(*Paper No. 3049.*)

"The Pressure of Grain."

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THE important problem of grain-pressure affords an interesting example of the application of the theory of semi-fluids. The Author knows of no reliable theoretical investigation relating to grain-pressure, although much needed, by reason of the large extent to which grain is stored, and the difficulty of making experiments except on a very small scale. Useful investigations on the pressure of wheat on the bottom of small bins have been published by Mr. Isaac Roberts [1] and by Mr. H. A. Janssen; [2] but the Author has not been able to find any Tables of the coefficients of friction of grain, either on grain, or on wood, iron, or brickwork, the materials of which bins are usually constructed. These coefficients enter most materially into any theory of the pressure of grain; and, with the kind assistance of Mr. F. E. Duckham, M. Inst. C.E., the engineer of the Millwall Docks, the Author has made the experiments, of which the results are given in a Table in Appendix II to determine them.

The grain is treated in this investigation as a semi-fluid, on the principle explained by Weisbach,[3] which assumes that the pressure on the side of a bin is the maximum pressure due to a wedge-shaped mass of the grain which may be supposed to separate from the general mass; and the angle of slope of the particular wedge-shaped mass which exerts the maximum pressure has to be determined. There is friction between the grain and the sides of the bin, because, as the grain is piled into the bin, it sinks together from the pressure, and so causes the friction. There is also friction between the grains of corn along the supposed plane of separation of the wedge-shaped mass. These frictions vary with

[1] Report of the British Association for the Advancement of Science, 1882, p. 678, Tract 8vo., vol. 354; and Proceedings of the Royal Society, vol. xxxvi. p. 225, Tract 8vo., vol. 473.

[2] Minutes of Proceedings Inst. C.E., vol. cxxiv. p. 553.

[3] "Mechanics of Machinery and Engineering," vol. ii.

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the nature of the grain and the material of the bin. Frequently the depth of a bin is much greater than its width, and, when it is filled with grain, the plane of separation of the mass of grain, which causes the maximum pressure on one side of the bin, meets the opposite side within the mass of grain. When there is only a small depth of grain in the bin, the plane of separation passes out of the mass before it meets the opposite side. These two cases, which have a common limit at a particular depth of grain, have been investigated separately in Appendix I.

The simple relation arrived at in Appendix I for rectangular bins of the same shape but different sizes, and for which $\frac{h}{b}$ is constant (where h is the height of the grain, and b the breadth of the bin), viz., that the pressures on the sides and bottom vary as b^3, much simplifies the computation of the pressures in any given case. For it is only necessary to prepare a Table (similar to that on p. 349) for any required shape of bin (defined by the proportion $\frac{l}{b}$), and for some standard value of b, say $b = 10$ feet, in order to determine the pressures in any bin of similar shape, almost by inspection. Suppose, for example, that the pressure on the sides of a bin 6 feet square, filled with wheat to a depth of 30 feet is required. In this case, $\frac{h}{b} = \frac{30}{6} = 5$; and referring to the Table on p. 349, which is computed for a bin 10 feet square, the pressure at 50 feet depth of grain (for which $\frac{h}{b} = \frac{50}{10} = 5$) is 535,440 lbs. Therefore the pressure on the sides of the 6-foot bin, filled to a depth of 30 feet $= \frac{6^3}{10^3} \times 535{,}440 = 115{,}655$ lbs.; and the pressure on each side is $\frac{115{,}655}{4} = 28{,}914$ lbs.

A good idea of the magnitude of the pressures of grain is formed by computing the pressure for a 10-foot square bin of smooth planks, with depths of wheat varying from 5 feet to 100 feet, the results of which are given in the following Table which was computed with the following values of the coefficients of friction of grain on grain (μ), and of grain against the sides of the bin (μ'): viz., $\mu = 0 \cdot 466$, and $\mu' = 0 \cdot 361$ (Table in Appendix II). The value of h, common to cases I and II, is $h = b \tan \theta = b \times \left\{\mu + \sqrt{\mu \frac{1 + \mu^2}{\mu + \mu'}}\right\}$ (Appendix I) $= 10 \times 1 \cdot 294 = 12 \cdot 94$ feet. Therefore for $h = 5$ feet,

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and $h = 10$ feet, the pressures have been calculated from the formulas of case I, and for $h = 15$ feet, and all higher values of h, from the formulas of case II. With the above values of the coefficients, the formulas used for the computation of the Table were—

$$\textit{Case I}\ \begin{cases}\tan\theta = 1\cdot294.\\ P = h^2 \times 8\cdot404.\end{cases}$$

$$\textit{Case II}\ \begin{cases}\tan\theta = \sqrt{h \times 0\cdot294 + 1\cdot481} - 1\cdot006\\ P = 250\,(2\,h - 10\tan\theta)\cdot\dfrac{\tan\theta - 0\cdot466}{0\cdot832 + 0\cdot827\tan\theta},\end{cases}$$

where θ is the angle which the plane of separation of the grain makes with the horizontal, and P is the pressure against the side of the bin, Appendix I.

TABLE OF THE PRESSURES ON THE SIDES AND BOTTOM OF A SMOOTH WOODEN BIN, 10 FEET SQUARE, HOLDING WHEAT WEIGHING 50 LBS. PER CUBIC FOOT, FOR WHICH THE COEFFICIENTS OF FRICTION ARE $\mu = 0\cdot466$ AND $\mu' = 0\cdot361$, APPENDIX II.

Depth of Grain in Bin h.	Values of $\tan\theta$ for Maximum Pressure on Sides of Bin.	Weight of Grain in Bin.	Pressures on Sides per Foot Run of Horizontal Circumference P.	Total Pressure of Grain on Sides of Bin P × 40.	Weight of Grain held up by the Friction on the Sides.	Weight of Grain Carried on the Bottom of Bin.
Feet.		Lbs.	Lbs.	Lbs.	Lbs.	Lbs.
5	1·294	25,000	210	8,404	3,034	21,966
10	1·294	50,000	840	33,616	12,136	37,864
15	1·422	75,000	1,878	75,120	27,118	47,882
20	1·708	100,000	3,169	126,760	45,760	54,240
25	1·967	125,000	4,625	185,000	66,785	58,215
30	2·205	150,000	6,214	248,560	89,730	60,270
35	2·427	175,000	7,900	316,000	114,076	60,924
40	2·635	200,000	9,657	386,280	139,447	60,553
45	2·832	225,000	11,488	459,520	165,887	59,113
50	3·019	250,000	13,386	535,440	193,294	56,706
55	3·198	275,000	15,331	613,240	221,380	53,620
60	3·369	300,000	17,305	692,200	249,884	50,116
65	3·535	325,000	19,332	773,280	279,154	45,846
70	3·694	350,000	21,385	855,400	308,799	41,201
75	3·848	375,000	23,503	940,120	339,383	35,617
80	3·997	400,000	25,617	1,024,680	369,909	30,091
85	4·142	425,000	27,773	1,110,920	401,042	23,958
90	4·283	450,000	29,937	1,197,480	432,290	17,710
95	4·420	475,000	32,119	1,284,760	463,798	11,202
100	4·555	500,000	34,326	1,373,040	495,667	4,333

The above Table shows that the pressure on the bottom attains a maximum value of 60,924 lbs., or 27·2 tons for a depth of

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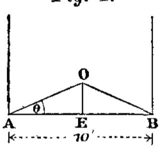
grain of 35 feet. This would hold good for all other cases of grain-pressure, but the depth of grain that would produce the maximum pressure would probably vary with the coefficients of the grain and the dimensions of the bin. The limiting pressure on the bottom of the bin is 3,883 lbs., being the weight of a pyramid of grain whose sides are at the greatest natural angle of slope of the grain; for this pyramid of grain cannot be supported by the friction on the sides, however great the pressure on the sides might become. The pressure on the sides continually increases with the depth of grain, and the possible amount of friction due to this pressure increases with it. But more of this friction cannot come into action than will support the weight of grain in the bin less the limiting weight of 3,883 lbs. on the bottom. It happens that the limiting weight of 3,883 lbs. is reached at a depth of grain of a little over 100 feet, as may be inferred from the last figures of the Table; so that for depths of grain greater than 100 feet, the pressure on the bottom will remain constant at 3,883 lbs. The quantity 3,883 lbs. is obtained thus:—

Fig. 1.



Let AOB, *Fig. 1*, be the pyramid of grain, of which the sides are sloped at the angle θ, for which $\tan\theta = \mu = 0\cdot466$. Then volume of pyramid $= \frac{1}{3} \times AB^2 \times OE = \frac{1}{3} \times AB^2 \times AE \tan\theta = \frac{1}{3} \times 10^2 \times 5 \times 0\cdot466 = 77\cdot66$ cubic feet; weight of pyramid $= 50 \times 77\cdot66 = 3{,}883$ lbs. The side pressures are those which are mostly to be guarded against, for they increase continually with the depth of grain; and although they are small compared with the pressures that would be produced by a perfect fluid, yet they are considerable. Thus at a depth of 80 feet, the side pressure per foot run of horizontal circumference is 25,617 lbs., or 11·4 tons. The corresponding side pressure due to a perfect fluid of the same weight as the grain, namely, 50 lbs. per cubic foot, would be 160,000 lbs. or 71·4 tons. This difference exhibits clearly the effect of the internal friction of grain in reducing the external pressure.

The pressure per square foot on the side of the bin at any required depth below the surface of the grain, can be easily obtained. Suppose, for example, there is a 60-foot depth of grain in the bin, and the pressure on the side of the bin for the foot lying between 60-foot depth and 59-foot depth is required. The total pressure per foot run of the horizontal circumference can be computed for each of the values 60 and 59, and their difference will be the pressure on the foot required. In the Table the

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pressures are computed for depths proceeding by 5 feet at a time; and the difference of pressure for the 5 feet between 60 feet and 55 feet is 1,974 lbs., so that approximately the pressure per square foot at this depth would be $\frac{1,974}{5} = 395$ lbs.

To ascertain the effect on the side and bottom pressures of altering the value of μ' (the coefficient of friction between the grain and the material of the bin), the pressures having been obtained as in the Table, for values $\mu = 0{\cdot}466$ and $\mu' = 0{\cdot}361$ (the values of the coefficients for wheat in a smooth wooden bin, Appendix II), find the effect on the pressures of using values $\mu = 0{\cdot}466$ and $\mu' = 0{\cdot}412$ (the values of the coefficients for wheat in a rough wooden bin) is required. Let h, the depth of grain, be 40 feet in both cases. Then from the Table and by direct computation the following results are obtained:—

Value of μ.	Value of μ'.	Weight of Grain in Bin.	Pressures on Sides per Foot Run of Horizontal Circumference.	Total Pressures on Sides.	Weight of Grain held up by the Friction on the Sides.	Weight of Grain Carried on the Bottom of the Bin.
		Lbs.	Lbs.	Lbs.	Lbs.	Lbs.
0·466	0·361	200,000	9,657	386,280	139,447	60,553
0·466	0·412	200,000	9,322	372,880	153,627	46,373

Showing that with the same grain, the smaller μ' is, or the smoother the bin, the greater are the pressures both on the sides and on the bottom.

For the same depth of grain, the side pressure per foot run of horizontal circumference P is greater with a large bin than with a small one. Therefore it is advantageous to store grain in bins of moderate size, rather than in large bins or in a single mass. Thus the Table shows that, for a depth of grain of 40 feet stored in a 10-foot bin, the pressure on the sides per foot run of horizontal circumference is 9,657 lbs.; and the pressure on one side is $9,657 \times 10 = 96,570$ lbs. But with the same depth of grain stored in a 20-foot bin, the pressure per foot run of horizontal circumference is 12,675 lbs., and the pressure on one side $12,675 \times 20 = 253,500$ lbs.

The method of conducting the experiments for ascertaining the coefficients of friction of different kinds of grain, together with the results obtained, are given in Appendix II.

Mr. Isaac Roberts has given the results of some experiments

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on the pressure of wheat on the bottom of small wooden bins, both square and hexagonal.[1] These experimental bins were very small, the largest hexagonal bin having a diameter for the inscribed circle of $20\frac{3}{4}$ inches. The corn was piled into this bin to the depth of 8 feet, or rather more than four and a half times the diameter. He also made a number of experiments on the central pressure on the bottom of large bins, as also on small portions of their sides; but these do not appear to be useful for comparison with the Author's results. Mr. H. A. Janssen has given the results of a number of experiments on the pressure of wheat on the bottom of small wooden bins.[2] These bins were all square, the largest being 2 feet square. The greatest depth of corn piled into this bin was 5·95 feet, or only about three times the length of a side of the bin.

In the Author's method, the aggregate pressure on the sides of a bin (from which the pressure on the bottom of the bin is deduced) has been found by multiplying the pressure on a vertical strip 1 foot wide by the horizontal circumference of the bin in feet. But this process takes no account of the jamming of the grain in the angles of a rectangular bin, which would hold up the grain to a certain extent and diminish the pressure on the bottom of the bin. With large bins, such as are used in practice, the effect of the angles on the bottom pressure would be only a very small part of the whole; but, with small experimental bins of 1 foot or 2 feet square, it would probably be a large proportion. Therefore it might be expected that the pressure on the bottom of small bins, obtained by experiment, would be less than the pressure deduced from the Author's method, which ignores the effect of the angles, and is applicable to large bins without serious error.

These remarks may be verified by a comparison of the results obtained by Roberts and Janssen respectively. Mr. Janssen, having piled 1,190 lbs. of wheat into a bin 2 feet square, obtained a pressure on the bottom of 408 lbs. The Author has found by trial that his method would produce this same result with values of the coefficients $\mu = 0{\cdot}466$ and $\mu' = 0{\cdot}412$. Moreover, from the reasoning in the preceding paragraph, he would expect that, if the angles did not exist (as in a circular bin), or were very much more obtuse (as in a hexagonal bin), the pressure on the bottom would have been greater. The Author's method would produce a greater pressure on the bottom with a smaller value of μ',

[1] Proceedings R. S., 31 January, 1884.

[2] Zeitschrift des Vereines deutscher Ingenieure, 1895, p. 1045.

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keeping $\mu = 0 \cdot 466$ as before, as indicated by the Table on p. 351 Mr. Roberts found that, when 1,014 lbs. of wheat were piled into his hexagonal bin, the pressure on the bottom was 224 lbs.; and the Author has found by trial that his method would produce this result with values of the coefficients $\mu = 0 \cdot 466$ and $\mu' = 0 \cdot 371$. Accordingly the value of μ', as deduced from Roberts's experiments, is considerably less than the value deduced from Janssen's experiments; and the Author has no doubt that this is due to the jamming of the grain in the angles of the rectangular bin used by Janssen. Probably the small experimental bins, both of Roberts and Janssen, were made of planed boards, for which the value of μ' would be $0 \cdot 361$, as found by the Author, Appendix II; and this compares well with the value $0 \cdot 371$ deduced from Roberts's experiments, especially as the angles of Roberts's hexagonal bin would produce some small effect, which would make the agreement still closer if taken into account. The Author considers that his formulas and coefficients are confirmed in a most satisfactory manner by these experiments.

Judging from the reckless manner in which large quantities of grain are frequently stored in brickwork buildings that are totally unfit to support much lateral pressure, and from the frequent reports that have reached him of the bulging and destruction of such buildings, the Author thinks that there exists a very general ignorance of the magnitude of the lateral pressure of grain. In a recent instance in Ipswich, a brickwork store about 60 feet by 40 feet, was filled to a depth of about 50 feet with wheat; the end wall was burst by the pressure, and the upper part of it was thrown down to within about 10 feet from the ground-level. Some 6,000 quarters of wheat ran out upon the quay, and a large portion was lost in the dock. The Author trusts that his investigations will render more certain the calculations necessary for the design of such buildings.

[APPENDIXES.

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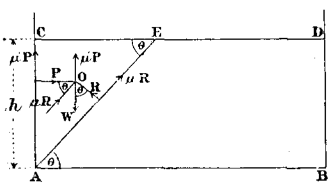
APPENDIXES.

APPENDIX I.

Case I.—To find the pressure on the sides and bottom of a bin, when the depth of grain in the bin is such that the plane of separation which causes maximum pressure on one side of the bin passes out of the grain before it meets the opposite side of the bin. Let C A B D be a vertical section of a bin, *Fig. 2*, and C D the surface of the grain. Let A E be the plane of separation. Then A C E is the wedge-shaped mass of grain which causes maximum pressure against the side A C, and O its centre of gravity. Let W be the weight of A C E, 1 foot thick, P the pressure against the side A C, μ' P the friction between the grain and the side A C, acting in the direction A C, μ' being the coefficient of friction, R the pressure of the mass A C E on the plane of separation A E, μ R the friction between the grain along the plane of separation A E, μ being the coefficient of friction, h the depth of grain A C in feet, θ the angle E A B which the plane of separation makes with the horizontal, and γ the weight of 1 cubic foot of the grain in lbs.

Fig. 2.



For convenience, the forces are all drawn as if acting at O in their proper directions; and resolving the forces that support the mass A C E parallel and perpendicular to A E,

$$\mu R + P \cos\theta = (W - \mu' P) \sin\theta$$

$$R - P \sin\theta = (W - \mu' P) \cos\theta$$

whence

$$\mu = \frac{(W - \mu' P) \sin\theta - P\cos\theta}{(W - \mu' P)\cos\theta + P\sin\theta};$$

and from this, by reduction,

$$P = W \cdot \frac{\tan\theta - \mu}{1 - \mu\mu' + (\mu + \mu')\tan\theta}.$$

But

$$W = \gamma \times \frac{1}{2} AC \times CE = \gamma \cdot \frac{1}{2} \cdot h \cdot \frac{h}{\tan\theta} = \gamma \cdot \frac{h^2}{2\tan\theta},$$

$$P = \gamma \cdot \frac{h^2}{2\tan\theta} \cdot \frac{\tan\theta - \mu}{1 - \mu\mu' + (\mu + \mu')\tan\theta}$$

$$= \gamma \cdot \frac{h^2}{2} \frac{\tan\theta - \mu}{(1 - \mu\mu')\tan\theta + (\mu + \mu')\tan^2\theta}.$$

To find the value of θ, that causes P to be a maximum, equate $\frac{dP}{d\theta}$ to zero.

Now

$$\frac{dP}{d\theta} = \gamma \cdot \frac{h^2}{2}\left\{\frac{1}{\cos^2\theta}\left[(1-\mu\mu')\tan\theta + (\mu+\mu')\tan^2\theta\right] - (\tan\theta - \mu)\right.$$

$$\left.\times\left[\frac{1}{\cos^2\theta}(1-\mu\mu') + 2\tan\theta \cdot \frac{1}{\cos^2\theta} \cdot (\mu+\mu')\right]\right\}$$

all divided by

$$\{(1-\mu\mu')\tan\theta + (\mu+\mu')\tan^2\theta\}^2,$$

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therefore

$$(1 - \mu\mu') \tan\theta + (\mu + \mu') \tan^2\theta - (1 - \mu\mu') (\tan\theta - \mu)$$
$$- 2 \tan\theta (\tan\theta - \mu) (\mu + \mu') = 0;$$

whence, by reduction, $\tan\theta = \mu + \sqrt{\mu \cdot \frac{1 + \mu^2}{\mu + \mu'}}$, since the part under the radical is greater than μ. If this value of $\tan\theta$ is substituted in the expression for P, the lateral pressure of the wedge A C E, 1 foot thick, will be obtained; and this result, multiplied by the horizontal circumference of the bin in feet, gives the total pressure on the sides of the bin. This, multiplied by μ', gives the vertical sustaining force of the side friction; and the pressure on the bottom is the total weight of the grain less the vertical sustaining force of the friction. Thus the fundamental equations for case I are—

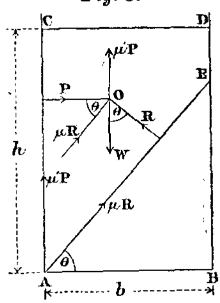
$$\tan\theta = \mu + \sqrt{\mu \cdot \frac{1 + \mu^2}{\mu + \mu'}},$$

and

$$P = \gamma \cdot \frac{h^2}{2 \tan\theta} \cdot \frac{\tan\theta - \mu}{1 - \mu\mu' + (\mu + \mu') \tan\theta}.$$

Case II.—To find the pressure on the sides and bottom of a bin, when the depth of grain is such that the plane of separation which causes maximum pressure on one side of the bin, meets the opposite side of the bin within the mass of grain. Let b be the breadth of the bin, *Fig. 3.* Then, as in case I,

Fig. 3.



$$P = W \cdot \frac{\tan\theta - \mu}{1 - \mu\mu' + (\mu + \mu') \tan\theta}.$$

But

$$W = \gamma \times 1 \times \text{area AEDCA} = \gamma \times CD \times \frac{AC + DE}{2}$$

$$= \gamma \cdot \frac{b}{2} \cdot (2h - b \tan\theta),$$

$$\therefore \qquad P = \gamma \cdot \frac{b}{2} (2h - b \tan\theta) \cdot \frac{\tan\theta - \mu}{1 - \mu\mu' + (\mu + \mu') \tan\theta}$$

$$\text{or } P = \gamma \cdot \frac{b}{2} \left\{ 2h \cdot \frac{\tan\theta - \mu}{1 - \mu\mu' + (\mu + \mu') \tan\theta} - b \cdot \frac{\tan^2\theta - \mu \tan\theta}{1 - \mu\mu' + (\mu + \mu') \tan\theta} \right\}$$

To find the value of θ which makes P a maximum, equate $\frac{dP}{d\theta}$ to zero.

Now

$$\frac{dP}{d\theta} = \gamma \cdot \frac{b}{2} \left\{ 2h \left[\frac{1}{\cos^2\theta} (1 - \mu\mu') + \frac{1}{\cos^2\theta} (\mu + \mu') \tan\theta \right.\right.$$
$$\left. - \frac{1}{\cos^2\theta} (\mu + \mu') \tan\theta + \frac{1}{\cos^2\theta} (\mu + \mu') \cdot \mu \right]$$
$$- b \left[\left(2 \tan\theta \cdot \frac{1}{\cos^2\theta} - \mu \cdot \frac{1}{\cos^2\theta} \right) \left\{ 1 - \mu\mu' + (\mu + \mu') \tan\theta \right\} \right.$$
$$\left.\left. - \frac{1}{\cos^2\theta} (\mu + \mu') (\tan^2\theta - \mu \tan\theta) \right] \right\}$$

all divided by $\qquad \{1 - \mu\mu' + (\mu + \mu') \tan\theta\}^2,$

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so that $2h(1+\mu^2) - b\{\tan^2\theta(\mu+\mu') + 2\tan\theta(1-\mu\mu') - \mu(1-\mu\mu')\} = 0,$

which by reduction gives finally

$$\tan\theta = \sqrt{\left\{\frac{2h}{b} \cdot \frac{1+\mu^2}{\mu+\mu'} + \frac{1+\mu^2}{\mu+\mu'} \cdot \frac{1-\mu\mu'}{\mu+\mu'}\right\}} - \frac{1-\mu\mu'}{\mu+\mu'}.$$

Substituting this value of $\tan\theta$ in the expression for P, the maximum pressure on the side of the bin of the wedge-shape mass A E D C A, 1 foot thick, is obtained; and the pressures on the sides and bottom of the bin can be deduced as before. Thus the fundamental equations for case II are :—

$$\tan\theta = \sqrt{\frac{2h}{b} \cdot \frac{1+\mu^2}{\mu+\mu'} + \frac{1+\mu^2}{\mu+\mu'} \cdot \frac{1-\mu\mu'}{\mu+\mu'}} - \frac{1-\mu\mu'}{\mu+\mu'},$$

and $$P = \gamma \cdot \frac{b}{2} \cdot (2h - b\tan\theta) \cdot \frac{\tan\theta - \mu}{1-\mu\mu' + (\mu+\mu')\tan\theta}.$$

It is seen from *Fig.* 2 that the value of h for which the plane of separation of the mass of maximum pressure meets the opposite side at the surface level of the grain, is determined by the condition $h = b\tan\theta$. If in the expression for $\tan\theta$ in case II, $b\tan\theta$ is substituted for h, then

$$\tan\theta = \sqrt{2\tan\theta \cdot \frac{1+\mu^2}{\mu+\mu'} + \frac{1+\mu^2}{\mu+\mu'} \cdot \frac{1-\mu\mu'}{\mu+\mu'}} - \frac{1-\mu\mu'}{\mu+\mu'},$$

or, $$\tan\theta = \mu + \sqrt{\mu \cdot \frac{1+\mu^2}{\mu+\mu'}},$$

the same expression as in case I. Similarly the value of P from case II becomes, by putting $h = b\tan\theta$ or $b = \frac{h}{\tan\theta}$,

$$P = \gamma \cdot \frac{h^2}{2\tan\theta} \cdot \frac{\tan\theta - \mu}{1-\mu\mu' + (\mu+\mu')\tan\theta},$$

the same expression again as was obtained in case I, showing that the equations for $\tan\theta$ and P in cases I and II are continuous.

A useful law connecting the pressures in bins of different sizes can be obtained immediately from the equations as follows:—In case I, taking the case of a square bin, the length of side of which is b, the aggregate pressure on the sides of the bin is $P \times 4b$. Calling this π, the second equation, case I, becomes

$$\pi = 2\gamma \cdot h^2 b \cdot \frac{\tan\theta - \mu}{(1-\mu\mu')\tan\theta + (\mu+\mu')\tan^2\theta},$$

which may be written

$$\pi = 2\gamma \cdot \frac{h^2}{b^2} \cdot b^3 \times \frac{\tan\theta - \mu}{(1-\mu\mu^1)\tan\theta + (\mu+\mu')\tan^2\theta},$$

therefore, in comparing the pressures on the sides of square bins of different sizes, if $\frac{h}{b}$ (the proportion of the depth of grain to the side of the bin) is the same for both, the pressure varies as b^3.

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In case II, with square bins as before, calling $\pi\,(= \mathrm{P} \times 4\,b)$ the aggregate pressure on the sides of the bin

$$\pi = 2\,\gamma \cdot b^3 \left(\frac{2\,h}{b} - \tan\theta\right) \cdot \frac{\tan\theta - \mu'}{1 - \mu\,\mu' + (\mu + \mu')\tan\theta}.$$

Therefore, in this case also, in comparing the aggregate pressures on the sides of square bins of different sizes, if $\frac{h}{b}$ is the same for both, the pressure varies as b^3. The total weight of grain in the bin is $\gamma\,h\,b^2$, and the pressure on the bottom of the bin is $\gamma\,h\,b^2 - \mu'\,\pi$. Then comparing square bins, for which $\frac{h}{b}$ is constant, say $\frac{h}{b} = \mathrm{C}$, the pressure on the bottom $= \gamma \cdot b^3\,\mathrm{C} - m\,b^3 = b^3\,(\gamma\,\mathrm{C} - m)$, m being a constant; therefore the pressure on the bottom also varies as b^3.

In the case of oblong rectangular bins, whose length is l and breadth b, comparing the pressures on the sides of bins of different sizes, but for both of which $\frac{h}{b}$ is the same, the values of $\tan\theta$ are the same, and the value of π varies as $b^2\,(b + l) = b^3\left(1 + \frac{l}{b}\right)$. Similarly the pressure on the bottom of the bin is $\gamma \cdot h \cdot b \cdot l - \mu'\,\pi$; and putting $\frac{h}{b} = \mathrm{C}$, this becomes

$$\gamma\,\mathrm{C}\,b^2 l - \mathrm{C}' \cdot b^3\left(1 + \frac{l}{b}\right) = b^3\left\{\gamma\,\mathrm{C}\,\frac{l}{b} - \mathrm{C}'\left(1 + \frac{l}{b}\right)\right\}$$

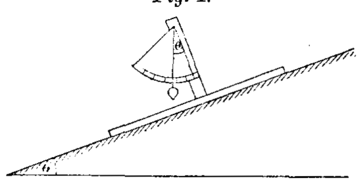
C' being a constant. Therefore, in oblong bins of the same shape, for which $\frac{l}{b}$ and $\frac{h}{b}$ are constant, the aggregate pressure on the sides and the pressure on the bottom of the bin both vary as b^3.

APPENDIX II.

Experiments for ascertaining the Coefficients of Friction of Grain.

The grain stored on floors at the Millwall Docks offered great convenience for experiment. A board, 3 feet long, 11 inches wide, and $1\frac{1}{2}$ inch thick, was fitted with a graduated sector-arc and a plumb-bob, so as to act as a clinometer. The instrument weighed from 15 to 20 lbs. according as it was shod, and was applied to determine the slopes of grain as in *Fig. 4*. The arc was adjusted so that when the board was placed on a horizontal surface, the plumb-line read zero. When the instrument was placed on the grain, the angle indicated by the position of the plumb-line was the slope of the grain surface to the horizontal. For determining μ, the grain was thrown up so as to stand as steep as it could, and was allowed to take its own slope. The surface was then gently smoothed with a straight-edge, and the angle of slope was read on the clinometer. The tangent of the angle was adopted as the value of μ.

Fig. 4.



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For determining μ', the coefficient of friction between the grain and the material of the bin, the surface of the board was prepared to represent the materials generally used in the construction of bins, as follows:—In the first set of experiments, the surface of the board was rough, just as it was bought in the deal; in the second set, the surface of the board was planed smooth; in the third set, the board was shod with a wrought-iron plate, fixed to it by screws whose heads were countersunk; in the fourth set of experiments, the iron plate was removed, and the board was coated with cement, which adhered by means of tacks in the board, around which it set. The slope of the grain surface was trimmed down with a long straight-edge, and the clinometer was tried upon it. This operation was repeated again and again, till the slope was found at which the board would just slip, and no more, on the surface of the grain. The angle of slope was then read off on the clinometer; and the tangent of the angle was adopted as the value of μ'. The results of the experiments are given in the following Table, and also the weight of a cubic foot of the different kinds of grain used:—

TABLE OF THE COEFFICIENTS OF FRICTION OF VARIOUS KINDS OF GRAIN.

—	Weight of a Cubic Foot loosely filled into Measure.	Coefficients of Friction.				
		Grain on Grain μ.	Grain on Rough Board μ'.	Grain on Smooth Board μ'.	Grain on Iron μ'.	Grain on Cement μ'
	Lbs.					
Wheat . .	49	0·466	0·412	0·361	0·414	0·444
Barley . .	39	0·507	0·424	0·325	0·376	0·452
Oats . . .	28	0·532	0·450	0·369	0·412	0·466
Maize . .	44	0·521	0·344	0·308	0·374	0·423
Beans . .	46	0·616	0·435	0·322	0·366	0·442
Peas . . .	50	0·472	0·287	0·268	0·263	0·296
Tares . . .	49	0·554	0·424	0·359	0·364	0·394
Linseed . .	41	0·456	0·407	0·308	0·339	0·414

If the measure is well shaken, and the grain is closely pressed into the measure, the weight of a cubic foot will in all cases be about 4 lbs. greater than the weights given in the Table. In all the calculations in this Paper relating to the pressure of wheat, the Author has adopted 50 lbs. for the weight of a cubic foot.

The cement surface was rather rough, and might be expected to give about the same coefficients as an ordinary brick wall. The coefficients for a smooth cement surface would be somewhat smaller than those given in the Table. During the experiments the weather was fairly dry; but in damp weather, nearly all the coefficients would be somewhat larger than those given in the Table.

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