the ordinary seminvariants; and for $m = 2$, $\mu = 4$, $\nu = 1$,

$$2(4ac - 5b^2),$$

$$96(a^2d - 3abc + 2b^3),$$

the pure reciprocal protomorphs; whilst, for $m = \mu = 1$, $\nu = 0$, we find

$$2ac - b^2,$$

$$2(3a^2d - 3abc + b^3),$$

the well-known non-unitary symmetric functions.

On the Theory of Screws in Elliptic Space. Fourth Note.

By A. BUCHHEIM, M.A.

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In this fourth, and, I hope, last Note on Screws in Elliptic Space, I extend some of the results of my Third Note to space of n dimensions. I have tried to vary the methods used as far as possible, in order that the extension should not consist merely in the addition of subscript indices and signs of summation, but should subserve an object that I have had in view all along, namely, the illustration and extension of Grassmann's methods. I have only succeeded in extending the results contained in the first part of my former Note. It does not seem possible to extend the application of Biquaternions to the representation of motions, for the simple reason that, as soon as we get beyond quaternions it becomes impossible to write the product of two quantities of the same kind as a new quantity of the same kind, unless we use one of Peirce's Algebras; but I do not think we should gain anything by doing so.

I.

In what follows I denote the complement of any quantity a by Ka , so that, if a is a product of units and the product of all the units is unity, we have $aKa = 1$. I denote by ω the matrix of any equation of the absolute, so that, if a' is the conjugate of a with respect to the absolute, and if ω denotes the matrix of the equation of the absolute

corresponding to the class of quantity to which a belongs (point equation if a is a point, line equation if a is a line, and so on), we have

$$a' = K\omega a.$$

I use Sab to denote $aKwb = bKwa$, and T^2a for Saa , and I do not use any special symbol for aKb when the matrix of the absolute is not the unit matrix.

If ϕ is any matrix there is a certain matrix ϕ' corresponding to it, and defined by the equation $Sx\phi'y = Sy\phi x$, x, y being any two quantities of the same kind. This matrix I call the conjugate of ϕ . If we suppose ϕ written in the form of a square array, there is another matrix corresponding to it, which is generally called its conjugate. I shall call this the reflection of ϕ , and denote it by ϕ'' ; the equation defining ϕ'' is

$$xK\phi''y = yK\phi x.$$

It is easy to show that we have $\phi' = \omega^{-1}\phi''\omega$, for we have

$$\begin{aligned} Sx\phi'y &= Sy\phi x \\ &= yK\omega\phi x \\ &= xK\phi''\omega y^* \\ &= xK\omega \cdot \omega^{-1}\phi''\omega y \\ &= Sx\omega^{-1}\phi''\omega y, \end{aligned}$$

and therefore $\phi' = \omega^{-1}\phi''\omega$.

If $\phi'' = \phi$, ϕ is symmetric; and if $\phi'' = -\phi$, ϕ is skew-symmetric; if $\phi' = \phi$ I call ϕ self-conjugate, and if $\phi' = -\phi$ I call ϕ skew-conjugate.

If ϕ is symmetric or skew-symmetric, $\omega^{-1}\phi$ is self-conjugate or skew-conjugate.

For

$$\begin{aligned} (\omega^{-1}\phi)' &= \omega^{-1}(\omega^{-1}\phi)''\omega \\ &= \omega^{-1}\phi''\omega^{-1}\omega \\ &= \omega^{-1}\phi'', \end{aligned}$$

and therefore, if $\phi'' = \pm \phi$, $(\omega^{-1}\phi)' = \pm \omega^{-1}\phi$.

It is easy to see that we have $(ab)' = b'a'$.

For

$$\begin{aligned} (ab)' &= \omega^{-1}(ab)''\omega \\ &= \omega^{-1}b''a''\omega \\ &= \omega^{-1}b''\omega \cdot \omega^{-1}a''\omega \\ &= b'a'. \end{aligned}$$

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* Since ω is symmetric.

II.

Let ϕ be any matrix, and let ω be the matrix of the point equation of the absolute, then the equation of the absolute is $xK\omega x = 0$. If we transform space by means of the operator ϕ , the point ϕx is on the absolute if $\phi x K \omega \phi x = 0$, that is, if $x K \phi'' \omega \phi x = 0$. Therefore, if the matrix ϕ is to represent a motion, that is, if it is not to alter the absolute, we must have $\phi'' \omega \phi = \lambda \omega$, where λ is a scalar which may always be supposed equal to unity. Now, if $\phi'' \omega \phi = \omega$, we have $\omega^{-1} \phi'' \omega \phi = 1$, that is, $\phi' \phi = 1$. It follows at once from this that, if

$$\psi = \frac{1 - \phi}{1 + \phi},$$

we have

$$\psi + \psi' = 0,$$

so that ψ is skew-conjugate. We see, therefore, that the matrix defining a motion can be written in the form $\frac{1 - \psi}{1 + \psi}$, where ψ is a skew-conjugate matrix. Moreover, we saw in (2) that, if χ is a skew-symmetric matrix, $\omega^{-1} \chi$ is skew-conjugate, and conversely; we can therefore write $\omega^{-1} \chi$ for ψ , and we get Prof. Cayley's expression for the automorphics of a quadric ω in the form

$$\frac{1 - \omega^{-1} \chi}{1 + \omega^{-1} \chi},$$

where χ is a skew-symmetric matrix.

III.

In an n -tuple space (n -point, space of $n-1$ dimensions) a screw is defined as a form of order $n-2$. If a is any screw, I write

$$a = \sum a_{ik} K e_i e_k,$$

where the summation extends to the different pairs that can be formed from the n indices 1, 2, ... n , and where we need not be careful as to the order of e_i, e_k , if we assume $a_{ik} = -a_{ki}$. If we multiply a point by a screw, we get a form of order $n-1$ whose n coefficients are derived from those of the point by a skew-symmetric matrix. For, let $x = \sum v_i e_i$ be any point; then we see that in the product xa the coefficient a_{ik} will occur twice, namely, once in the term $a_{ik} x_i e_i K e_i e_k$, and again in the term $a_{ik} x_k e_k K e_i e_k$. Now $e_i [e_k K e_i e_k] = 1$, and $e_k [e_i K e_i e_k] = -1$. We see, therefore, that the two terms in which a_{ik} occurs are $-a_{ik} x_i K e_k$ and $a_{ik} x_k K e_i$ respectively, and that, if we write $xa = \sum \xi_i K e_i$, the coefficient of x_k in the value of ξ_i is a_{ik} , while

the coefficient of x_i in the value of ξ_k is $-a_{ik} = a_{ki}$; it follows that we have $(\xi_1 \dots \xi_n) = (A \mathfrak{X} x_1 \dots x_n)$, where $A = \|a_{ik}\|$, and is skew-symmetric.

I call A the matrix appertaining to the screw a , and a the screw appertaining to the matrix A .

The analytical connection between a screw and a motion can now be stated as follows:—Starting with the screw a , we have a skew-symmetric matrix A appertaining to it, and the matrix defining the motion

$$\text{is} \quad \phi = \frac{1 - \omega^{-1}A}{1 + \omega^{-1}A}.$$

If we start with a motion defined by a matrix ϕ , we have a skew-symmetric matrix

$$\omega \frac{1 - \phi}{1 + \phi},$$

and the screw appertaining to this matrix is the screw defining the motion.

IV.

We have now to reduce the motion to a canonical form. In my former Note, I started with a skew-conjugate matrix and reduced this to its canonical form. In the present Note, I start with the (orthogonal) matrix ϕ . Let $\lambda_1 \dots \lambda_n$ be its latent roots, supposed unequal, and let $e_1 \dots e_n$ be the latent points appertaining to them, so that

$$\begin{aligned} \phi e_i &= \lambda_i e_i, \\ \phi' e_i &= \phi^{-1} e_i \\ &= \lambda_i^{-1} e_i. \end{aligned}$$

We see that, unless $\lambda_i = \pm 1$,

$$T^2 e_i = 0.$$

For we must have

$$\begin{aligned} \lambda_i T^2 e_i &= S e_i \phi e_i \\ &= S e_i \phi' e_i \\ &= \lambda_i^{-1} T^2 e_i. \end{aligned}$$

Moreover,

$$\begin{aligned} S e_i \phi e_k &= \lambda_k S e_i e_k \\ &= S e_k \phi' e_i \\ &= \lambda_i^{-1} S e_i e_k, \end{aligned}$$

and therefore, unless

$$\begin{aligned} \lambda_i \lambda_k &= 1, \\ S e_i e_k &= 0. \end{aligned}$$

It follows that all points appertaining to latent roots not equal to unity are on the absolute, and that any two points appertaining to latent roots whose product is not unity are conjugate with respect to the absolute, so that if they are both on it the line joining them is a generator. Now it is easy to see, by considering the form of ω , that unless the absolute is a degenerate quadric it is impossible to take n asyzygetic points all on the absolute, such that the lines joining any two of them are generators,* and that if one of a set of n asyzygetic points is on the absolute there must be at least one other point of the set which is not on the same generator.

It follows from this that, in general, if $n = 2\mu$, the latent points group themselves in pairs answering to latent roots that are reciprocals, and that if $n = 2\mu + 1$ we have a latent root unity,† and the points appertaining to the remaining latent roots group themselves in pairs.

If $n = 2\mu$, I write, for the canonical form of ϕ ,

$$\begin{aligned}\phi e_i &= \lambda_i e_i, \\ \phi e_{\mu+i} &= \lambda_i^{-1} e_{\mu+i} \quad (i = 1 \dots \mu).\end{aligned}$$

If $n = 2\mu + 1$, I write

$$\begin{aligned}\phi e_\infty &= e_\infty, \\ \phi e_i &= \lambda_i e_i, \\ \phi e_{\mu+i} &= \lambda_i^{-1} e_{\mu+i} \quad (i = 1 \dots \mu),\end{aligned}$$

and we have $Se_i e_k = 0$, unless $k = \mu + i$, and $T^2 e_i = 0$ unless $i = \infty$. We have of course $e_i = e_k$ if $i \equiv k \pmod{2\mu}$.

V.

I shall now transform the equations of the absolute, taking $e_\infty \dots e_{2\mu}$ as points of reference.‡ I shall consider the case in which n is odd, since we can always pass from $n = 2\mu + 1$ to $n = 2\mu$, by omitting the terms involving e_∞ .

We have at once, if x is a point,

$$\begin{aligned}T^2 x &= T^2 (x_\infty e_\infty + x_1 e_1 + \dots + x_{2\mu} e_{2\mu}) \\ &= x_\infty^2 T^2 e_\infty + 2 \sum x_i x_k S e_i e_k \\ &= x_\infty^2 T^2 e_\infty + 2 \sum x_i x_{\mu+i} S e_i e_{\mu+i},\end{aligned}$$

* For $T^2 (x_1 e_1 + \dots + x_n e_n) = \sum x_i T^2 e_i + 2 \sum x_i x_k S e_i e_k \equiv 0$ if $T^2 e_i, S e_i e_k$ vanish for all values of i, k , that is, all points of space would be on the absolute.

† If a latent root is -1 , we can take $-\omega$ instead of ω . If $+1$ and -1 are latent roots, the motion is a particular case, and the screw belonging to it is indeterminate.

‡ These transformations are effected in an unnecessarily clumsy way in Section III. of my third Note.

since all the other terms disappear in virtue of the relations connecting the e 's. I write (∞) for $T^2 e_\omega$, and (i) for $Se_i e_{\mu+i}$, and we therefore have

$$T^2 x = x_\omega^2 (\infty) + 2 \sum (i) x_i x_{\mu+i}.$$

To get the remaining equations, I use the theorem

$$S[a_1 a_2 \dots a_k][\beta_1 \beta_2 \dots \beta_k] = \text{Det. } |Sa_i \beta_k| . *$$

We get
$$S[e_i e_k][e_l e_m] = \begin{vmatrix} Se_i e_l & Se_i e_m \\ Se_k e_l & Se_k e_m \end{vmatrix} = 0,$$

unless
$$\left. \begin{aligned} l &= \mu + i \\ m &= \mu + k \end{aligned} \right\},$$

or
$$\left. \begin{aligned} l &= i = \infty \\ m &= \mu + k \end{aligned} \right\},$$

and is $= (i)(k)$, or $(\infty)(k)$, in these two cases.

$$T^2[e_i e_k] = \begin{vmatrix} T^2 e_i & Se_i e_k \\ Se_i e_k & T^2 e_k \end{vmatrix} = 0,$$

unless
$$k = \mu + i,$$

and
$$= -(i)^2,$$

if this condition is satisfied.

I write $[i, k, l \dots]$ for the coefficient of $[e_i e_k e_l \dots]$ in any quantity. We see that, if x is a line,

$$T^2 x = - \sum_i (i)^2 [i, \mu + i]^2 + 2 \sum_{i, k=1 \dots \mu} (i)(k) [i, k] [\mu + i, \mu + k] + 2(\infty) \sum_{i, k=1 \dots \mu} (i) [\infty, i] [\infty, \mu + i].$$

In general, we get

$$S[e_i e_k e_l \dots e_m][e_{\mu+i} e_{\mu+k} e_{\mu+l} \dots e_{\mu+m}] = (i)(k)(l) \dots (m),$$

$$S[e_\infty e_k e_l \dots e_m][e_\infty e_{\mu+k} e_{\mu+l} \dots e_{\mu+m}] = (\infty)(k)(l) \dots (m),$$

$$T^2[e_i e_{\mu+i} \dots e_m e_{\mu+m}] = (-)^{i(r-1)} (i)^2 (k)^2 \dots (m)^2,$$

$$T^2[e_\infty e_i e_{\mu+i} \dots e_m e_{\mu+m}] = (-)^{i(r-1)(r-2)} (\infty)^2 (i)^2 \dots (m)^2,$$

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* Grassmann, *Ausdehnungslehre* (1862), § 175. Grassmann only proves this for the case $\omega = 1$, but the extension is obvious.

where r is the order of the form, and these are the only terms that do not vanish. We have therefore, for forms of order $2r$,

$$\begin{aligned} T^2 x &= (-)^r \Sigma (i)^2 (k)^2 \dots [i, \mu+i, k, \mu+k \dots]^2 \\ &\quad + 2 \Sigma (i) (k) \dots [i, k \dots] [\mu+i, \mu+k \dots] \\ &\quad + 2 (\infty) \Sigma (i) (k) \dots [\infty, i, k \dots] [\infty, \mu+i, \mu+k \dots], \end{aligned}$$

and for forms of order $2r+1$

$$\begin{aligned} T^2 x &= (-)^r \Sigma (\infty)^2 (i)^2 (k)^2 \dots [\infty, i, \mu+i, k, \mu+k \dots]^2 \\ &\quad + 2 \Sigma (i) (k) \dots [i, k \dots] [\mu+i, \mu+k \dots] \\ &\quad + 2 (\infty) \Sigma (i) (k) \dots [\infty, i, k \dots] [\infty, \mu+i, \mu+k \dots], \end{aligned}$$

where the indices take the values 1, 2, ... μ .

VI.

I now return to the motion. Let x be the old position of any point, x' its new position; then we have

$$\begin{aligned} x &= \Sigma (x_i e_i + x_{\mu+i} e_{\mu+i}) + x_{\infty} e_{\infty}, \\ x' &= \Sigma (\lambda_i x_i e_i + \lambda_i^{-1} x_{\mu+i} e_{\mu+i}) + x_{\infty} e_{\infty}, \end{aligned}$$

and therefore

$$\begin{aligned} Sxx' &= \Sigma (i) x_i x_{\mu+i} (\lambda_i + \lambda_i^{-1}) + (\infty) x_{\infty}^2, \\ Tx' &= Tx, \end{aligned}$$

and therefore

$$\begin{aligned} \cos (xx') &= \Sigma \frac{(i) x_i x_{\mu+i} (\lambda_i + \lambda_i^{-1})}{T^2 x} + \frac{(\infty) x_{\infty}^2}{T^2 x} \\ &= \Sigma \frac{2 (i) x_i x_{\mu+i}}{T^2 x} \cos \theta_i + \frac{(\infty) x_{\infty}^2}{T^2 x}, \end{aligned}$$

if $\cos \theta_i = \frac{\lambda_i + \lambda_i^{-1}}{2}.$

Now
$$T^2 [x_i e_{\mu+i}] = \begin{vmatrix} T^2 x, & Sx e_i, & Sx e_{\mu+i} \\ Sx e_i, & 0, & (i) \\ Sx e_{\mu+i}, & (i), & 0 \end{vmatrix}$$

$$= (i)^2 \{ 2 (i) x_i x_{\mu+i} - T^2 x \},$$

and

$$T^2 [e_i e_{\mu+i}] = -(i)^2.$$

Therefore
$$\sin^2(xe_i, e_{\mu+i}) = \frac{T^2 [xe_i, e_{\mu+i}]}{T^2_x T^2 [e_i, e_{\mu+i}]} \\ = \frac{T^2 x - 2(i)x_i x_{\mu+i}}{T^2_x},$$

and
$$\cos^2(xe_i, e_{\mu+i}) = \frac{2(i)x_i x_{\mu+i}}{T^2_x}.$$

Moreover,
$$Sx_{e_\infty} = (\infty)x_\infty,$$

and therefore
$$\cos^2(xe_\infty) = \frac{x_\infty^2}{T^2_x}.$$

Remarking that, if a point is on $e_i, e_{\mu+i}$, it obviously remains on this line and is moved through a distance θ_i , and that e_i remains fixed, we get the following theorem:—

Every motion of a rigid body in a space of $2\mu-1$ dimensions consists of translations along μ axes at right angles to each other; if θ_i is the distance through which a point on one of the axes is moved, we have $\theta_i = \cos^{-1} \frac{(\lambda_i + \lambda_i^{-1})}{2}$, λ_i being a latent root of the matrix defining the motion. The general expression for the distance through which any point is moved is $\cos^{-1} \sum \cos^2 D_i \cos \theta_i$, D_i being the distance of the point from the i^{th} axis. Every motion of a rigid body in a space of 2μ dimensions consists of a rotation about a fixed point, and translations along 2μ lines. A point moves through a distance given by the equation $\cos \theta = \sum \cos^2 D_i \cos \theta_i + \cos^2 D$, where D is its distance from the fixed point, D_i its distance from an axis points on which move through the distance $\theta_i = \cos^{-1} \frac{(\lambda_i + \lambda_i^{-1})}{2}$, λ_i being a latent root (not unity) of the matrix defining the motion.

VII.

I shall now show how the preceding investigations lead to a reduction of the screw defining the motion to a canonical form.

I write ν for the latent roots of the skew-conjugate matrix appertaining to the motion. Then, since the latent points of this skew-conjugate matrix ψ are the same as those of the orthogonal matrix ϕ , for

$$\phi = \frac{1-\psi}{1+\psi},$$

and since, Δ being the matrix of the screw α defining the motion, we

have $\psi = \omega^{-1}A$, we see that the points e_i satisfy the equations

$$\omega^{-1}Ae_i = \nu_i e_i,$$

$$\omega^{-1}Ae_{\mu+i} = -\nu_i e_{\mu+i},$$

$$\omega^{-1}Ae_\infty = 0,$$

or

$$Ae_i = \nu_i \omega e_i,$$

$$Ae_{\mu+i} = -\nu_i \omega e_{\mu+i},$$

$$Ae_\infty = 0;$$

and therefore, by what was proved in (2),

$$e_i a = \nu_i K \omega e_i = (i) \nu_i K e_{\mu+i},$$

$$e_{\mu+i} a = -\nu_i K \omega e_{\mu+i} = -\nu_i (i) K e_i,$$

$$e_\infty a = 0,$$

since it follows, from the form of ω referred to $e_\infty, e_i, \dots, e_{2\mu}$, that $\omega e_i = (i) e_{\mu+i}$. We see, then, that we must have

$$a = -\sum \nu_i (i) K [e_i e_{\mu+i}],^*$$

whether $n = 2\mu$ or $2\mu + 1$, and we can, if we choose, omit the negative sign. This is the canonical form of the screw a . It is obvious that we might have started by reducing $\omega^{-1}A$ to a canonical form, thus determining the axes $e_i, e_{\mu+i}$ of the screw; and that we should have found that the motion consisted of translations along the axes of the screw, accompanied in the case of a space of an even number of dimensions by a rotation about a fixed point; and we see (since the fixed point belongs to the screw as much as the axes) that for space of any number of dimensions a motion consists of a twist about a screw.

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* Since we can divide the points e_i by any scalars, we see that we can make all the (i) equal to unity. If we do this, the product of all the units, which is $\sqrt{[(i)(k) \dots]}$, will be unity.