)} \cdot p S(a_m r_m^2) & = & S a_m \cdot S(u_p^2) \end{array}$$

Secondly, from terms in $\cos \theta$ and $\sin \theta$.

$$\begin{array}{l} r \dots p S (a_m r_m^{2n-1} \cos \theta_m) = S a_m \cdot S (u_p^{2n-1} \cos \omega_p) \\ r^3 \dots p S (a_m r_m^{2n-3} \cos \theta_m) = S a_m \cdot S (u_p^{2n-3} \cos \omega_p) \\ r^5 \dots p S (a_m r_m^{2n-5} \cos \theta_m) = S a_m \cdot S (u_p^{2n-5} \cos \omega_p) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ r^{2n-1} \cdot p S (a_m r_m \cos \theta_m) = S a_m \cdot S (u_p \cos \omega_p) \end{array}$$

and

$$\begin{array}{l}
r \dots p S(a_m r_m^{2n-1} \sin \theta_m) = S a_m \cdot S(u_p^{2n-1} \sin \omega_p) \\
r^3 \dots p S(a_m r_m^{2n-3} \sin \theta_m) = S a_m \cdot S(u_p^{2n-3} \sin \omega_p) \\
r^5 \dots p S(a_m r_m^{2n-5} \sin \theta_m) = S a_m \cdot S(u_p^{2n-5} \sin \omega_p) \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
r^{2n-1} \cdot p S(a_m r_m \sin \theta_m) = S a_m \cdot S(u_p \sin \omega_p)
\end{array}$$

Thirdly, from terms in $\cos 2\theta$ and $\sin 2\theta$.

$$\begin{array}{l} r^2 \dots p S(a_m r_m^{2n-2} \cos 2 \theta_m) = S a_m \cdot S(u_p^{2n-2} \cos 2 \omega_p) \\ r^4 \dots p S(a_m r_m^{2n-4} \cos 2 \theta_m) = S a_m \cdot S(u_p^{2n-4} \cos 2 \omega_p) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ r^{2n-2} \cdot p S(a_m r_m^2 \cos 2 \theta_m) = S a_m \cdot S(u_p^2 \cos 2 \omega_p) \end{array}$$

and

$$\begin{array}{rcl} r^2 \dots p S(a_m r_m^{2n-2} \sin 2\theta_m) & = & S a_m \cdot S(u_p^{2n-2} \sin 2\omega_p) \\ r^4 \dots p S(a_m r_m^{2n-4} \sin 2\theta_m) & = & S a_m \cdot S(u_p^{2n-4} \sin 2\omega_p) \\ . & . & . \\ r^{2n-2} . p S(a_m r_m^2 \sin 2\theta_m) & = & S a_m \cdot S(u_p^2 \sin 2\omega_p) \end{array}$$

Proceeding in this way, we arrive at,

Lastly, from terms in $\cos n \theta$ and $\sin n \theta$.

$$\begin{aligned} r^n \dots p S(a_m r_m^n \cos n \theta_m) &= S a_m \cdot S(u_p^n \cos n \omega_p) \\ r^n \dots p S(a_m r_m^n \sin n \theta_m) &= S a_m \cdot S(u_p^n \sin n \omega_p) \end{aligned}$$

Now, to consider these in relation to the porism, we must find the number of equations which are produced in this series. The forms of them are given above; but it will be more convenient to recur to the original development in *Lemma i.*, in order to discover this object of our inquiry.

1. From the terms clear of the θ , we have the alternate terms of a binomial (disregarding the special co-efficients in all cases, which do not affect the actual number of terms) of the $2n$ th degree: that is, exclusive of the term r^{2n} which cancels in the porism, we have n terms.

2. In that which involves $\cos \theta$, we have the terms which were omitted in the first line, viz. n terms: and a similar number in that which involves $\sin \theta$.

3. In each of those involving $\cos 2\theta$ and $\sin 2\theta$ we have all those of the first line except the first and last; or $n-1$ in each case.

4. In each of those involving $\cos 3\theta$ and $\sin 3\theta$ we have all those of the second line except the first and last; or $n-2$ in each case.

Proceeding thus, we find at each successive step a diminution of one in the number of terms belonging respectively to the cosines and the sines of the multiple arcs: till in the last line we get one term involving $\cos n\theta$ and another involving $\sin n\theta$.

Again, the number of lines which involve cosines and sines is the same as the number of multiples of θ which are involved: that is, there are n lines which give the double of the number of powers of r in the co-efficients. Whence, including the first line, there will be $2(1+2+3+\dots+n)+n=n(n+1)+n=n(n+2)$ equations of condition involved in the statement of the porism.

Now, the determination of a point involves two conditions: viz. such as will enable us to find u_p and ω_p . Whence these $n(n+2)$ equations will require that $\frac{1}{2}n(n+2)$ points should be porismatised, instead of $n+1$ as stated by Dr STEWART.

Again, *except n be even*, the number of conditional equations will be *odd*; and hence there will not be the requisite conditions for the determination of any number of points—either giving one condition too few, which would render the first point indeterminate, or one too many, which may be (and, generally, would be) contradictory amongst themselves.

We thus see that except n be even, the porism cannot be true, which is a limitation not laid down in the ‘*General Theorems*’: and we see that when n is even, the number of lines to be porismatised is $\frac{1}{2}n(n+1)=p$, and not $n+1$ as there stated. The relation to be observed between m and n , will be discussed hereafter, when we come to consider the structure and solution of the equations of conditions themselves. It will hence be unnecessary to discuss this question further in this place; though we may remark, that this result, when applied to a former case (*Props.* 34, 35), is in keeping with the conclusions there obtained. For, in that case $n+1$ instead of $\frac{1}{2}n(n+2)$, n being equal to 2, or 3 instead of 4 points, are stated to be determinable.

The next eight porismatic propositions (46 to 53 inclusive) which close the series of porisms respecting points and lines, are, in fact, but varieties of the same general proposition, according to the positions of the given lines, the form of the number n , and the relative values of the magnitudes $a_1, a_2, \dots, a_m$.

PROPOSITIONS XLVI. TO LIII. PORISMS.

Let there be given m lines, and as many magnitudes $a_1, a_2, \dots, a_m$: then there can be found p other lines, such, that if from any point whatever Z , there be drawn perpendiculars $ZA_1, ZA_2, \dots, ZA_m$ to the given lines, and $ZB_1, ZB_2, \dots, ZB_p$ to those found, we shall have, for values of n , subject to conditions which may be determined

$$p \ S(a_m \cdot Z A_m^n) = S a_m \cdot S(Z B_p^n)$$

The perpendiculars being respectively,

$$Z A_m = \pm \{ p_m - r \cos(\theta - \theta_m) \}$$

$$Z B_p = \pm \{ q_p - r \cos(\theta - \omega_p) \}$$

we shall form the expression by means of *Lemma ii.* Also, as the process is general, we may, in conformity with previous practice, omit the double sign of these perpendiculars.

Taking, then, the conditional equations furnished by the developments of *Lemma ii.*, we shall have in succession:—

First, from terms clear of θ .

$$\begin{aligned} r^0 \dots p \cdot S(a_m p_m^n) &= S a_m \cdot S(q_p^n) \\ r^2 \dots p \cdot S(a_m p_m^{n-2}) &= S a_m \cdot S(q_p^{n-2}) \\ r^4 \dots p \cdot S(a_m p_m^{n-4}) &= S a_m \cdot S(q_p^{n-4}) \\ . & \end{aligned}$$

Second, from terms in $\cos \theta$ and $\sin \theta$.

$$\begin{aligned} r \dots p \cdot S(a_m p_m^{n-1} \cos \theta_m) &= S a_m \cdot S(q_p^{n-1} \cos \omega_p) \\ r^3 \dots p \cdot S(a_m p_m^{n-3} \cos \theta_m) &= S a_m \cdot S(q_p^{n-3} \cos \omega_p) \\ . & \end{aligned}$$

and

$$\begin{aligned} r \dots p \cdot S(a_m p_m^{n-1} \sin \theta_m) &= S a_m \cdot S(q_p^{n-1} \sin \omega_p) \\ r^3 \dots p \cdot S(a_m p_m^{n-3} \sin \theta_m) &= S a_m \cdot S(q_p^{n-3} \sin \omega_p) \\ . & \end{aligned}$$

Proceeding thus through $2\theta, 3\theta, \dots, n\theta$ we get at last to

$$r^{n-1} \dots S(a_m p_m \cos(n-1)\theta_m) = S a_m \cdot S(q_p \cos(n-1)\omega_p)$$

$$r^{n-1} \dots S(a_m p_m \sin(n-1)\theta_m) = S a_m \cdot S(q_p \sin(n-1)\omega_p)$$

and, lastly;

$$r^n \dots S(a_m \cos n\theta_m) = S a_m \cdot S(\cos n\omega_p)$$

$$r^n \dots S(a_m \sin n\theta_m) = S a_m \cdot S(\sin n\omega_p)$$

Our business is, in the first place, to find the value of p corresponding to the different forms of n . Now, it is familiarly known that for all the discussions regarding multiple arcs, all integer values of n may be considered under the forms of $4\mu, 4\mu+1, 4\mu+2$, and $4\mu+3$: but our purpose in the present instance will be effected with equal completeness by considering n to exist under the forms 2ν and $2\nu+1$ whilst the length of the discussion will be diminished about one-half.

1. *Let $n=2\nu$.*

The first line has all the even powers of r (r^0 included): the second has all the odd powers: the third has all the even powers but the lowest, r^0 : the fourth all the odd powers but the lowest, r^1 : the fifth has all the even powers but the two lowest, r^0 and r^2 : the sixth has all the odd powers but the two lowest, r^1 and r^3 : and so on.

Now, in the case supposed ($n=2\nu$) the first line will have $\nu+1$ terms, r^0 being even. But by the general structure of these theorems, the last term is cancelled from both sides of the equation of the porism: whence the number of terms in the first line is ν .

Pursuing the enumeration in the same manner as was done in the preceding proposition, we shall find that:

the second line has ν terms

... third ν ...

... fourth $\nu-1$...

... fifth $\nu-1$...

... sixth $\nu-2$...

... seventh $\nu-2$...

and so on.

Again, there will be $2\nu+1$ lines; for all the multiple cosines are found in them from 0 to 2ν inclusive. Leaving out of view, for the moment, the first line, we shall have 2ν lines, which, in pairs, contain the same number of terms. The last two of these will be

$$L r^{2\nu-1} p \cos(2\nu-1)(\theta-\theta_m),$$

$$M r^{2\nu} p \cos 2\nu(\theta-\theta_m).$$

and each of these several equations will be doubled by the expansion of the cosines. Whence, for all the lines except the first we have $2(1+2+\dots+\nu)\frac{2\nu}{2} = 2(\nu+1)\nu$: or for the whole number of conditional equations $2\nu(\nu+1)+\nu=\nu(2\nu+3)$.

We are, therefore, in this case, led to the conclusion, that *except ν be even*, there will be either an indeterminateness or contradiction in the results of the hypotheses of the porism: that is, n if even must be of the form 4μ ; and when this is fulfilled, $p=\mu(4\mu+3)$, instead of $4\mu+1$ as stated in the '*General Theorems*.'

When $\mu=1$ or $\nu=2$, we have $1\{4+3\}=7$, as before determined in reference to *Props.* 35–38.

2. Let $n=2\nu+1$.

The first line will have $\nu+1$ terms

... second		$\nu+1$	...
... third		ν	...
... fourth		ν	...
... fifth		$\nu-1$	...
... sixth		$\nu-1$	...

and so on.

Whence, as there are $2\nu+2$ lines, having all the multiple cosines from 0 to $2\nu+1$, we shall have in all $(\nu+1)(\nu+2)$ terms. Also, with the exception of the first, they are doubled by the expansions of the cosines, and the entire number of equations of condition will be

$$2(\nu+1)(\nu+2)-(\nu+1)=(\nu+1)(2\nu+3)$$

When ν is *even*, this condition cannot hold in reference to Dr STEWART'S theorem; for it will give either an indeterminate or an impossible condition, as before. That is, the form $n=4\mu+1$ is precluded from the enunciation.

When ν is *odd*, the condition is capable of fulfilment: that is, when $n=4\mu+3$. In this case, if we denote ν by $2x+1$, we shall have $p=(x+1)(4x+5)$.

When $\nu=1$, the determination agrees with what was found in *Props.* 24–25; for then $\frac{1}{2}(1+1)(2\cdot 1+1)=5$.

We have thus obtained the number of equations to be fulfilled for the general forms of n , as well as the general forms of those equations themselves: and have shewn that the number of lines porismatised is erroneously in all the cases, and impossibly in some of them, laid down by Dr STEWART. For the case of n odd, we see that there can be no number of lines porismatised, except $n=4\mu+3$, in Propositions 50–53: and in 46–49, except $n=4\mu$.

Particular conditions will lessen this number. In 46–47 this takes place in the two cases in each proposition.

First, let the given lines be all parallel.

The equations in this case reduce to

$$\begin{aligned} p \cdot S(a_m p_m^{2n}) &= S a_m \cdot S(q_p^{2n}) \\ p \cdot S(a_m p_m^{2n-1}) &= S a_m \cdot S(q_p^{2n-1}) \\ p \cdot S(a_m p_m^{2n-2}) &= S a_m \cdot S(q_p^{2n-2}) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p \cdot S a_m p_m &= S a_m \cdot S(q_p) \end{aligned}$$

which are $2n$ in number, determining $p=2n$, and all the lines found, parallel to the given ones.

Second, let the given lines meet in a point.

In this case $p_1=p_2=\dots p_m=0$, and we have only those terms left which involve the absolute terms of the co-efficients of the multiple cosines. Whence the equations become

$$\begin{aligned} p \cdot S(a_m \cos 2n \theta_m) &= S a_m \cdot S(\cos 2n \omega_p) \\ p \cdot S(a_m \cos 2(n-1) \theta_m) &= S a_m \cdot S(\cos 2(n-1) \omega_p) \\ p \cdot S(a_m \cos 2(n-2) \theta_m) &= S a_m \cdot S(\cos 2(n-2) \omega_p) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p \cdot S(a_m \cos 2 \theta_m) &= S a_m \cdot S(\cos 2 \omega_p) \end{aligned}$$

and

$$\begin{aligned} p \cdot S(a_m \sin 2n \theta_m) &= S a_m \cdot S(\sin 2n \omega_p) \\ p \cdot S(a_m \sin 2(n-1) \theta_m) &= S a_m \cdot S(\sin 2(n-1) \omega_p) \\ p \cdot S(a_m \sin 2(n-2) \theta_m) &= S a_m \cdot S(\sin 2(n-2) \omega_p) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p \cdot S(a_m \sin 2 \theta_m) &= S a_m \cdot S(\sin 2 \omega_p) \end{aligned}$$

Thus again, we have in this case $p=2n$; all the porismatic lines passing through the same point as the given ones.

The two conclusions in the latter particular cases agree with those which were found in the still more special case of *Props. 15 and 19*. For then $n=1$; and either Dr STEWART's general form, or the one here deduced, as applied to that case, is precisely the same. In his view, $1+1=2$: in what is here found, $2 \times 1=2$.

SECTION III. INDETERMINATE THEOREMS.

These are divisible, with a few miscellaneous exceptions already proved by Dr STEWART in the work, into three Classes; and which we shall take in order.

1. Perpendiculars on the sides of a regular circumscribed polygon.
2. Lines drawn to the angular points of a regular inscribed polygon.
3. Perpendiculars upon lines which pass through a point and make equal angles with each other.

CLASS I.—*Regular Circumscribed Polygons.*

PROPOSITIONS V., XXIX., XL.

Let there be a regular polygon circumscribed about a circle whose radius is ρ , and let n be any number less than m ; also, from any point Z , whose distance from the centre of the circle is r , let perpendiculars $Z A_1, Z A_2, \dots, Z A_m$ be drawn to the sides of the polygon: then, if $t_0, t_1, t_2, \dots, t_n$ denote the co-efficients of a binomial of the n^{th} degree, we shall have

$$S (Z A_m^{2n}) = m \left\{ t_0 \rho^{2n} + \frac{1}{2} \cdot t_2 \rho^{2n-2} r^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot t_4 \rho^{2n-4} r^4 + \dots \right\}.$$

Without sacrificing generality, we may somewhat simplify our expressions by taking the line from Z to the centre of the circle as angular origin, and the centre itself as polar origin. Then, for the formation of the equations of the sides we shall obviously have,

$$p_1 = p_2 = \dots = p_m = \rho$$

$$\theta_2 = \frac{2\pi}{m} - \theta_1, \theta_3 = \frac{4\pi}{m} - \theta_1, \dots, \theta_m = \frac{2(m-1)\pi}{m}$$

or the angles $\theta_1, \theta_2, \dots, \theta_m$ in arithmetical progression whose common difference is $\frac{2\pi}{m}$.

Whence, since the general form of $Z A_k^{2n}$ is

$$Z A_k^{2n} = \left\{ \rho - r \cos \left(\frac{2k\pi}{m} - \theta_1 \right) \right\}^{2n}$$

we shall find upon forming the sum of them by the expansion in *Lemma ii.*, that all the terms involving the cosines disappear by virtue of *Lemma iv.* This sum is, therefore, reduced to the first line of the expansion for each of the perpendiculars; and these are all equal, and as many in number as there are perpendiculars; that is, the sum is m times the first line of the development, as stated in the proposition.

The conclusion just obtained is Dr STEWART'S 40th proposition. If we put $n=2$ it becomes the 29th; and if also in this case $r=\rho$, we obtain the 26th. Also, finally, if $n=1$, we obtain the 5th proposition.

[See also note F.]

PROPOSITIONS XXII., XXVIII., XXXIX.

Let there be a regular polygon of m sides circumscribed about a circle, whose radius is ρ , and let n be any number less than m : then, if from any point in the circumference of the circle perpendiculars $Z A_1, Z A_2 \dots Z A_m$ be drawn to the sides of the figure, we shall have

$$S (Z A_m^n) = m \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} \cdot \rho^n.$$

For, in this case, we have the general form of the n th power of the perpendicular

$$Z A_k^n = \rho^n \left\{ 1 - \cos \left(\frac{2k\pi}{m} - \theta_1 \right) \right\}^n.$$

Expand each term, for all values of k from 1 to m , by *Lemma iii.*, and we find from *Lemma iv.* that all the terms involving the cosines vanish of themselves, and the expansion is reduced for each perpendicular to its absolute term. As these are all equal, and m in number, we get at once the general form given by Dr STEWART.

The proposition just proved is Dr STEWART'S 39th. When $n=4$, it becomes the 28th; and when $n=3$, the 22d.

PROPOSITION III.

Let perpendiculars $Z A_1, Z A_2 \dots Z A_m$ be drawn from any point Z to the sides of a regular polygon of m sides described about a circle whose radius is ρ : then if the distance of Z from the centre of the circle be r , we shall have

$$S (Z A_m) = m \rho.$$

For the general form of $Z A_k$ is $\rho - r \cos \left(\frac{2k\pi}{m} - \theta_1 \right)$; and we shall have, as in former cases, all the cosines mutually cancelling, giving the proposition stated.

PROPOSITIONS XXII., XXIII.

From any point $Z, (r \theta)$, draw perpendiculars $Z A_1, Z A_2, \dots Z A_m$, to a regular polygon of m sides circumscribed about a circle whose radius is ρ : then we shall have

$$2 S (Z A_m^3) = 2 m \rho^3 + 3 m \rho r^2.$$

For as before, expanding, and recollecting that $\theta_1, \theta_2, \dots, \theta_m$ are in arithmetical progression, having the common difference $\frac{2\pi}{m}$, the angular functions will vanish of themselves from the expression of the sum of the cubes: hence we have simply $\rho^3 + \frac{3\rho r^2}{2}$ for each perpendicular; and hence again

$$S(Z A_m^3) = m \left\{ \rho^3 + \frac{3\rho r^2}{2} \right\},$$

which is equivalent to STEWART'S form.

This is *Prop. 23*; but putting $r=\rho$ we have another proof of *Prop. 22*, viz.:

$$2 S(Z A_m^3) = 5 m \rho^3.$$

CLASS II.—Regular Inscribed Polygons.

PROPOSITIONS IV., XXVII., XLIV.

Let there be drawn from any point Z ($r \theta$) lines to all the angular points of a polygon of m sides inscribed in a circle whose radius is ρ , namely, $Z A_1, Z A_2, \dots, Z A_m$: then we shall have (n being any integer less than m , and $t_0, t_1, \dots, t_n$, as before)

$$S(Z A_m^{2n}) = m \{ t_0^2 \rho^{2n} + t_2^2 \rho^{2n-2} r^2 + t_4^2 \rho^{2n-4} r^4 + \dots \}.$$

The general form of these values is

$$Z A_k^{2n} = \{ \rho^2 - 2\rho r \cos(\theta - \theta_k) + r^2 \}^n.$$

Expand and add; then, since $\theta_1, \theta_2, \dots, \theta_m$ are in arithmetical progression, having the common difference $\frac{2\pi}{m}$, all the angular functions will vanish from the sum. Wherefore only the first line of the expansion in *Lemma i.* will remain, and that one for each $Z A$. Wherefore the sum of them is m times that line; and the Proposition 42 follows at once.

When $n=2$ it becomes *Proposition 27*: and when also, $n=\rho$ it becomes *Proposition 4*.

Dr STEWART remarks, p. 38, that *Prop. 2* is a particular case of the porismatic *Prop. 9*. This will be apparent if we consider that the origin is, in this case, the centroid of the angular points; and hence that all the cosines involved in the mutual expressions cancel among themselves. We shall, however, return to this subject hereafter.

PROPOSITIONS XXVI., XLI.

Let there be a regular polygon of m sides inscribed in a circle whose radius is ρ , and let n be any number less than m ; and from any point Z in the circumference of the circle, let lines be drawn to all the angular points of the polygon: then we shall have

$$S(Z A_m^{2n}) = m \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} \cdot 2^n \cdot \rho^{2n}.$$

For in this case $r=\rho$; and the expression for $Z A_k^{2n}$ takes the form

$$\begin{aligned} Z A_k^{2n} &= \{2\rho^2 - 2\rho^2 \cos(\theta - \theta_k)\}^n \\ &= 2^{2n} \rho^{2n} \{1 - \cos(\theta - \theta_k)\}^n. \end{aligned}$$

Expanding the binomial factor by *Lemma* iii., and keeping in view that $\theta_1, \theta_2, \dots, \theta_m$ are in arithmetical progression, having the common difference $\frac{2\pi}{m}$, we see that all the angular functions vanish, by *Lemma* iv. Whence the sum is reduced as before to m times the term clear of the angles, and we thus have

$$\begin{aligned} S(Z A_m^{2n}) &= \frac{1}{2^n} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} \cdot 2^{2n} \rho^{2n} \\ &= \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} \cdot 2^n \rho^{2n}. \end{aligned}$$

This is *Proposition* 41, as proposed by Dr STEWART. When $n=2$ it becomes *Prop.* 26.

CLASS III.—Lines making equal Angles.

PROPOSITIONS XIV., XXXIV., XLV.

Let there be m lines meeting in a point, and making all the angles round the point equal, and let n be any number less than m : then if from any point Z , whose distance from the common point of section is r , perpendiculars be drawn to all the lines, $Z A_1, Z A_2, \dots, Z A_m$, we shall have

$$S(Z A_m^{2n}) = \frac{m}{2^n} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} \cdot r^{2n}.$$

For simplification, take the line from Z to the given intersection as angular origin, that intersection being the polar origin. Then $\theta_1, \theta_2, \dots, \theta_m$ of the general equation are the angles made with this line by the perpendiculars to the given lines. Also, in the equations of the given lines $p_1 = p_2 = \dots = p_m = 0$; and $\theta_1, \theta_2, \dots, \theta_m$ are in arithmetical progression, having the common difference $\frac{2\pi}{m}$. Wherefore, the general form of the expression for $Z A_m^{2n}$ is

$$Z A_m^{2n} = r^{2n} \cos^{2n} \frac{2k\pi}{m}.$$

Expand, therefore, in the ordinary form, and bearing in mind that $2n$ is even, and the angular functions vanish, we have for each value of k from 1 to m inclusive, the expression

$$\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} r^{2n}.$$

and as these are m lines, we have at once the expression in question.

This is *Prop.* 45: also, when $n=2$, it becomes *Prop.* 34; and when $n=1$ it becomes *Prop.* 14.

NOTES UPON THE PRECEDING DISCUSSION.

NOTE A, page 574.

The first case of the solution of any one of Dr STEWART'S *General Theorems* being published, that I remember to have met with, is that of the 41st Proposition, by Professor PLAYFAIR, in his paper on the Arithmetic of Imaginary Quantities, *Phil. Trans.* 1777; and I am not aware, impressed as he was with the great beauty of these propositions, that he published anything more on the subject, and even this one is taken up incidentally.

Dr SMALL, in the *Edinb. Trans.*, vol. ii., gave solutions of Propositions 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 30, 31.

Professor LOWRY gave solutions of 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, in vols. i. ii., O.S., of Leybourn's Repository; and annexed to them several interesting, but tolerably obvious deductions.

Mr SWALE of Liverpool also gave, in the same work, solutions of 15, 16, 17, 18, 19, 20, 21.

The three last-mentioned authors treat the subject by methods purely geometrical; but Dr SMALL has been inconsiderately censured for a "lack of geometrical purity," merely because he *mentions* the "centre of gravity," notwithstanding he builds nothing upon it deduced from its *physical* character. (*Repos.* i. p. 131.) Geometrical purity, however, is not vitiated by the use of an injudicious term, but by the employment of *methods* which are unrecognised by geometry, or inconsistent with those which are recognised. The standard itself was originally arbitrary; but being once recognised and generally admitted, it is the proper criterion by which to judge of the purity of geometrical processes. Taking, however, the strictest view of the subject, I confess I know of no flaw in Dr SMALL'S argument; and there is no doubt that to Dr SMALL'S paper we are in reality indebted for all that has been effected concerning the porismatic part of Dr STEWART'S Theorems. It may be further remarked, too, that the ordinary sense of the word *geometrical* is altogether inapplicable to the greater part of theorems themselves, since they relate to magnitudes which the ancient geometry does not recognise (*viz.*, to fourth and higher powers of lines); and hence, as far as purity is concerned, it seems to savour of the "gnat and camel" character, to affect a rigid adherence to even the forms of the ancient geometry in attempting their solution. Dr

STEWART did not set such an example by attempting to antiquate the forms of their enunciation; and he does not even call them *geometrical theorems*, though, as they relate to the properties of geometrical figures, he might have done so without impropriety. Whether he even *obtained* them by geometrical considerations, in the first place, is open to question, when we turn to what Professor PLAYFAIR remarks (*Ed. Trans.*, i. p. 60) concerning the probable origin of his inquiries on this subject. In fact, his being in possession of SIMSON's definition of the porism (which may be inferred from his formally distinct enunciations of the porismatic part of his propositions), I do not think there is a single step in the present paper, which it would have been at all improbable that Dr STEWART could readily take. Any slight mistake in estimating the number of final equations would create no surprise, when we recollect how much he was in the habit of "thinking out" his conclusions without the aid of writing. Possibly, therefore, Dr STEWART's investigations were not much unlike those of the present paper, in their essential character. But to return:—

Mr GLENIE, of the Royal Artillery, gave, in the *Edinburgh Transactions* (vol. vi.), demonstrations of nearly all the *indeterminate theorems*, except those proved by Dr STEWART himself, and the 41st theorem. Of this last, he afterwards gave a proof in a small tract (1813.)

Mr GLENIE's course of investigation is remarkably elegant, and it discloses many curious and interesting properties of the circle, of which uses, generally unsuspected, may yet be made. This able geometer has, however, fallen into the prevailing error on the subject of geometrical purity: that $\frac{r^n}{r^n-3}$ expresses an idea recognised by the ancient geometry; but it in no degree vitiates his reasonings as demonstrations, though it throws them, as all demonstrations of these theorems *must* be thrown, into the domain of algebra.

Mr BABBAGE, after paying a high and deserved tribute to the genius of Dr STEWART, gave (*Quarterly Journal of Science*, vol. i.) investigations of nearly all the *indeterminate theorems*, by means of trigonometry. They differ from the investigations given in this paper mainly in the forms of expansion employed; and I gladly avail myself of the opportunity of acknowledging my obligations to that paper, for some useful modifications of my own primary processes respecting this class of propositions.

The late Mr THOMPSON published (in the *Newcastle Magazine*, 1826–27–28) investigations of a considerable number of these theorems. He adopted the mixed method—that is, the employment of the ancient geometry, algebra, and trigonometry, as best suited his immediate purpose. These solutions bear ample testimony to the mathematical powers of their author; but they are, unfortunately, so disfigured by an awkward and irregular notation, and bad style of printing, as to be not only generally unintelligible to the ordinary reader, but, in many cases, to almost defy interpretation by those who look into the subject with the greatest care. I was much struck, when I first met with this work (three or four months ago), with the near approach made, in some of his demonstrations, to the use of the principle which forms one of the foundations of this paper—the formation of conditional equations. He seems, however, to use it as a matter of convenience rather than as a principle—as something that may answer a special purpose, rather than as a general method founded in the nature of the algebraic analysis, and applying to all possible cases.

Owing to the circumstances before mentioned, had my own investigations been in a less complete state than they were at that time, Mr THOMPSON's papers could have afforded me no assistance; but I gladly embrace this opportunity of publicly recording the high estimate

I form of the abilities of the author, as evinced by his varied correspondence with the most valuable English periodicals devoted to mathematical researches.

My old friend and former pupil, Mr LESLIE ELLIS (in the *Cambridge Mathematical Journal*, May 1841), has proved a considerable number of the indeterminate theorems which relate to inscribed and circumscribed polygons, by means of a very remarkable theorem (which he has there investigated), which reduces the sums of the specified powers of the lines, or perpendiculars, to a definite integral. The following is his theorem; but, for the manner of employing it, reference must be made to the paper itself.

$$f\phi + \dots + f\left(\phi + \frac{n-1}{n}2\pi\right) = \frac{n}{2\pi} \int_0^{2\pi} f\phi d\phi.$$

In a letter to me, Mr ELLIS also suggests the application of the same process to spherical polygons. It may also, evidently, with slight modifications, be applied to regular polyhedrons, inscribed and circumscribed to a sphere.

Should any other English authors have discussed these theorems, their works are unknown to me, and that, after taking much trouble to discover all that had been attempted relative to them. I am not aware of any writer on the continent who has distinctly dwelt upon them, except M. CHASLES, in his *Aperçu Historique des Méthodes en Géométrie*, though LHUILLIER, CARNOT, and many other distinguished continental geometers have made occasional reference to SIMSON'S PORISMS and STEWART'S THEOREMS. Both works are, however, by them considered merely as relating to indeterminate theorems. CHASLES forms the highest estimate of STEWART'S researches; but as his view of the ancient porism is so opposed to Dr SIMSON'S, he was not likely to be led to any method of investigation adapted to the discussion of these propositions:—in fact, he does not offer any. He, however, enunciates the “extension” of two of the general theorems (pages 353–54), namely, the 44th and 53d. These extensions will be true only when the original theorems respecting the numbers of porismatic points and lines, as given by Dr STEWART, are corrected. In this case, all the equations of condition for different values of δ are contained amongst those for $\delta=0$; and hence the porismatic lines, which fulfil the conditions for the value $\delta=0$, fulfil those, also, for all higher values within the prescribed limits; although the conditions of the porisms might be fulfilled with a smaller number of porismatic points and lines for those cases.

CHASLES'S extensions are, in our notation,

$$\begin{aligned} p. S(a_m \cdot Z A_m^{2(n-\delta)}) &= S a_m \cdot S(Z B_p^{2(n-\delta)}); \\ p. S(a_m p_m^{(n-2\delta)}) &= S a_m \cdot S(q_p^{(n-2\delta)}); \end{aligned}$$

the former being that of *Prop.* 44, and the latter that of *Props.* 49 and 53, as n is odd or even; and δ being taken from 0, 1, 2, . . . $\frac{n+1}{2}$ in the former case, and from 0, 1, 2, $\frac{n-2}{2}$ in the latter.

Finally, it may be remarked that nearly all the general theorems have analogous ones in respect to points, lines, or planes situated arbitrarily in space, the numbers of porismatic ones being properly chosen. Several of them, too, have corresponding properties (with a different porismatic number, of course), with respect to the hyperboloid of one sheet; but this is not the proper time for details on any collateral subject.

NOTE B.

ON LEMMAS I, II, III, pp. 578–81.

I am not aware that the general forms of the co-efficients of the multiple cosines have been given by any author ; and, indeed, since they are not required in any class of inquiries which fall under the general objects of science, it is scarcely likely that they should have been an object of research. Cases, however, having much analogy to them, are sufficiently well known ; and the method by which they are treated, naturally pointed in the direction by which these could be obtained when occasion called for them.

By taking $r_1 = r$ in *Lemma* i., and $p = r$ in ii., and comparing the terms in the two cases clear of the cosines, with the corresponding ones in iii., we obtain two elegant formula, first given by EULER in the *Acta Acad. Petropolitaneæ*, 1781, viz. :—

$$t_0^2 + t_1^2 + t_2^2 + \dots + t_{n-1}^2 + t_n^2 = \frac{2n(2n-1)(2n-2)\dots(n+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}$$

$$t_0 + \frac{1}{2}t_2 + \frac{1 \cdot 3}{2 \cdot 4}t_4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}t_6 + \dots = \frac{2n(2n-1)(2n-2)\dots(n+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}$$

NOTE C.

ON PROPS. IX, X, p. 582.

This theorem is well known, and has been demonstrated in a great variety of ways, *as a property of the centroïd* ; though, probably, it has not been before considered in the light of a porism, nor, consequently, investigated as such. It was suggested by CARNOT (*Géom. de Pos.* p. 326), that it would offer some advantage to take this point as the origin of co-ordinates ; and as it will illustrate the manner in which, under particular circumstances, the porismatic proposition may become an ordinary indeterminate, the same property is here investigated, with the centroïd taken as origin of polar co-ordinates.

$$\begin{aligned} a_1 \cdot A_1 Z^2 &= a_1 r^2 - 2 a_1 r r_1 \cos (\theta - \theta_1) + a_1 r_1^2 \\ a_2 \cdot A_2 Z^2 &= a_2 r^2 - 2 a_2 r r_2 \cos (\theta - \theta_2) + a_2 r_2^2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_m \cdot A_m Z^2 &= a_m r^2 - 2 a_m r r_m \cos (\theta - \theta_m) + a_m r_m^2 \end{aligned}$$

Now, since the origin is the centroïd, the middle vertical column on the right side is zero ; and hence

$$S(a_m \cdot A_m Z^2) = S a_m \cdot r^2 + S(a_m r_m^2)$$

NOTE D.

ON PROPS. XI, XII, p. 583.

These may be proved, as, indeed, most of the *earlier* propositions can be, very neatly, but at greater length, by means of rectangular co-ordinates.

Let $\alpha_1 \beta_1, \alpha_2 \beta_2, \dots, \alpha_m \beta_m$ denote the given points, xy the arbitrary point Z , $\alpha \beta$ the

NOTE E.

ON PROPS. XXXV, XXXVIII, p. 590.

TO remove any latent suspicions that may be entertained of the correctness of the determination of the value of p , in consequence of any one of the former equations (29–32) being virtually contained in the other three, it will be desirable to examine the consequences of such an hypothesis in detail.

On the hypothesis of $p=3$, and the fourth equation being contained in the three others, we can find the value $q_1^4 + q_2^4 + q_3^4$ by means of the other three equations; and if that hypothesis be correct, we ought to obtain the same value of this function as is given in (32).

Put S_1, S_2, S_3, S_4 , for the sums of the first, second, third, and fourth powers of the roots of the equation which results from the elimination of q_2 and q_3 from (29, 30, 31). Then it is sufficiently well known that

$$6 S_4 = 3 S_1 S_3 + 3 S_2 (S_2 - 2 S_1^2) + S_1^4$$

Moreover, as the origin of co-ordinates is altogether arbitrary in the investigation by which those equations are obtained, we are at liberty to take it so as to fulfil the condition

$$q_1 + q_2 + q_3 = \frac{3 S(a_m p_m)}{S a_m} = 0; \text{ or } S_1 = 0.$$

This will convert the equation above into

$$6 S_4 = 3 S_2^2, \text{ or } 2 S_4 = S_2^2.$$

Substituting in this the values of S_2 and S_4 from (30, 32), we have

$$2 S a_m \cdot S(a_m p_m^4) = \{S(a_m p_m^2)\}^2$$

an equation which is manifestly incorrect.

The same general result might have been obtained without any hypothesis regarding the origin of co-ordinates; but the expressions would have been more complex, and the examination of the results more troublesome.

We are hence (even without an actual solution of the several equations) entitled to infer, that the case of three lines only (or $p=3$), as propounded by Dr STEWART, fulfilling the conditions of the porism, is inaccurate. Moreover, as a general theorem ought to be true in all its particular cases, it follows that the general statement given amongst the “*Theorems*” is also inaccurate.

NOTE F.

ON PROPS. XL. AND XLII, p. 594.

THE connection between the porismatic and indeterminate proposition is capable of a striking exemplification, by a comparison of these propositions with the equations of the corresponding porisms. Owing, however, to the already extended space required for printing the present part of the discussion, that exemplification must be deferred, as well as some other necessary remarks on the porismatic proposition. See, however, Note C., which is a case in point.