On Play "à outrance." By Major P. A. MACMAHON, R.A.

[Read March 14th, 1889.]

Suppose two players *A* and *B* to have *a* and *b* counters respectively and to play a number of games, the winner receiving a counter from the loser until one of the players is deprived of all of his counters. We may consider the probability of all of the counters remaining finally with *A*, being given the probability of his winning any one game. When the probability of *A* winning each game is equal to a constant *p*, and no game can be drawn, the problem has been solved. (See Todhunter's *History of the Theory of Probability*.) I introduce here the consideration of the effect of supposing that the probability of a player winning a certain game depends in some manner upon the number of counters which he possesses at the commencement of that game.

In regard to the game which is commenced when the players *A* and *B* have respectively *x* and *n* - *x* counters (where *a* + *b* = *n*), let the chances of winning be as $\phi(x)$ to $\phi(n-x)$, where $\phi(x)$ is a known function of *x*, which is finite for all integer values of the argument from 1 to *n*. Further, let u_x denote the chance of *A* winning the "partie" when he is in possession of *x* counters at the commencement of a game.

The chance of *A* winning the next game is then

$$\frac{\phi(x)}{\phi(x) + \phi(n-x)},$$

and the chance of his losing it is

$$\frac{\phi(n-x)}{\phi(x) + \phi(n-x)};$$

hence the difference equation

$$u_x = \frac{\phi(x)}{\phi(x) + \phi(n-x)} u_{x+1} + \frac{\phi(n-x)}{\phi(x) + \phi(n-x)} u_{x-1},$$

which may be written $\frac{\phi(x)}{\phi(n-x)} = \frac{u_x - u_{x-1}}{u_{x+1} - u_x},$

and thence we obtain

$$\frac{\phi(n-1)}{\phi(1)} \cdot \frac{\phi(n-2)}{\phi(2)} \cdots \frac{\phi(n-s)}{\phi(s)} = \frac{u_{n-s} - u_{n-s-1}}{u_n - u_{n-1}}.$$

$$\text{Put } 1 + \frac{\phi(n-1)}{\phi(1)} + \frac{\phi(n-1)\phi(n-2)}{\phi(1)\phi(2)} + \dots \\ + \frac{\phi(n-1)\phi(n-2)\dots\phi(n-s)}{\phi(1)\phi(2)\dots\phi(s)} = \Phi(s+1),$$

$$\text{so that } \frac{u_n - u_{n-s-1}}{u_n - u_{n-1}} = \Phi(s+1),$$

and thence the two results,

$$\frac{u_n - u_x}{u_n - u_{n-1}} = \Phi(n-x),$$

$$\frac{u_n - u_0}{u_n - u_{n-1}} = \Phi(n);$$

$$\text{from these, by division, } \frac{u_n - u_x}{u_n - u_0} = \frac{\Phi(n-x)}{\Phi(n)}$$

The conditions of the problem are

$$u_n = 1, \quad u_0 = 0;$$

$$\text{hence } 1 - u_x = \frac{\Phi(n-x)}{\Phi(n)},$$

and since, by definition,

$$\Phi(n-x) + \Phi(x) = \Phi(n),$$

we obtain

$$u_x = \frac{\Phi(x)}{\Phi(n)};$$

this is the solution of the difference equation subject to the given conditions; that is to say, the probability u_x is the ratio of the sum of the first x terms to the sum of all of the terms of the series—

$$1 + \frac{\phi(n-1)}{\phi(1)} + \frac{\phi(n-1)\phi(n-2)}{\phi(1)\phi(2)} + \dots + \frac{\phi(n-1)}{\phi(1)} + 1.$$

This reminds one of the binomial series,

$$(1+1)^{n-1},$$

with which it becomes identical when

$$\phi(x) = x.$$

When, in general,

$$\phi(x) = x^m,$$

where m may be any number, positive, zero, or negative, commensurable or not, we are concerned with powers of binomial coefficients.

If we write

$$\frac{\phi(x)}{\phi(x) + \phi(n-x)} = f(x),$$

so that

$$f(x) + f(n-x) = 1,$$

the difference equation becomes

$$u_x - u_{x-1} = f(x)(u_{x+1} - u_{x-1}),$$

and the solution may be written,

$$u_x = \frac{F(x)}{F(n)},$$

where
$$F(x) = 1 + \frac{f(n-1)}{f(1)} + \frac{f(n-1)f(n-2)}{f(1)f(2)} + \dots$$

$$\dots + \frac{f(n-1)f(n-2)\dots f(n-x+1)}{f(1)f(2)\dots f(x-1)}$$

I now suppose that when A and B have respectively x and $n-x$ counters the chances of winning the next game are as $\phi(x)$ to $f(n-x)$, the most general case.

The difference equation becomes

$$u_x = \frac{\phi(x)}{\phi(x) + f(n-x)} u_{x+1} + \frac{f(n-x)}{\phi(x) + f(n-x)} u_{x-1},$$

or

$$\frac{\phi(x)}{f(n-x)} = \frac{u_x - u_{x-1}}{u_{x+1} - u_x},$$

putting

$$x = n-s,$$

$$\frac{\phi(n-s)}{f(s)} = \frac{u_{n-s} - u_{n-s-1}}{u_{n-s+1} - u_{n-s}},$$

thence
$$\frac{\phi(n-1)\phi(n-2)\dots\phi(n-s)}{f(1)f(2)\dots f(s)} = \frac{u_{n-s} - u_{n-s-1}}{u_n - u_{n-1}},$$

put
$$\left(\frac{\phi}{f}\right)_{s+1} = 1 + \frac{\phi(n-1)}{f(1)} + \frac{\phi(n-1)\phi(n-2)}{f(1)f(2)} + \dots$$

$$\dots + \frac{\phi(n-1)\dots\phi(n-s)}{f(1)\dots f(s)},$$

and then

$$\frac{u_n - u_{n-s-1}}{u_n - u_{n-1}} = \left(\frac{\phi}{f}\right)_{s+1},$$

which gives

$$\frac{u_n - u_x}{u_n - u_{n-1}} = \left(\frac{\phi}{f}\right)_{n-x},$$

and

$$\frac{u_n - u_0}{u_n - u_{n-1}} = \left(\frac{\phi}{f}\right)_n,$$

and remembering that $u_n = 1$, $u_0 = 0$, we obtain, by division,

$$1 - u_x = \frac{\left(\frac{\phi}{f}\right)_{n-x}}{\left(\frac{\phi}{f}\right)_n},$$

or

$$u_x = \frac{\left(\frac{\phi}{f}\right)_x}{\left(\frac{\phi}{f}\right)_n},$$

showing that to obtain u_x we have to divide the sum of the first x terms of the series,

$$\left(\frac{\phi}{f}\right)_n;$$

by the series itself:

Similarly, if v_x be the chance that B wins the partie when he has x counters,

$$v_x = \frac{\left(\frac{f}{\phi}\right)_x}{\left(\frac{f}{\phi}\right)_n},$$

and since

$$u_x + v_{n-x} = 1,$$

we obtain the interesting identity,

$$\frac{\left(\frac{\phi}{f}\right)_x}{\left(\frac{\phi}{f}\right)_n} + \frac{\left(\frac{f}{\phi}\right)_{n-x}}{\left(\frac{f}{\phi}\right)_n} = 1,$$

and the value of u_x may be presented in either of the forms shown by

$$u_x = \frac{\left(\frac{\phi}{f}\right)_x}{\left(\frac{\phi}{f}\right)_n} = \frac{\left(\frac{f}{\phi}\right)_n - \left(\frac{f}{\phi}\right)_{n-x}}{\left(\frac{f}{\phi}\right)_n}.$$

This result may be of service when it is necessary to take account of the courage of the two players. Thus putting

$$f(x) = px^a,$$

$$\phi(x) = qx^{-\beta},$$

where α and β are small quantities determined from experience, we would have the circumstance of A playing better when he is winning and of B playing better when he is losing; and doubtless a more correct estimate of the probability of either player winning the partie would thus be formed.