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On the Rate of Interest yielded by Foreign Government Loans.
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FROM the fact that a very large amount of capital has been of recent years invested in loans to foreign governments, the question of sinking funds and other kindred matters has come prominently into notice; and matters which at one time were only looked upon as belonging to the theory of compound interest, are now become of great moment to thousands of persons who have invested, or purpose to invest, their money in the above-mentioned class of security. Various causes appear to have led to the large increase in the popularity of foreign loans among certain classes of investors. No doubt, the rapid growth of unemployed capital has considerably reduced the rate of interest realizable on first-class home securities, and has induced persons anxious to obtain a high rate of interest to invest their money elsewhere. From the nature of the case, however, even a fairly high rate of interest would not generally be sufficient inducement for an investor to lend money to a perfectly irresponsible foreign government, which might at any time, if it thought fit, renounce all its past financial obligations. Whether the general practice of providing for the repayment of

loans has arisen from the scruples of foreign governments to accumulate a large national debt like our own, or whether, on the other hand, it has been found necessary to give an undertaking to repay the loan over a term of years, in order to make the loan at all palatable to investors, I cannot say; but the fact none the less remains, that very generally provision appears to be made, in the case of loans by foreign governments, for repayment in a shorter or longer period of time. It may be stated that, for all practical purposes, this is equivalent to borrowing money by means of terminable annuities, and undoubtedly there are many cases where the revenue of certain duties, taxes, or other dues specially hypothecated in favour of the said loan, is sufficient to provide the amount of the annual charge. Judging from appearances, however, it is found in practice that a fairly high rate of interest, with repayment sooner or later of the money invested, is not the most attractive form for investors. The form which these loans usually take is that of a moderately high rate of interest coupled with a considerable bonus on repayment, and it may naturally be concluded that this is the form most suited to the taste of investors. By bringing a loan before the public in this shape, two purposes at least are served—one is, that the actual terms on which the loan is obtained are to some extent concealed, otherwise, it is possible that the extravagant terms offered in some cases would act as a caution to intending investors, and instead of serving as a bait would have the contrary effect; and another reason is that, by offering a bonus on repayment, a slight gambling element is introduced, which is an almost irresistible attraction to great numbers of investors. The prospect of making 5 to 7 or 8 percent interest, and when your bond is drawn for repayment receiving a bonus of, say 15 to 25 or 30 percent, is undoubtedly a pleasant one, and the parties who bring out these foreign loans are well aware of the fact. Many persons, however, have recently learned that this pleasing prospect does not always become a reality. The disclosures already made before the Parliamentary Committee now sitting are of a very startling nature, and this investigation will doubtless check for a time the fondness of investors for the class of security in question. Two very important lessons have, at any rate, already been taught—namely, first, that the respectability of the firm or company acting as agent for the loan is no criterion whatever of the soundness of the loan itself; and, secondly, that an official quotation on the Stock Exchange must be taken *quantum valeat*, and no more. It is not intended, however, in the present paper, to enter further

into this question, but rather to take matters as they are, and discuss certain points more strictly within the actuary's province.

As has been already pointed out, there is generally a provision made for repayment with a bonus, the amount of such bonus being a fixed quantity, but the time at which it will be received being generally uncertain; and it has become an important practical question to determine the exact effect of this bonus, both from the borrower's point of view and the investor's or lender's. In the former case, as is well known, the problem is generally an extremely simple one theoretically. In the common case where the loan takes the form of a terminable annuity, or, as it is usually put, where the loan is redeemable by an accumulative sinking fund, we have to determine the actual rate of interest (x) paid by the borrower the equation

$$1-p=(r+s)\frac{1-(1+x)^{-n}}{x},$$

the nominal amount of the loan being 1, the price of issue $1-p$, so that p is the bonus received on repayment, r the nominal rate of interest, s the amount redeemed the first year, where $s \cdot \frac{(1+r)^n-1}{r} = 1$.

We have from the above

$$\frac{1-p}{r+s} = \frac{1-(1+x)^{-n}}{x} = a, \text{ (say).}$$

There exists no means of solving exactly this equation in x , and various methods of approximation have therefore been resorted to, which it may be useful to consider on the present occasion.

Professor De Morgan, in a very valuable paper in vol. 8 of the *Journal*, p. 61, "On the determination of the Rate of Interest of an Annuity", in speaking of the solution of the above equation, says, "The problem is, I doubt not, of infrequent occurrence. It never occurred to myself to want more than a look at the tables would furnish; but it may be suspected that, as the requirements of business become closer, inverse problems will acquire more importance than they now possess". This prognostication has undoubtedly been fulfilled with regard to the problem we are now dealing with.

If i be a near value of x less* than the true value, and $\frac{1-(1+i)^{-n}}{i} = a'$, then we have

$$(1). \quad x = i \left(1 + \frac{a' - a}{a - nv^{n+1}} \right).$$

* This limitation is incorrect, as pointed out by Mr. Baden.—W.S. (See p. 91.)

This is the formula demonstrated by Baily, in his *Doctrine of Interest and Annuities*, and is also given in Jones, vol. i, p. 41.

$$(2). \quad x = i \left(1 + \frac{a' - a}{a' - nv^{n+1}} \right).$$

This is Barrett's improvement on formula (1); and the earliest demonstration of it with which I am acquainted is given in a letter signed "M" in vol. vi, p. 56 of the *Journal*, where use has been made of Taylor's theorem in the Differential Calculus for the purpose. Recently it has been demonstrated in precisely the same way by Mr. Makeham, as a particular case of a more general formula, in his highly suggestive paper on "Loans repayable by Instalments"—see vol. xviii, p. 136. Still more recently another method of obtaining the formula has been given by Mr. McLauchlan—see vol. xviii, p. 297.

(3). A nearer approximation than the preceding:—

$$x = i \left\{ 1 + \frac{a' - a}{a - nv^{n+1} + \frac{n \cdot n + 1 i (a' - a) v^{n+2}}{2(a - nv^{n+1})}} \right\}$$

This is given by the late Professor De Morgan in vol. viii, p. 67 of the *Journal*, but without demonstration. In Mr. McLauchlan's contribution already referred to, it is shown how the formula may be obtained; but the method employed by De Morgan, it is probable, was simpler than that used by Mr. McLauchlan.

$$\text{Thus we have} \quad a = \frac{1 - (1 + i + \rho)^{-n}}{i + \rho},$$

where i is an approximate value obtained by inspection, and $x = i + \rho$; then the above gives us approximately,

$$\frac{n \cdot n + 1}{2} v^{n+2} \rho^2 + (a - nv^{n+1}) \rho - (a' - a) i = 0;$$

or writing $\rho \times \frac{i(a' - a)}{a - nv^{n+1}}$ for ρ^2 , which is approximately true,

since $\rho = \frac{i(a' - a)}{a - nv^{n+1}}$ from (1),

this gives us at once

$$\rho = \frac{(a' - a) i}{a - nv^{n+1} + \frac{n \cdot n + 1}{2} v^{n+2} \frac{i(a' - a)}{a - nv^{n+1}}}$$

(4). A nearer approximation still is

$$x = i \left\{ 1 + \frac{a' - a}{a - nv^{n+1} + \frac{n \cdot n + 1}{2} \cdot \frac{i(a' - a) v^{n+2}}{a' - nv^{n+1}}} \right\},$$

which differs from the preceding in the second term of the denominator, where $a' - nv^{n+1}$ takes the place of $a - nv^{n+1}$. Mr. McLauchlan has, in the previously quoted article, given two demonstrations of this formula. The quadratic equation in ρ where $\rho = x - i$, which he obtains, follows at once by an application of Taylor's theorem in the Differential Calculus, just in the same way as formula (2) is obtained, only taking three terms of the expansion instead of two.

Denoting the value of the annuity certain for n years at the rate of interest i by $\phi(i)$ we have, using the previous notation, $\phi(x) = a$ and $\phi(i) = a'$,

$$\therefore \phi(x) = \phi(i + \rho) = \phi(i) + \rho \frac{d\phi}{di} + \frac{\rho^2}{2} \frac{d^2\phi}{di^2} \text{ approximately. . . (a)}$$

$$\text{Now } \phi(i) = \frac{1 - v^n}{i};$$

$$\therefore \frac{d\phi}{di} = -\frac{1 - v^n}{i^2} + \frac{nv^{n+1}}{i} = -\frac{1}{i} (a' - nv^{n+1})$$

$$\begin{aligned} \text{and } \frac{d^2\phi}{di^2} &= -\frac{1}{i} \left\{ -\frac{1 - v^n}{i^2} + \frac{nv^{n+1}}{i} \right\} + \frac{1 - v^n}{i^3} - \frac{nv^{n+1}}{i^2} - \frac{n \cdot n + 1 v^{n+2}}{i} \\ &= -\frac{1}{i^2} \left\{ n \cdot n + 1 \cdot i v^{n+2} - 2 \left(\frac{1 - v^n}{i} - nv^{n+1} \right) \right\}, \end{aligned}$$

and substituting these results in (a) we have

$$\frac{1}{i^2} \left\{ \frac{n \cdot n + 1 v^{n+2}}{2} - (a' - nv^{n+1}) \right\} \rho^2 + \frac{1}{i} (a' - nv^{n+1}) \rho - (a' - a) = 0,$$

which is the equation for determining $x - i$ or ρ obtained by Mr. McLauchlan.

A much shorter way of obtaining this formula is by using the method adopted for (3), taking as the first approximation to the value of ρ , $\rho = \frac{i(a' - a)}{a' - nv^{n+1}}$, given by (2). Writing, therefore,

$$\rho \times \frac{i(a' - a)}{a' - nv^{n+1}} \text{ for } \rho^2 \text{ in the equation}$$

$$\frac{n \cdot n + 1}{2} v^{n+2} \rho^2 + (a - nv^{n+1}) \rho - (a' - a) i = 0,$$

we get

$$\rho = \frac{i(a' - a)}{a - nv^{n+1} + \frac{n \cdot n + 1}{2} v^{n+2} \frac{i(a' - a)}{a' - nv^{n+1}}},$$

which is Mr. McLauchlan's result.

(5.) Professor De Morgan, on p. 67 of vol. viii, gives a formula still more exact than the preceding.

This formula is as follows :—

$$\begin{aligned} \text{Let } p &= (a' - a)i & s &= k + \frac{n+2}{3}q \\ q &= a - nv^{n+1} & t &= \frac{vp}{q^2}, \\ k &= n \cdot n + 1v^{n+1} \end{aligned}$$

then
$$\rho = \frac{p}{q} \left\{ 1 - \frac{kt}{2} (1-st) \right\}.$$

This formula gives very nearly indeed the correct value of ρ . In the example given by De Morgan, where $n=100$, $a=9.5233704$, the result taking $i=.10$ is $i+\rho$ or the actual rate $=.10499991$, while it should be $.105$ —an approximation certainly sufficiently near for ordinary purposes.

This formula, like (3), is not demonstrated by De Morgan; but may be thus obtained :—

From
$$a = \frac{1 - (1+i+\rho)^{-n}}{i+\rho}$$

we have

$$\frac{n \cdot n + 1 \cdot n + 2}{2 \cdot 3} v^{n+3} \rho^3 - \frac{n \cdot n + 1}{2} v^{n+2} \rho^2 - (a - nv^{n+1}) \rho + (a' - a) i = 0;$$

or, using the above notation,

$$\frac{k}{2} \frac{(s-k)}{q} v^2 \rho^3 - \frac{k}{2} v \rho^2 - q \rho + p = 0.$$

Putting, as before,

$$\rho^2 = \rho \times \frac{p}{q},$$

and

$$\rho^3 = \rho \times \frac{p^2}{q^2},$$

the equation becomes

$$\frac{k}{2} \frac{(s-k)}{q} v^2 \rho \times \frac{p^2}{q^2} - \frac{k}{2} v \rho \times \frac{p}{q} - q \rho + p = 0;$$

or
$$\rho \left\{ q + \frac{k}{2} \times \frac{pv}{q} - \frac{k}{2} (s-k) q t^2 \right\} = p;$$

or
$$\begin{aligned} \rho &= \frac{p}{q + \frac{k}{2} \left(\frac{pv}{q} - (s-k) q t^2 \right)} \\ &= \frac{\frac{p}{q}}{1 + \frac{k}{2} \left\{ t - (s-k) t^2 \right\}} \\ &= \frac{p}{q} \left\{ 1 - \frac{kt}{2} (1-st) \right\} \text{ approximately.} \end{aligned}$$

In the same paper, Professor De Morgan gives an interesting *résumé* of the various attempts that have been made to obtain an approximate solution of the equation, which, as there pointed out, is a problem of some two centuries' standing. Before anything like extensive sets of annuity values had been calculated and published, mathematicians appear to have confined their attention to obtaining a formula not involving the use of existing tables to get an approximate value. The formula of this nature, given by Professor De Morgan, on p. 65, vol. viii, affords apparently a good approximation to the true value, but as the process of computation is intricate, it will never probably be much employed.

Besides the methods already mentioned of determining the actual rate of interest paid in the case of a loan issued at one price and repaid by drawings at another, several others have from time to time been suggested. One method is, to ascertain the "probable epoch"—that is, the time which will elapse before just half the loan is repaid, and to consider the whole loan as repaid at that date, and then ascertain the actual rate of interest which would be paid on this assumption. This method is clearly incorrect in theory, and as practically used gives results wide of the mark.

In symbols, it is as follows:—

If s be the sinking fund per l, set aside the first year, and n the duration of the loan,

$$\text{then} \quad s \frac{1+i|^n - 1}{i} = 1$$

$$\text{and if} \quad s \frac{(1+i)^t - 1}{i} = \frac{1}{2}$$

t is the "*probable epoch*"

$$\text{To find } t \text{ we have} \quad s \frac{1+i|^t - 1}{i} = \frac{1}{2}$$

$$\text{but} \quad s = \frac{i}{1+i|^n - 1}$$

$$\therefore \frac{1+i|^t - 1}{1+i|^n - 1} = \frac{1}{2}$$

$$\text{or} \quad 1+i|^t = \frac{1+i|^n + 1}{2},$$

whence t can be found.

Denoting by x the actual rate of interest paid, we have, if loan is redeemed at par and issued at discount of D , say

$$D = \frac{1 - (1+x)^{-t}}{x} (x-i),$$

The value of x would be at once given by the use of Lieut.-Colonel Oakes's *Debenture Tables*. It may be remarked, however, in passing, that these tables are of no use, in cases by no means rare as regards debenture bonds, where the bonds are purchased at a premium.

Another method, suggested as an improvement upon the preceding, is as follows:—The loan being repayable in n yearly instalments, increasing year by year in the ratio 1 to $1+i$ (i being the nominal rate of interest at which the loan is issued), then denoting the amount paid off the first year by s , the following amounts will be

$$s(1+i), s(1+i)^2, \dots s(1+i)^{n-1};$$

and the average amount paid off in a year

$$= \frac{s + s(1+i) + \dots + s(1+i)^{n-1}}{n} = \frac{1}{n},$$

and putting $s(1+i)^{t-1} = \frac{1}{n}$, then at the end of the t th year $\frac{1}{n}$ th of the whole loan will be repaid. This year is called the “epoch of mean probability”. Then an approximation is made to the actual rate of interest paid on the loan by considering the whole loan as repaid t years from date of issue.

This method, like the preceding, is manifestly incorrect, but it gives values nearer to the true values than the former. The first of the two just-mentioned methods of approximation is in actual use; and for the second, which has been suggested as an improvement on the first, I am indebted to Messieurs Vintégoux and de Reinach, *Formules et Tables d'Intérêts composés et d'Annuités*, (Paris 1874), pp. 34–36.

Besides the question of the rate of interest paid by the borrower, there is the no less important question of the rate of interest realized by the investor. Of course, if the whole loan were taken up by a single individual, the rate of interest would in each case be the same. In practice it may be said this is never the case, and as far as I can gather, the general impression seems to be that, in the case, say, of the purchaser of a single bond, it is impossible to say what rate of interest he will realize on his investment, as his bond may be paid off in any year of the n years which repay-

ment extends, the rate of interest he realizes becomes, in the general case of repayment, at par, the bond being bought at a discount, less and less as repayment is deferred; and on the contrary, where the bonds are issued or bought at a premium, the rate of interest realized by the investor is greater the longer repayment is deferred.

Now, taking the general case, where the bonds are issued at a discount, the investor is in an exactly similar position to an office purchasing the reversion to a sum S , payable on the death of the life tenant of, say 60, for the sum $(A_{60})_s \times S$. Using the Carlisle Table, and assuming that the office has no other reversions, the office will make 6 percent on its investment if the tenant for life dies about the age of 73; if he dies before that age the office will make more than 6 percent, and if after that age less than 6 percent. Of course, if the office owned a large number of such reversions of equal amount in each case, it may be safely assumed that, on the average, the rate of interest realized would be about 6 percent; and just in the same way, if an investor held a considerable number of bonds in a foreign loan, he might fairly reckon that his bonds would be drawn for repayment year by year in the proportion the total amount of ~~them~~ ^{the loan} bore to the total amount of the loan. Assuming this to be the case, therefore, the investor would realize the same rate of interest as that paid by the borrower.

Returning to the case of the purchaser of a single bond, let us consider the matter from another point of view. If l be the amount of the bond, L the amount of the loan, P the portion of the loan repaid by the accumulative sinking fund the first year, i being the nominal rate of interest on the loan, then $\frac{P}{L}$ = chance of the particular bond in question being drawn for repayment the first year,

$$(1+i) \frac{P}{L} = \text{chance for 2nd year} = (1+i)s \text{ (say)}$$

$$(1+i)^2 \frac{P}{L} = \quad \text{,,} \quad \text{3rd} \quad \text{,,} \quad = (1+i)^2 s,$$

and so on.

Now at the end of the first year the investor receives his dividend i , and has a chance s of receiving l in repayment of his bond, and the value of this chance being $s \times l$ or sl , he may be considered as receiving $i + s$ at the end of the first year. Similarly

at the end of the second year he will receive a dividend i if his bond was not drawn for repayment in the previous year, and the chance of this being $1-s$, the value of such dividend may be taken as $(1-s)i$, and the chance of the bond being drawn for repayment being $s(1+i)$, the total value of his expectation at end of second year $= (1-s)i + s(1+i) = i + s$. Similarly the value of his expectation at the end of the third year

$$\begin{aligned} &= \{1-s-s(1+i)\}i + s(1+i)^2 \\ &= i(1-s) - s(1+i)(i-1-i) \\ &= i(1-s) + s(1+i) \\ &= i + s, \end{aligned}$$

and so on for other years.

In other words, the purchaser of a single bond is in exactly the same position as if he bought an annuity certain of $i+s$ to run for the term of the loan. Assuming, therefore, that the price of issue or purchase was $1-p$ per 1, we have

$$1-p = (i+s) \frac{1-(1+x)^{-n}}{x},$$

x being the rate of interest at which the various payments are discounted, that is the rate of interest realized by the purchaser, and we see that the same formula obtains as in the case of the borrower.

It has occurred to me that it would be of interest, in more ways than one, to apply some of the methods shown above to the case of existing foreign loans; and I have therefore, in the table which follows, taken a number of the principal foreign loans brought out in England, and have calculated the actual rate of interest paid by the borrower in each case, first without allowing for the redemption of the loan at par, and secondly allowing for redemption at par. The list of loans, and the various particulars relating to them, are taken from Mr. Bernard Cracroft's *Weekly Stock and Share List*, for 3 April 1875, under the heading "Foreign Government Loans and Securities guaranteed by Government".

With regard to the above table, various points have to be attended to. In the first place, it will be noticed that the sinking fund is generally quoted as percentage in round numbers, so that the time required for redemption of the loan will not generally be an exact number of years. This will not in practice have any considerable effect upon the results, which are obtained by taking

Name of Loan.	Date of Issue.	Amount.	Price of Issue.	Nominal Interest percent.	Sinking Fund, percent per Annum.	Actual Rate of Interest percent paid by borrower, assuming			
						(t) Repayment at Issue Price.	By Formula (t).	By Formula (s).	By Formula (d).
Argentina . . .	1868	2,500,000	72½	6	2½ accum.	8,2759	10,2005	10,1966	10,1087
Do. (Hard dollars)	1871	6,122,400	88½	6	"	6,7797	7,5021	7,5021	7,5021
Bolivia . . .	1872-4	p	76	6	1	7,8947	8,6070	8,6056	8,6055
Chili . . .	1872	1,700,000	78	6	2	7,6923	8,9439	8,9436	8,9438
Do. . .	1866	1,120,920	92	6	2	7,6087	7,9734	7,9733	7,9734
Do. . .	1867	2,000,000	84	6	2	7,1429	8,0329	8,0328	8,0329
Do. . .	1873	2,270,500	94	5	2	5,3191	5,7420	5,7419	5,7420
Egypt . . .	1862	3,292,800	{ 82½ } { 84½ }	7	'960 "	{ 8,4848 } { 8,2840 }	{ 8,9780 } { 8,7133 }	{ 8,9780 } { 8,7132 }	{ 8,9780 } { 8,7133 }
Do. . .	1864	5,704,200	93	7	3,87 "	7,5269	8,0071	8,0071	8,0071
Do. (Pasha)	1866	3,387,300	90	7	3,87 "	7,7778	8,5494	8,5491	8,5493
Do. . .	1867	2,080,000	90	9	3,41 "	10,0000	10,8875	10,8151	10,8404
Do. . .	1868	11,890,000	75	7	1	9,3333	10,1315	10,1298	10,1311
Do. (Khedive)	1870	7,142,860	78½	7	2,36 "	8,9172	10,2231	10,2182	10,2209
Do. . .	1873	32,000,000	84½	7	1	8,2840	8,7681	8,7681	8,7681
Honduras . . .	1870	2,500,000	80	10	3	12,5000	15,0263	13,3450	13,9518
Paraguay . . .	1871	1,000,000	80	8	2	10,0000	11,9689	11,9689	11,1389
Do. . .	1872	2,000,000	85	8	2	9,4118	10,2512	10,2450	10,2485
Peru . . .	1872	36,800,000	77½	5	2	7,7285	7,7285	7,7285	7,7285
Russian (Anglo-Dutch)	1864	6,000,000	85	5	1	5,8823	6,3308	6,3297	6,3304
Do. (Nicol. Railway)	1867-9	23,110,000	{ 61 } { 63 }	4	'15 "	{ 6,5574 } { 6,3492 }	{ 6,7774 } { 6,5576 }	{ 6,7773 } { 6,5576 }	{ 6,7774 } { 6,5575 }
Do. (Orel Railway)	1867	4,500,000	77½	5	'083 "	6,4516	6,5263	6,5262	6,5263
Do. (Moscow)	1868	1,920,000	78	5	'13 "	6,4103	6,5193	6,5192	6,5193
Do. (Char. Azof.)	1868	4,349,280	80	5	'083 "	6,2500	6,3175	6,3167	6,3174
Do. . .	1870	12,000,000	80	5	'088 "	6,2500	6,3284	6,3274	6,3283
Do. . .	1871	12,000,000	81½	5	'008 "	6,1300	6,2076	6,2073	6,2076
Do. . .	1872	15,000,000	89	5	'098 "	5,6180	5,6620	5,6607	5,6619
Do. . .	1873	15,000,000	93	5	'098 "	5,3763	5,4048	5,4046	5,4046
San Domingo . . .	1869	757,700	70	6	1,750 "	8,5714	10,1824	10,1792	10,1813
Turkish . . .	1854	3,000,000	80	6	1	7,5000	8,0757	8,0750	8,0754
Do. . .	1858-9	4,380,000	85	6	1	{ 7,0588 } { 9,6000 }	{ 7,4713 } { 10,7741 }	{ 7,4713 } { 10,7741 }	{ 7,4713 } { 10,8224 }
Do. . .	1862	8,000,000	68	6	2	8,4000	10,7712	10,7160	10,7497
Do. . .	1863	8,000,000	71½	6	2	8,4000	10,0928	10,0928	10,0933
Do. (Muttons)	1865	6,000,000	66	6	2,44 "	9,0909	11,5527	11,3657	11,4820
Do. (Cohens)	1869	22,222,220	60½	6	1	9,9174	11,2603	11,1193	11,1015
Do. (Tribute)	1871	5,700,000	73	6	1	8,2192	9,0372	9,0370	9,0371
Do. . .	1873	27,777,780	58½	6	1	10,2564	11,6728	11,4645	11,5007
Uruguay . . .	1871	3,500,000	72	6	2½ "	8,3333	10,3023	10,2935	10,2985

the term of redemption to be the nearest number of complete years.

Another point is, that in the majority of the cases the interest and drawings are payable half-yearly, and in some few cases quarterly; but this, too, will be found to be not of material consequence.

Still another point to consider is, that as the bonds are usually for fixed amounts of £20, £50, £100, £500, or £1000, the actual amount of any drawing will not in general be the exact theoretical amount.

A point of great importance to remember in considering on what terms government loans are obtained, is that in no case does the borrower receive the sum which would arise from the issue of the loan at the price of issue. Besides the absolutely necessary expenses attendant upon all transactions of this nature, there are other charges almost indispensable, such as agent's commissions, &c., which must have the effect of raising the actual rate of interest paid by the borrower. We need not say here anything of the many other financial "arrangements" which appear to form a by no means unimportant feature in bringing out a loan, and which in fact bear an exceedingly close resemblance to the practices in vogue in "floating" a company. It will be sufficient for our purpose to state generally that such "arrangements" have for their effect to increase very materially the actual rate of interest paid by the borrower, so that in practice a purchaser of the whole loan at the issue price would *not* make the same rate of interest as the loan costs the borrower.

The following account of the discussion on the paper is taken in part from the *Insurance Record*, Mr. Baden supplying the figures.

MR. BADEN said that an exact solution of the problem discussed by Mr. Sutton could not be obtained, and actuaries must therefore remain satisfied with some approximation to the truth. There are, however, two ways of going to work to approximate to the true rate of interest,—a scientific and an unscientific way. The scientific way had been very clearly set forth by Mr. Sutton in the presentation of a number of more or less intricate formulas, but he (Mr. Baden) invariably adopted the unscientific method, making a series of direct trials for the rate; and he now wished to justify that method, as the best practical way of arriving at the solution wanted. He then proceeded

—With this object, I have jotted down a few figures. I have taken the case of a loan issued at the price of 78 per-cent, and redeemable at par by an accumulative sinking fund in 30 years. If this is calculated at 5 per-cent interest, the borrower will pay an annuity of 6·505144 for 30 years, in consideration of the advance of 78 (*Jones On Annuities*, p. 98). This example appears to me to afford the best test that can be got of the respective advantages of the two methods. In questions of this kind we acquire, after a little practice, a sort of tact in guessing, which enables us, using, to start from, the ordinary tables given in works on compound interest, at once to fix an approximate rate not very far from the mark. In this case it appeared to me that a good approximate rate would be ·0734. I try it, having first of all obtained the actual value of the annuity, 11·99051, as a test. The value brought out by my trial is 11·9967, differing by 6 in the third place of decimals. These are the figures:—

Test value of annuity.

$$\begin{array}{r}
 \frac{p}{a} \cdot 06505144 \cdot 7800000 (11 \cdot 99051 = \text{true value}) \\
 \underline{6505144} \\
 1294856 \\
 \underline{6505144} \\
 6443416 \\
 \underline{58546296} \\
 5887864 \\
 \underline{58546296} \\
 332344 \\
 \underline{32525720} \\
 708680
 \end{array}$$

Trial rates.

$$\begin{array}{r}
 i' = \cdot 0734 \qquad \frac{1 - (1 \cdot 0734)^{-30}}{\cdot 0734} \\
 \cdot 03076159 \qquad \cdot 0734 \cdot 8805593 (11 \cdot 9967) \\
 \underline{30} \qquad \underline{734} \\
 \cdot 9228477 \qquad 1465 \\
 \underline{1 \cdot 0771523} \qquad \underline{734} \\
 1498 \qquad 7315 \\
 \underline{25} \qquad 6606 \\
 \cdot 1194407 \qquad 7099 \\
 \underline{\cdot 8805593} \qquad \underline{6606} \\
 \qquad \underline{4933} \\
 \qquad \underline{4404} \\
 \qquad \underline{529} \\
 \qquad \underline{5138} \\
 \qquad \underline{152}
 \end{array}$$

The CHAIRMAN—Have you not got at your result from previous cases which you have worked out, so that your power of guessing is due to your having worked out a great many cases?

Mr. BADEN—Yes; you acquire by practice a tact in guessing the proper quantity to interpolate in the tables you use.

The CHAIRMAN—I do not call that guessing. It is using knowledge previously obtained.*

Mr. BADEN—Well, my second trial brings out the annuity value equal to 11.9908, the true value being 11.9905. Now, as that would evidently bring the rate per-cent within a fraction of a penny, I thought that was quite near enough, and I left off. Here are the figures:—

Trial rates.

$i = .07345$	$.07345) \cdot 880726 (11.9908$
	<u>7345</u>
$\cdot 03078182$	<u>14622</u>
<u>30</u>	<u>7345</u>
$\cdot 9234546$	<u>72776</u>
$\cdot 10765454$	<u>66105</u>
<u>312</u>	<u>6671</u>
<u>142</u>	<u>66105</u>
$\cdot 119274$	<u>60500</u>
<u>$\cdot 880726$</u>	<u>58760</u>
	<u>1740</u>
	True value . . . 11.9905
	Approximate value . . . 11.9908
	Difference . . . <u>.0003</u>

This is a fair specimen of the amount of trouble which I had in order to arrive at the result, and the above include all the figures I have had to employ in the work.

I have not had time to go through all the formulas presented to us by Mr. Sutton, but I have taken the first and the last. The first is one with which we are probably all well acquainted, being the one given by Bailly in his *Interest and Annuities*. But I have two remarks to make upon it. The first is, that Mr. Sutton has presented to us the formula in so very different a form, that it does not look like Bailly's. It is not a mere change of symbols, but it is a different disposition of symbols altogether, which makes it another formula; and I consider that course undesirable. My second remark is that Mr. Sutton says that Bailly's formula is given by Jones on p. 42 of his first volume. Now the fact is that the formula there given by Jones is equivalent to the formula given by Bailly in Note B of his Appendix, but is not the same as that given by Bailly in the body of his work. Here is the working according to Bailly's formula:—

Bailly's Formula—Note B, Appendix. (Jones, p. 42.)

$$z = \frac{1 - \frac{p^i}{a} - (1+i)^{-n}}{\frac{p}{a} - n(1+i)^{-(n+1)}}$$

* Mr. Baden explains to us that his first approximate value is obtained by a mental estimate based on the examination of the annuity values in Rance's tables, which proceed by differences of $\frac{1}{4}$ per-cent in the rate of interest. We find that, interpolating by first differences, Rance's tables at once give the approximate rate .0734645. If the usual tables are employed, which proceed by differences of 1 per-cent in the rate of interest, the approximate value obtained is less accurate, being .073636.—ED. J. I. A.

Formula with i' below true rate: $i = i' + z = .07 + z$.

·12277301	·06505144)	7800000 (11·99051
30	6505144	·07
3·6881903	1294856	·8393357
11·99051	6505144	1313671
<u>8·30732</u>	6443416	·9707028
	58546296	<u>·0292972</u>
	5887864	
	58546296	
	332344	
	32525720	
	<u>708680</u>	

8·30732)·02929720 (	·003527
2492196	·07
437524	<u>·07353</u>
4153660	
221580	
1661464	
<u>554336</u>	

Trial of rate brought out by formula.

1 - (1·07353) ⁻³⁰
·07353
·03081419
30
·9244257
<u>1·0755743</u>
470
273
·1190075
·07353)·8809925(11·9814
7353
14569
7353
72162
66177
59855
58824
1031
7353
2957
29412
True value . . . 11·9905
Approximate value . . . 11·9814
Difference . . . <u>·0091</u>

You see that the result I have obtained by my unscientific method is nearer to the truth, although I have used a smaller number of figures.

Mr. Sutton has, I think, unduly limited the application of Baily's formula, when he says "if i be a near value of x , less than the true value"; for there is no necessity that the value of i , the approximate quantity chosen, should be less than the true value. It may be taken as more than the true value, and the only alteration so introduced is that the difference which it is the object of the equation to calculate, would have to be added to the approximate rate instead of being

deducted, in order to get the true rate. In order to show this, I have worked out the problem in two ways, first, as above, by taking the quantity less than the true value, and next by taking the quantity greater than the true value. These are the figures:—

Baily's Formula—Note B.

$$1 - \frac{pi'}{a} - (1+i')^{-n}$$

$$\frac{p}{a} - n(1+i')^{-(n+1)}$$

Formula with i' above true rate: i=i'—z=.075—z.

$ \begin{array}{r} .06505144 \cdot 78000000 \quad (\quad 11.99051 \\ \underline{6505144} \qquad \qquad \qquad 570 \\ 1294856 \qquad \qquad \qquad 8393357 \\ \underline{6505144} \qquad \qquad \qquad 599526 \\ 6443416 \qquad \qquad \underline{.8992883} \\ 58546296 \\ 5887864 \\ \underline{58546296} \qquad \qquad \qquad .8992883 \\ 332344 \qquad \qquad \qquad .1142210 \\ \underline{32525720} \qquad \qquad \underline{1.0135093} \\ 708680 \qquad \qquad \underline{-.0135093} \end{array} $	
--	--

$ \begin{array}{r} .1062521 \\ \underline{30} \\ -3.187563 \\ 11.99051 \\ 8.802947 \cdot 013509300 \quad (\quad .075 \\ \underline{8802947} \qquad \qquad \underline{.001534} \\ 4706853 \qquad \qquad \underline{.073466=i} \\ 44014735 \\ 3048795 \\ 26408841 \\ \underline{4079109} \end{array} $	
---	--

Trial of rate brought out by formula.

$1 - (1.073466)^{-30}$	
$ \begin{array}{r} .03078587 \\ \underline{242} \\ .03078829 \\ \underline{30} \\ .9236487 \\ \underline{1.0763513} \\ 491 \\ \underline{22} \\ .1192206 \\ \underline{.8807794} \end{array} $	$ \begin{array}{r} .073466 \cdot 8807794 (11.9889 \\ \underline{73466} \\ 146119 \\ \underline{73466} \\ 726534 \\ \underline{661194} \\ 65340 \\ 587728 \\ 65672 \\ 587728 \\ 68992 \\ \underline{661194} \\ 28726 \end{array} $

True value . . .	11.9905
Approximate value	11.9889
Difference	.0016

This latter calculation brings out a more accurate result, for the simple reason that the approximate rate assumed in it is rather nearer to the true rate than the one taken in the previous calculation. I pass over all the other formulas; but, as a crucial instance of what I believe to be the practical undesirability of using these formulas, I take Professor De Morgan's last. The formula, though looking extremely simple, is anything but simple:—

$$\rho = \frac{p}{q} \left\{ 1 - \frac{kt}{2} (1-st) \right\};$$

for you have to make an equation for every one of the symbols;

$$p = (a' - a)i = .07(a' - a), \quad q = a - nv^{n+1},$$

$$k = n(n+1)v^{n+1}, \quad s = k + \frac{n+2}{3} \cdot q, \quad t = \frac{vp}{q^2}.$$

Here are the figures:—

(1) *Determination of Symbols.*

12.40904 = a'	122773 = v^{n+1}	122773 = v^{n+1}	332
11.99051 = a	30	31	$\frac{10.6667}{8.30732} = \frac{n+2}{3}$
.41853	3.68819	122773	$\frac{766601}{8307320} = q$
.07	11.99051	368819	766601
<u>.0292971 = p</u>	8.30732 = q	3.805963	8307320
	0.9194609 = $\log q$	30	498439
	2	114.17889 = k	49844
	1.8389218 = $\log q^2$		4984
	69.01154 = q^2		581
			88.61168
			114.17889 = k
			<u>202.79057 = s</u>
	.0292971 = p		
	4975439		
	2636739		
	87891		
	11719		
	1465		
	204		
	26		
	1		
69.01154 = q^2	0.2738045 = vp	0.00396752 = t	
	20703462		
	6676988		
	62110386		
	4659494		
	41406924		
	5188016		
	48308078		
	3572082		
	34505770		
	<u>1215050</u>		

(2) Working of Formula.

$$\begin{array}{r}
 \frac{q}{8.30732} \cdot \frac{p}{0.0292971} (\cdot 0035267) = \frac{p}{q} \\
 \underline{2492196} \\
 437514 \\
 \underline{4153660} \\
 221480 \\
 \underline{1661464} \\
 553336 \\
 \underline{4984392} \\
 548968
 \end{array}$$

$$\begin{array}{r}
 114.17889 = k \\
 \underline{257693000} \\
 34253667 \\
 10276100 \\
 685073 \\
 79925 \\
 5709 \\
 228 \\
 \underline{045300702} = kt
 \end{array}$$

$$\begin{array}{r}
 202.79057 = s \\
 \underline{257693} \\
 60837171 \\
 18251151 \\
 1216743 \\
 141953 \\
 10140 \\
 406 \\
 \underline{080457564} = st \\
 \underline{919542436} = 1 - st
 \end{array}$$

$$\begin{array}{r}
 \cdot 022650351 = \frac{kt}{2} \\
 \underline{634245919} \\
 203853159 \\
 2265035 \\
 2038532 \\
 113252 \\
 9060 \\
 453 \\
 91 \\
 7 \\
 1 \\
 \underline{0208279590} \\
 \underline{979172041} \\
 7625300 \\
 \underline{29375161} \\
 4895860 \\
 195834 \\
 58750 \\
 6854 \\
 \underline{0034532459} = \rho \\
 \cdot 07 \\
 \underline{0734532} = i + \rho
 \end{array}$$

(3) Verification.

$$\begin{array}{r}
 1 - (\cdot 0734532)^{-30} \\
 \underline{0734532}
 \end{array}$$

$$\begin{array}{r}
 0.0307831 = \log(i + \rho) \\
 \underline{30} \\
 \cdot 923493 \\
 \underline{1.076507} \\
 \underline{1.1192634} = (i + \rho)^{-30} \\
 \underline{8807366}
 \end{array}$$

$$\begin{array}{r}
 \cdot 0734532) 8807366(11.99044 \\
 \underline{734532} \\
 1462046 \\
 \underline{734532} \\
 7275140 \\
 \underline{6610788} \\
 6643520 \\
 \underline{6610788} \\
 3273200 \\
 \underline{2938128} \\
 3350720 \\
 \underline{2938128} \\
 412592
 \end{array}$$

$$\begin{array}{r}
 \text{True value} \quad . \quad . \quad . \quad 11.99051 \\
 \text{Approximate value} \quad . \quad . \quad . \quad 11.99044 \\
 \text{Difference} \quad . \quad . \quad . \quad \underline{0.00007}
 \end{array}$$

This certainly gives a very accurate result; but look at the quantity of figures, and consider the great risk of error in the long series of complicated calculations. If you compare them with the comparatively few figures, and the simple process which I call the mechanical one, working by the common direct formula for the value of an annuity, that will sufficiently prove that, as a practical matter, it is not desirable to use these scientific formulas, however valuable they may be to us from a scientific point of view, in showing how nearly we can arrive at a solution of the problem. I would only make one more observation upon this paper, and that is with regard to the last part of it. In reality there is only one question asked by financiers who bring out these things. They do not say anything about the rate to the borrower. They send you, at the last moment, a prospectus with certain blanks, asking you to fill up the figure in the statement "which will furnish the investor a rate of (so much) per-cent". It simply means the interest which the borrower pays, supposing that he pays nothing more than is down on his prospectus as the price of the loan; and it means also the interest which would be paid to any investor if he were the purchaser of all the bonds that are issued; and, in the language of financiers, that interest is the *average* rate of interest which they want; and that is the sole question with which we have any concern in matters of this kind.

MR. HARDY—Two questions have been brought before us tonight. One is determining the actual rate of interest paid by the borrower, which, I think, cannot be solved exactly, for it involves the solution of an equation of the *n*th degree, for which no general method has yet been obtained. The other question seems to be the rate yielded to the individual purchaser of a bond. I get that question put to me sometimes, and I always give two rates. I say, "You will get so much if the bond is paid, and so much if it is not". I recollect it being suggested by a friend, that we should ascertain the cost of insuring bonds being drawn; and if you take that off, it will give you the rate of interest. I cannot understand why some syndicate or company has not been got up in order to carry that into practical operation. There is no doubt whatever that these things could be made absolute investments to yield so much money on the account.

MR. WALFORD—In reference to what Mr. Hardy has just stated, it may interest some persons to know that the suggestion which he has made, is not a mythical one at all. Some time ago a company was formed—in the year 1866—with the intention of underwriting those loans and bonds and stocks generally. The scheme was a good one, and was worked out with much skill and care; but the Overend and Gurney failure put a stop to it, and I do not think that it has been resumed.

MR. W. B. HODGE—Individually I was not very curious to know to a penny what rate of interest will be paid by the dividend and bonus upon foreign loans, because the great question to my mind is whether they will be paid at all. We have had in the last 50 years a most remarkable increase of foreign debts. There are loans in Japan, in China, in Turkey, in Egypt, and in every part of the world. I find, taking some of the great States, that in the year 1822, the debts of Russia, France, Spain, and the United States, amounted

to £264,000,000. They amount now to £2,120,000,000; and we have Peru, Austria, and other States, eight in number, with debts amounting to £3,200,000,000. I suppose that about the year 1822, the National Debt of England was about two thirds of the whole national debts of the world. If they amounted to £1,200,000,000, it would be about the outside, the debt of England being £800,000,000. There are now, I should think, between four and five thousand millions of public debts. Now, I apprehend that instead of England being, as it was, the principal debtor, owing about two thirds of the debts of the world, its debt is only about one sixth. France has now £900,000,000; at the battle of Waterloo it had only £64,000,000.

The CHAIRMAN—Mr. Baden's statement about approximations is very interesting, but it seems almost as if he got by intuition what was the result of his formula. I remember a remark of the late Mr. Peter Hardy's, who was a great advocate of approximations of all kinds. He once said to me that they are not only valuable, but more accurate than exact calculations, on account of the many conflicting elements which you have. In fact, even in calculating approximate annuities on three or four lives, you get better results by approximation than if you worked them out exactly. As to the result, it does not matter whether your calculations are a penny too much or a penny too little, but that they involve such enormous sums. I have been asked for a result for a bond of £100, and I have found afterwards that it was going to be for a loan of £100,000, or perhaps a million.

Mr. SUTTON, in reply, said—Recently I have had to make many calculations of this kind, and I find that there are cases where a rule-of-thumb method is not applicable.

Mr. BADEN—When I speak of the rule-of-thumb, I do not mean to say that I neglect any mathematical consideration on the subject. I merely mean that I use the direct method of trial. That is, I use the formula for an annuity, and try a series of rates.

Mr. SUTTON—Some loans are brought out in this way. Repayment does not commence until some years after the issue of the loan. That is a case where the ordinary method of approximation does not readily apply. It is all very well for gentlemen of great experience in these matters to use a tentative method like Mr. Baden's; but there are many people who only occasionally come across such questions, and it is for them, mostly, that formulas like these are wanted. I agree with Mr. Baden that in most questions of this kind the people asking for information do want the exact penny, the reason, as stated by our Chairman, being that the amounts dealt in are very large; and I am informed that even private individuals will sometimes take £100,000 or £150,000 worth of stock. Banks and stockbrokers frequently have inquiries made by their clients as to what rate of interest such and such an investment will yield. Indeed it would seem that private investors are getting as much interested in the matter as financiers. I have seen many prospectuses, but very few in which the actual rate of interest is put in. I believe that the common practise is that, in bringing out a loan, the financiers like to have these figures for their own guidance; but I have not seen them, as a rule, published in the prospectus itself. As to the remarks of Mr. Hodge

about the increase of these foreign debts, it certainly is very wonderful. Another point is also very striking, and I intend to bring it out more fully when the paper is completed. That is, how, notwithstanding the increase in the indebtedness of some countries, they are able from time to time to borrow, and to make fresh loans on easier terms than the preceding ones.