ART. XXXVI.—*Gibbs' Geometrical Presentation of the Phenomena of Reflection of Light*; by ARTHUR W. EWELL.

IN his lectures at Yale upon the Electromagnetic Theory of Light, Prof. Gibbs briefly outlined a geometrical method of representing and interpreting the phenomena of total and metallic reflection, which, to the writer's knowledge, has never been published, and is not included in his collected papers. The writer has found this method so valuable in teaching and in experiments upon metallic reflection, that he believes others, outside the small number who were privileged to hear Prof. Gibbs, will find it a clear, simple and practical method of representing the relative amplitude and phases and optical constants, the analytical expressions for which are complicated.

The foundations of this paper are lecture notes of the years 1898–99. In place of Prof. Gibbs' vector notation, the more common scalar quantities will be used, and the writer has added

extensions of this method to the simpler cases of reflection, applications to the determination of refractive indices, coefficients of absorption, and reflecting power, and numerical illustrations.

The method, in brief, consists in representing by lines the reciprocals of the velocities of light in the two media between which reflection occurs, and also their components parallel and perpendicular to the surface. These lines are so chosen that very simple geometrical quantities represent Fresnel's Laws, which are assumed, together with the ordinary laws of reflection and refraction. As the lines and angles vary with the angle of incidence and the optical constants of the media, the relative amplitudes, phases, etc., are given by simple geometrical relations.

Light is reflected at the surface, separating medium (1) from medium (2). Let C_1 and C_2 be the components, normal to the surface, of the reciprocal velocities, $\frac{1}{v_1}$ and $\frac{1}{v_2}$ (fig. 1). Let i be the angle of incidence, or reflection, and r the angle of refraction. Since the angles of incidence and reflection are equal and since the ratio of the sines of the angles of incidence and refraction is $v_1 \div v_2$, the components, A , of $\frac{1}{v_1}$ and $\frac{1}{v_2}$, parallel to the surface, are equal.

For simplicity, we will take as unity the components of the incident light vibration, parallel and perpendicular to the plane of incidence (i. e., the light is polarized at 45°). Let the amplitude of the component of the reflected vibration, parallel to the plane of incidence, be R_p , and of the perpendicular component, R_s . Fresnel's Laws of Reflection are then :

$$R_s = -\frac{\sin(i-r)}{\sin(i+r)}, \quad \frac{R_p}{R_s} = -\frac{\cos(i+r)}{\cos(i-r)}, \quad R_p = \frac{\tan(i-r)}{\tan(i+r)}$$

From the above figure and notation,

$$\sin i = \mathbf{A}v_1, \cos i = \mathbf{C}_1v_1, \sin r = \mathbf{A}v_2, \cos r = \mathbf{C}_2v_2$$

Substituting :

$$R_s = \frac{C_1 - C_2}{C_1 + C_2}, \quad \frac{R_p}{R_s} = \frac{A^2 - C_1 C_2}{A^2 + C_1 C_2}$$

The latter is of course the more interesting and important. The product of the two will give R_p .

[illegible]

These quantities will be represented graphically. Horizontal distances we will consider real, and positive if to the right.

6 moves to the right and as before, at the Polarizing Angle $\overline{14}$ becomes zero. Fig. 3 is drawn for light incident at 30° upon a glass ($n=1.53$) — air surface.

$$\frac{R_p}{R_s} = \frac{\overline{14}}{\overline{12}} = -.185, \quad R_s = \frac{\overline{32}}{\overline{34}} = .528$$

As the angle of incidence is still more increased, $\overline{16} A^2$ exceeds $\overline{15} = \frac{1}{v_2^2}$ and $\overline{65} = C_2^2$ becomes negative, and hence C_2 is imagin-

ary. At the point where 6 is at 5, $A^2 = \frac{1}{v_2^2}$ or the emerging waves, travel parallel to the surface and the angle of incidence is the Critical Angle. Beyond this angle, conditions are represented by fig. 4, which may be taken as typical of total reflection. Since $\overline{62} (=C_1 C_3)$ and $\overline{64}$ are mean proportionals of $\overline{65}$ and $\overline{63}$ and C_3 is imaginary, 2 and 4 must lie on a circle whose diameter is $\overline{53}$. Evidently numerically,

$\frac{R_p}{R_s} = 1 = R_s = R_p$, and the angle between $\overline{12}$ and $\overline{14}$ gives the phase difference between the two components and the angle between $\overline{32}$ and $\overline{34}$, the phase difference between R_s and the

incident, perpendicular, component. The latter phase difference is evidently zero when total reflection begins and π at grazing incidence. The difference of phase between the two components, angle $\angle 12$, is seen to be zero at both extremes and a maximum when $\overline{12}$ and $\overline{14}$ are tangents. For this case, by geometry,

$$\overline{52}^2 : \overline{23}^2 = \overline{56} : \overline{63} = \overline{15} : \overline{13} = \frac{1}{v_2^2} - \frac{1}{v_1^2} \therefore \overline{52} : \overline{23} = v_1 : v_2 \therefore$$

the maximum angle $\angle 12 = \pi - \angle 235 = \pi - 4 \tan^{-1} \frac{v_1}{v_2}$

Fig. 4 is drawn for light incident upon a glass ($n=1.53$)—air surface at 48° . The angles of phase difference are: angle 412 $\left(\frac{R_p}{R_s}\right) = 46^\circ$, angle 234 $(R_s) = 56^\circ$. The maximum difference of phase between the two components $= \pi - 4 \tan^{-1} \frac{1}{1.53} = 47^\circ$.

We have hitherto considered perfectly transparent bodies. We will now consider the general case, so called Metallic Reflection. The general expression for the light disturbance may be written:

$$X = \Lambda e^{-\frac{2\pi k w}{\lambda}} e^{2\pi i \left(\frac{t}{T} - \frac{w}{\lambda} \right)}$$

where X is the instantaneous force, Λ = amplitude, λ = wave length, T = period, t = time, w = coördinate for the direction in which the light waves are advancing, and k =

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Coefficient of Absorption, i. e., the amplitude is reduced to $\frac{1}{e}$ of its initial value in a distance $\frac{\lambda}{2\pi k}$. If we write the above equation in the form:

$$\bar{X} = \Lambda e^{2\pi i \left(\frac{t}{\lambda} - \frac{w}{\lambda} (1 - ik) \right)}$$

and determine the reciprocal velocity of the light waves, we obtain the complex expression: $\frac{T}{\lambda} (1 - ik)$. The real part, $\frac{T}{\lambda}$, is the ordinary reciprocal velocity, $\frac{1}{v}$, or proportional to

the refractive index, and the imaginary portion is proportional to the product of the refractive index and the coefficient of absorption. If we chose as unity the velocity in pure ether, the real part is the Refractive Index, n , and the imaginary portion is nk .

In total reflection, C^2 , as we have seen, is a pure imaginary, i. e., no waves are propagated in the second medium perpendicular to the surface. In the general case of light meeting the surface separating two media, waves are propagated in the second medium with some absorption, i. e., C^2 is complex. Fig. 5 represents a typical case of so-called Metallic Reflection. We will suppose that the first medium has no appreciable absorption. As in previous figures, $\overline{13} = \frac{1}{v_1^2}$, $\overline{16} = A^2$, $\overline{63} = C_1^2$,

$\overline{65} = C_2^2$, $\overline{15} = \frac{1}{v_2^2} \cdot C_2^2$, ($\overline{65}$), being complex, the mean propor-

tional between it and $\overline{63}$ will be $\overline{62}$, whose length is the arithmetical mean proportional and such that the angle 263 is half the angle 563.

Since $\frac{R_p}{R_s} = \frac{\overline{14}}{\overline{12}}$, the phase difference must decrease from π at perpendicular incidence (6 at 1), to zero at grazing incidence (6 at 3), while the phase of $R_s = \frac{\overline{32}}{\overline{34}}$ varies from zero to π . At the Principal Incidence, I, the phase difference is $\frac{\pi}{2}$ or $\overline{14}$ and $\overline{12}$ are at right angles. The numerical value of $\frac{\overline{14}}{\overline{12}}$ is the Principal Azimuth.

To determine the refractive index, n , and the coefficient of absorption, k , we extract the square root of $\frac{1}{v_2^2}$, ($\overline{15}$). The real part of the root is n ($\frac{1}{v_1}$ being unity) and the imaginary portion is nk .

Fig. 5 represents reflection from copper in air, when the angle of incidence is $71^\circ 35'$, the Principal Incidence (Drude). The velocity of light in air, v_1 , is taken as unity and $\frac{1}{v_2}$ is represented by $\overline{13}$. $\overline{16} = A^2 = \sin^2 71^\circ 35'$, and $\overline{63} = C_1^2 = \cos^2 71^\circ 35'$. The Principal Azimuth of copper is $38^\circ 57'$. Since at the principal Incidence, I, the difference of phase is $\frac{\pi}{2}$, $\overline{12}$

and $\overline{14}$ are drawn at right angles, in the ratio $\overline{14} : \overline{12} = \tan 38^\circ 57'$, and such that $\overline{24}$ is bisected at 6. From the preceding, $\overline{62} = C_1 C_2$ and $\overline{64} = -C_1 C_3$. $C_2^2 = \overline{65}$ is now constructed. Since $\overline{62}$ is a mean proportional between $\overline{63}$ and $\overline{65}$, the angle 563 is made twice angle 263 and the length of $\overline{65}$ is such that $\overline{62}^2 = \overline{63} \times \overline{65}$. $\frac{1}{v_2^2} = A^2 + C_2^2$ is then $\overline{15}$. The point 5

being determined, the relative amplitudes $\left(\frac{\overline{14}}{\overline{12}}\right)$ and phases (angle 412) may be determined for any angle of incidence, i , since $\sin^2 i = \overline{16}$ and $\overline{62} = -\overline{64}$ is a geometric mean proportional between $\overline{63}$ and $\overline{65}$.

Let $\overline{18}$ be the geometric square root of $\overline{15}$ (or the numerical square root at an angle $816 = \frac{1}{2}$ angle 516). The real part, $\overline{19}$, is the refractive index, n , and the imaginary part, $\overline{98}$, is nk , where k is the coefficient of absorption. Fig. 5 gives, for copper, $n = .62$, $nk = 2.6$, hence $k = 4.1$.

It is not difficult to show also that k is the tangent of twice angle 124, which is half the angle 263 (Drude's Q).

$R_s \left(= \frac{32}{34} \right)$ is numerically .94 and the angular difference of phase (angle 234) is 167° .

The intensity of the reflected light is:

$$R_s^2 + R_p^2 = \left(\frac{32}{34}\right)^2 \left[1 + \left(\frac{14}{12}\right)^2 \right]$$

(which equals 1.46 for the above illustration, the incident intensity being 2 or 73 per cent), and evidently increases rapidly with increase of the imaginary portion of $\frac{1}{v_2^2}$, i. e., of k , thus illustrating Selective Reflection.

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