5. Dead gray metal from Terre Noire, near Lyons	7·014
6. Grayish iron from Tollend, S. Staffordshire	7·144
7. Mottled iron from Staffordshire	7·329
8. White metal forced out from cast-iron pigs at Alais	7·339
9. Whitish gray cast iron from Alais	7·404
10. White cast iron from the Marley Iron-works, South Wales..	7·407
11. White cast iron from Alais, (fine granulated)	7·442
12. Mottled iron from Staffordshire	7·451
13. White crystalline iron.....	7·472
14. Bar iron heated in a reheating furnace	7·487
15. Wootz from Bombay	7·508
16. No. 12. as plate metal	7·513
17. Cement steel made from English iron (highest concentration)	7·519
18. Cement steel from Swedish iron (double bullet) middling hard	7·563
19. Cast iron, silvery white, extremely large-grained, from Alais	7·582
20. Cement steel from Swedish iron (<i>Koop L.</i>), highest conversion	7·621
21. White-charcoal iron from Fourchambault (France)	7·709
22. Cold-short bar iron from Staffordshire	7·753
23. Cement steel made from bar iron, prepared from Staffordshire iron according to my method of puddling (highest conver- sion).....	7·755
24. Swedish iron called <i>Koop L.</i> in soft gray leaves	7·761
25. Iron melted by me with 3 per cent. of sugar coal (it hardened not at all)	7·764
26. Swedish iron, called double bullet, fracture fine-granulated..	7·810
27. Bar iron, according to my puddling process for steel (fine-gra- nulated)	7·841
28. The same, the granulations interrupted by fibres.....	7·845
29. The same, the fibres more developed	7·840
30. No. 23. in cast-steel ingots	7·892
31. The foregoing tilted into small bars for razors.....	7·900

Of the lowest specific gravity, it will be seen, was gray brilliant cast iron from Vienne, department d'Iserre in France, equal to 6·898; and of the greatest specific gravity appears tilted razor-steel, prepared, according to my method, from English iron at Mr. Solly's works at Dudley, equal to 7·900.

[To be continued.]

LXIII. *Memoir on Rational Derivation from Equations of Co-existence, that is to say, a new and extended Theory of Elimination.* By J. J. SYLVESTER, F.R.S., Professor of Natural Philosophy in University College, London*. (Part I.)

ANY number of equations existing at the same time and having the same quantities repeated, may be termed equations of coexistence: in the *present* paper we consider only the case of *two* algebraical equations:

$$x^m + a_1 x^{m-1} + a_2 x^{m-2} + \dots + a_m = 0$$

$$x^n + b_1 x^{n-1} + b_2 x^{n-2} + \dots + b_n = 0$$

* Communicated by the Author.

The above being "equations of coexistence," x is called "the repeating term."

If we suppose the equation

$$c_0 x^r + c_1 x^{r-1} + c_2 x^{r-2} + \dots + c_r = 0$$

to be capable of being deduced from the two above, and, therefore, necessarily implied by them, this will be called "a Particular Derivative" from the equations of coexistence of the r^{th} degree, (r being supposed less than m and n *, and the coefficients being *rational* functions of the coefficients of the equations of coexistence.)

There will be an *indefinite* number in general of such derivatives, and the form involving arbitrary quantities which includes them *all* is called "the general derivative of the r^{th} degree."

Any "Particular Derivative," in which the terms are all integral, numerically as well as literally speaking, is called an "Integral derivative."

That "Integral Derivative" of any given degree in which the literal parts of the coefficients are of the *lowest possible dimensions*†, and the numerical parts as low as they can be made, is called the "Prime Derivative" of that degree. So that there is nothing left ambiguous in the prime derivative save the sign.

The "Derivative by succession" is that particular derivative which is obtained by performing upon the equations of coexistence, the process commonly employed for the discovery of the greatest common measure and equating the *successive remainders* to zero.

To express the product of the sums formed by adding each of one row of quantities to each of another row, we simply write the one row above the other; a notation clearly capable of extension to any number of rows, which would not be the case if we spoke of *differences* instead of *sums*‡.

Theorem (1.)

Let $h_1, h_2, \dots h_m$, be the roots of one equation of coexistence,

* This restriction upon the value of r is not essentially requisite, and is only introduced to keep the attention fixed upon the particular objects of this 1st Part.

† Of course the dimensions of the coefficients in the equations of coexistence are to be understood as denoted by the indices subscribed.

‡ The wider views which I have attained since writing the above, and which will be developed in a future paper, lead me to request that this notation may be considered only as temporary. It would have been more in accordance with these views to have used the two rows to denote products of differences than of sums. But a change now in the text would be very apt to cause errors in printing.

$k_1, k_2, \dots k_n$, the roots of the other. The general derivative of the r^{th} degree is represented by

$$\Sigma \left(\text{SR } (h_1 h_2 h_3 \dots h_r) \{ \overline{x-h_1} \cdot \overline{x-h_2} \dots \overline{x-h_r} \} \right. \\ \left. \times \left\{ \begin{matrix} h_{r+1} h_{r+2} \dots h_m \\ -k_1 -k_2 \dots -k_n \end{matrix} \right\} \right) = 0$$

$\text{SR } (h_1 \cdot h_2 \cdot h_3 \dots h_r)$ denoting any *symmetrical rational* (integral or fractional) function of $h_1 h_2 \dots h_r$;

$$\left\{ \begin{matrix} h_{r+1} \cdot h_{r+2} \dots h_m \\ -k_1 \cdot -k_2 \dots -k_n \end{matrix} \right\} \text{ being to be inter-}$$

preted as above explained, and Σ of course including as many terms as there are ways of putting n things r and r together*.

A form tantamount to the above, and which may be substituted for it is its analogue,

$$\Sigma \left(\text{SR } (k_1 \cdot k_2 \dots k_r) \cdot \{ \overline{x-k_1} \cdot \overline{x-k_2} \dots \overline{x-k_r} \} \right. \\ \left. \times \left\{ \begin{matrix} k_{r+1} \cdot k_{r+2} \dots k_n \\ -h_1 \cdot -h_2 \dots -h_m \end{matrix} \right\} \right) = 0$$

When $r = 0$ the theorem gives simply

$$\left\{ \begin{matrix} h_1 h_2 \dots h_m \\ -k_1 -k_2 \dots -k_n \end{matrix} \right\} = 0 \text{ and is coincident}$$

with that given by Bezout in his *Theory of Elimination*.

Subsidiary theorem (A).

If $h_1 h_2 \dots h_m$ be the roots of the equation

$$x^m + a_1 \cdot x^{m-1} + a_2 x^{m-2} + \dots + a_m = 0$$

and if $c^m + a_1 c^{m-1} + a_2 c^{m-2} + \dots + a_m - u = 0$ then

$$\Sigma \frac{h_1^r}{(h_1-h_2) \cdot (h_1-h_3) \dots (h_1-h_m)} = \frac{1}{r+1} \cdot \frac{d}{d u} \cdot \Sigma (c^{r+1}), u$$

being made zero after differentiation.

Cor. If $R h_1$ denote any integral rational function of h_1 ,

then $\Sigma \frac{R(h_1)}{(h_1-h_2)(h_1-h_3) \dots (h_1-h_m)}$ is *always* integral and is zero when the dimensions of $R(h)$ fall short of $(m-1)$.

* The general derivative may clearly be expressed also by the sum of any two particular derivatives affected respectively with arbitrary rational coefficients. The equivalency of an arbitrary function to two arbitrary multipliers is very remarkable, and analogous to what occurs in the solution of certain differential equations.

Subsidiary theorem (B).

$$\Sigma \frac{\text{SR } (h_1 h_2 \dots h_r)}{\left\{ \frac{h_1 \cdot h_2 \dots h_r}{-h_{r+1} \cdot -h_{r+2} \dots -h_m} \right\}}$$

can be expressed by the sum of terms, each of which is the product of series of the form

$$\Sigma \frac{\text{R } (h_1)}{(h_1 - h_2) \cdot (h_1 - h_3) \dots (h_1 - h_m)}$$

it is *always* integral, and when the dimensions of the numerator fall short of $(m - r)$ it vanishes*.

Subsidiary theorem (C).

The *only* modes of satisfying the equation

$$\Sigma \left(f(h_1 \cdot h_2 \dots h_r) \times \text{SR } (h_1 \cdot h_2 \dots h_r) \right) = 0$$

for all forms of the latter factors short of $\overline{m-r} \cdot \overline{n-r}$ dimensions, is to put $f(h_1 h_2 \dots h_r) = 0$, or else

$$f(h_1 h_2 \dots h_r) = \frac{\text{constant}}{\left(\frac{h_1 h_2 \dots h_r}{-h_{r+1} \cdot -h_{r+2} \dots -h_m} \right)}$$

Theorem (2.)

By virtue of the subsidiary theorem (B), the two equations

$$\begin{aligned} & \pm \Sigma \left((x - h_1) (x - h_2) \dots (x - h_r) \right. \\ & \quad \times \left. \frac{\left\{ \frac{h_{r+1} \cdot h_{r+2} \dots h_m}{-k_1 \cdot -k_2 \dots -k_n} \right\}}{\left\{ \frac{h_{r+1} \cdot h_{r+2} \dots h_m}{-h_1 \cdot -h_2 \dots -h_n} \right\}} \right) = 0 \\ & \pm \Sigma \left((x - k_1) (x - k_2) \dots (x - k_r) \right. \\ & \quad \times \left. \frac{\left\{ \frac{k_{r+1} \cdot k_{r+2} \dots k_n}{-h_1 \cdot -h_2 \dots -h_m} \right\}}{\left\{ \frac{k_{r+1} \cdot k_{r+2} \dots k_n}{-k_1 \cdot -k_2 \dots -k_r} \right\}} \right) = 0 \end{aligned}$$

are each integer derivatives of the r^{th} degree.

* It may be remarked also in passing, that any term in the numerator which contains *any* one power not greater than $m - 2r$ may be neglected and thrown out of calculation. Moreover, an analogous proposition may be stated of fractions in the denominators of which *any number* of rows are written one under the other: see the last note to page 429.

Theorem (3.)

And by virtue of the subsidiary theorem (C), the above two equations are the "Prime Integer Derivatives," and are exactly identical with each other.

Cor. 1. The leading coefficient of the "prime derivative" of the r^{th} degree is always of $(m-r)(n-r)$ dimensions.

Cor. 2. If P_r be called the prime derivative of the r^{th} degree and if $(X=0 \ Y=0)$ be the two equations of coexistence, and $\lambda_r \ \mu_r$ the two "prime constituents of multiplication" to the said derivative, *i. e.* if λ_r and μ_r satisfy the equation $\lambda_r \cdot X + \mu_r \cdot Y = P_r$, then the coefficient of the leading terms in λ_r and in μ_r is of $(m-r-1)(n-r-1)$ dimensions.

Theorem (4.)

The "Prime Derivative" of any given degree is an exact factor of the "derivative by succession," of the same degree. The quotient resulting from striking out this factor is called "the quotient of succession."

Theorem (5.)

If $L_1 \ L_2 \ L_3$, &c. be the leading coefficients of the derivatives occurring first, second, third, &c., in order after the equations of coexistence, and if $Q_1 \ Q_2 \ Q_3$, &c. represent the first, second, third, "quotients of succession" reckoned in the same order, then

$$\begin{aligned} Q_1 &= 1 \\ Q_2 &= \frac{1}{L_1^2} \\ Q_3 &= \frac{L_1^4}{L_2^2} \\ Q_4 &= \frac{L_2^4}{L_1^4 \cdot L_3^2} \end{aligned}$$

and in general

$$\begin{aligned} Q_{2n} &= \frac{L_2^4 \cdot L_4^4 \dots L_{2n-3}^4 \cdot L_{2n-2}^4}{L_1^4 \cdot L_3^4 \dots L_{2n-1}^4 \cdot L_{2n-1}^4} * \\ Q_{2n+1} &= \frac{L_1^4 \cdot L_3^4 \dots L_{2n-3}^4 \cdot L_{2n-1}^4}{L_2^4 \cdot L_4^4 \dots L_{2n-2}^4 \cdot L_{2n}^4} \dagger. \end{aligned}$$

* That the appearance of the index 4 may not startle, let my reader bear in mind that there are what may be termed secondary derivatives of succession for every degree appearing in the process of successive division.

† The prime derivatives must be capable of yielding an *internal* evidence of the truth of Sturm's theorem. In fact, for the case of all the roots be-

Cor. Hence, in place of Sturm's auxiliary functions, we may substitute the functions derived from the equations of coexistence $\left(fx = 0, \frac{dfx}{dx} = 0 \right)$ according to theorem (2.) due regard being had to the sign.

Scholium. Hitherto it has been supposed that the values of the coefficients in the equations of coexistence are independent of one another, but particular relations may be supposed to exist which shall cause the leading terms given by theorem (2.) to vanish, giving rise to anormal or singular primes, as they may be called, of the degree r of fewer than $(m - r)$. $(n - r)$ dimensions. The theory of this, the failing case, (so to say) is highly interesting, and I have already discovered the law of formation for the quotients of succession on the supposition of *any number* of primes vanishing consecutively; but I forbear to vex the patience of my reader further, the more so, as I hope soon to be able to present a complete memoir, with all the steps here indicated filled up, and numerous important additions, (the perfect image of which this is but a rough mould,) as homage to the learned and illustrious society which has lately done me the honour of admitting me into its ranks.

Why this has not already been done must be excused, by the fact of the theory having suggested itself abroad in the intervals of sickness*. Yet thus much will I add in general terms, viz. that as many primes as vanish consecutively, so many units must be added to the index 2 of the accessions received in the numerator and denominator of the subsequent quotient; and in the quotient after that, it is not the square of the leading term of the penultimate prime,—but the product of this

ing possible, a little consideration will serve to show that the leading term of each prime derivative of the $= n \left\{ fx \frac{dfx}{dx} \right\} = 0$ will consist of a series of fractions, each of which fractions is, *numerically* speaking, of the *same sign*.

* The reflections which Sturm's memorable theorem had originally excited, were revived by happening to be present at a sitting of the French Institute, where a letter was read from the Minister of Public Instruction, requesting an opinion upon the expediency of forming tables of elimination between two equations as high as the 5th or 6th degree containing one repeating term. The offer was rejected, on the ground of the excessive labour that would be required. I think that this has been very much overrated; and probably many will be of the same opinion who have dwelt upon the fact that no numerical quantity will occur in the result higher than the highest index of the repeating term. Would it not redound to the honour of British science that some painstaking ingenious person should gird himself to the task? and would not this be a proper object to meet with encouragement from the Scientific Association of Great Britain?

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term by the leading term of that anormal prime of the same degree which has the lowest dimensions,—that finds its way into the numerator. The rest of the formation remaining undisturbed, *unless and until* a new failure have taken place.

Note on Sturm's Theorem.

When one of the equations of coexistence is the differential coefficient with respect to the repeated term of the other, the prime derivatives given in theorem (2.) which coincide in this case with Sturm's auxiliary functions *reduced* to their lowest terms, may be exhibited under an integral aspect.

Let SPD intimate that the squared product of the differences is to be taken of the quantities which follow it.

Let S_1 indicate the sum of the quantities to which it is prefixed.

S_2 the sum of the binary products.

S_3 the sum of the ternary products, and so on.

Let $h_1 h_2 \dots h_n$ be the roots of any equation.

Then Sturm's last auxiliary function may be *replaced* by SPD ($h_1 h_2 \dots h_n$).

The last but one may be replaced by

$\Sigma \text{SPD} (h_1 h_2 \dots h_{n-1}) x + \Sigma S_1 (h_2 h_3 \dots h_{n-1}) \text{SPD} (h_1 h_2 \dots h_{n-1})$.

The one preceding by

$\Sigma \text{SPD} (h_1 h_2 \dots h_{n-2}) x^2 + \Sigma S_1 (h_1 h_2 \dots h_{n-2}) \text{SPD} (h_1 h_2 \dots h_{n-2}) x + \Sigma S_2 (h_1 h_2 \dots h_{n-2}) \text{SPD} (h_1 h_2 \dots h_{n-2})$ and so on.

Thus then Sturm's rule for determining the absolute number of real roots in an equation is based wholly and solely upon the following

ALGEBRAICAL PROPOSITION.

If there be n quantities, real and imaginary, the imaginary ones entering in pairs, as many changes of sign as there are in the terms

$$\left. \begin{array}{l} \Sigma \text{SPD} (h_1 h_2) \\ \Sigma \text{SPD} (h_1 h_2 \dots h_3) \\ \dots \dots \dots \\ \Sigma \text{SPD} (h_1 h_2 \dots h_{n-1}) \\ \text{SPD} (h_1 h_2 \dots h_n) \end{array} \right\} \begin{array}{l} \text{so many in number are these} \\ \text{pairs.} \end{array}$$

Query (1.) Is there no proposition applicable to any (n) quantities *whatever*?

Query (2.) Is there no faintly analogous proposition applicable to higher powers than the squares?

Query (3.) Seeing that in forming the coefficients in the equation to the squares of the differences, we pass from (n) functions of the roots, to $n \frac{n-1}{2}$ and not n functions, of their

squared differences, does not a natural passage to the former lie through n functions of the squared differences?

In other words, may not the quantities Σ SPD ($h_1 h_2 \dots h_n$), &c., serve as natural and valuable intermediaries between the coefficients of an equation involving simple quantities and the coefficients of the equation involving the squares of their differences?

P.S. In the next part I trust to be able to present the readers of this Magazine with a *direct* and *symmetrical* method of eliminating any number of unknown quantities between any number of equations of any degree, by a newly invented process of symbolical multiplication, and the use of *compound* symbols of notation.

I must not omit to state that the constituents of multiplication λ_r and μ_r explained in Cor. 2. to Theorem (3.) are equal to the expressions

$$\Sigma (x - k_1) (x - k_2) \dots (x - k_{m-r-1}) \frac{\begin{pmatrix} k_1 \cdot k_2 \dots k_{m-r-1} \\ -h_1 -h_2 \dots -h_n \end{pmatrix}}{\begin{pmatrix} k_1 \cdot k_2 \dots k_{m-r-1} \\ -k_{m-r} \dots k_m \end{pmatrix}}$$

and its analogue respectively.

University College, London, October 15, 1839.

[To be continued.]

LXIV. *Observations on Professor Plateau's Defence of his Theory of accidental Colours.* By Sir DAVID BREWSTER, K.H., V.P.R.S. Ed.*

In the Numbers of this Journal for May and June last, Professor Plateau has published an elaborate defence of his "Theory of accidental Colours," against certain observations which have been made upon it in England. The authors to whose observations he replies are an anonymous writer in the Edinburgh Review † and myself; and as I have no hesitation in acknowledging that I am the anonymous reviewer, I feel myself bound to bear the whole burden of examining the able defence which Professor Plateau has now made. Having the honour of being personally acquainted with this distinguished individual, and admiring his character as well as his talents, I regret that my opinion should have been placed in collision with his; but from the temperate and truly philosophical manner in which he has conducted the discussion, and which in so far as the first of these qualities is concerned I shall not fail to imitate, I trust the interests of science and of

* Communicated by the Author.

† April 1834, p. 160.