Reflexion and Transmission by Condensers of Electric Waves along Wires

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XIX. *Reflexion and Transmission by Condensers of Electric Waves along Wires.* By EDWIN H. BARTON, D.Sc., F.R.S.E., A.M.I.E.E., Senior Lecturer in Physics at University College, Nottingham, and LOUIS LOWNDS, B.Sc., 1851 Exhibition Science Research Scholar*.

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INTRODUCTION.

1. IN 1892 J. Ritter von Geitler† experimentally detected the reflexion by a condenser of electric waves along wires. He also examined qualitatively the effects of condensers of various sizes; but no attempt was made to give a complete theory of the various phenomena involved, and by careful quantitative observations compare experiment with theory. This fuller examination one of us, who followed von Geitler under the late Professor Hertz, at Bonn, determined to attempt when opportunity offered. The present paper embodies the result of the work carried out during the session 1898-99 at University College, Nottingham, and advances the problem another stage towards complete solution.

* Read January 26, 1900.

† "Inaugural Dissertation," Bonn, January 1893. Wied. Ann. xlix. pp. 184-195 (1893).

2. The phenomena are more complicated than might be at first supposed. This is chiefly owing to the fact that the wave-train, reflected by the condenser under examination, passes and repasses between the point of reflexion and the oscillator at the beginning of the line. Hence the effect to be observed, owing to reflexion at any condenser, is a function not only of the coefficient of that reflexion but also of the other constants of the circuit. To avoid unnecessary complication, the wave-train transmitted by the condenser is always completely absorbed by a suitable bridge at the end of the line.

3. By placing the electrometer beyond the condenser, the wave-train transmitted is determined, and by placing the condenser *far* beyond the electrometer the wave-train reflected is measured. In order to determine the phase-lag introduced in the reflected wave-train, the condenser is placed in a number of positions, each of which is only a short distance beyond the electrometer. We thus obtain stationary waves by the interference of the incident and reflected wave-trains, and these, explored by the electrometer, exhibit the phase-change sought. To bring this change of phase into clearer relief, stationary waves are also obtained by a resistance-bridge, whose reflexion coefficient is the same as that of the condenser, but whose phase-change is π simply.

4. The results obtained with the three condensers used are all in fair accord with theory, both as to phase-lag and coefficient of reflexion and transmission. The experimental discrepancy, if such it can be called, is in the direction of less reflected and less transmitted than the long-wave theory predicts.

It may be mentioned here that the experiments, though given in this paper after the theory, were in practice finished before the theory was fully worked out. The experimental results are thus free from any bias due to anticipation based on theory. Each set of readings was accepted simply on the ground of consistency of the separate readings. Further, the experimental curves were in all cases plotted before superposition of the theoretical ones.

EXPERIMENTAL ARRANGEMENTS.

5. The arrangement of the apparatus adopted for observing the reflected wave-trains is diagrammatically represented in fig. 1. In this figure, A denotes the battery of two storage-

Fig. 1.—Circuit arranged for Reflected System and for Interference Phenomena.

cells, B and C are the primary and secondary coils respectively of the induction-coil, of which C has a resistance of 3000 ohms and an inductance of about 20 henries. G is the spark-gap, which was adjusted to 2 mm., the balls being of brass, highly polished before each occasion of experimenting. The wire PGP', measured along the semicircle, is about 2 metres long. PP' are the edge-views of the condenser-plates of zinc 40 cm. diameter placed opposite to and 30 cm. distant from the precisely similar disks SS'.

6. The line is represented by SEDTT'D'E'S', and consists of two parallel copper wires, 1.5 mm. diam., and kept 8 cm. apart by wood separators at intervals of about 1 metre. These wood separators are about 12 cm. long, 2.5 cm. wide, and 1.2 cm. thick, and were well saturated before use in melting paraffin-wax. Their resistance was thus raised to something over 60,000 megohms each, which thus renders the leakage of the line quite negligible.

7. EE' denotes the electrometer, which has a single plane needle initially uncharged, and suspended by a fine quartz fibre between two disks attached to the line at E and E'. The needle is therefore electrified by induction whenever a wave-train passes EE', and its ends are consequently attracted to the disks whatever the sign of their potential-difference. The first throw of the electrometer is thus proportional to the time-integral of the square of the potential-difference. The needle is provided with weak magnetic control and a plane mirror by Adam Hilger, its indications being read by

a distant telescope and scale. The induction-coil is put on end to avoid any magnetic effect on the electrometer-needle. The leads also of the coil are kept close together, and, before taking a set of readings, the electrometer is tested as to the adequacy of these precautions against possible disturbance. The disks and the ends of the needle are each about 2 cm. diam. and 1 cm. apart. It will be shown later that the reflexion of waves at the electrometer, regarded as a condenser on the line, is extremely small and quite negligible.

8. TT' denotes the terminal bridge which is used to absorb the waves on arrival at the end of the line. It consists of graphite markings on a disk of ground glass, 3 inches diameter, and before each set of readings is adjusted to the resistance which corresponds to complete absorption; namely, for the line in question, 560 ohms*.

DD' denotes one of the condensers in use to reflect the waves. For the transmitted system they are removed from DD' and inserted at FF'. They were all three made of disks of zinc 1 cm. apart, their radii being respectively 15, 9, and 5 cm. The two disks were kept at the right distance apart, and insulated by pieces of wood saturated in paraffin-wax, notched and secured with shellac.

9. The total length of the line from S to T is of the order 166 metres. The distance SE is 116 metres, SF is of the order 60 metres. ED varies from about 75 metre to 50 metres. Exact details will be given in connexion with the various experiments.

10. The waves generated by the primary oscillator are of the damped simple harmonic type 8.5 metres long, their logarithmic decrement per wave being 0.6. Thus the wave-train launched upon and propagated along the line with the speed of light is a damped train with its large end or head leading, the tail, after about 10 or 12 waves, being almost negligible. As the waves proceed along the line they suffer attenuation, also some loss on reflexion at the oscillator. The constants expressing these effects were previously determined experimentally on the line now in use, and their numerical values will be introduced into the theory later.

* "Absorption of Electric Waves, &c.," Phil. Mag., January 1897.

THEORY.

11. *Preliminary.*—The plan adopted throughout the experiments is to take (1) readings of the electrometer with no condenser on the line, but only the absorbing-bridge at the end; alternated with (2) electrometer-readings with a condenser on the line at an intermediate position, as well as the absorbing-bridge at the end. The mean ratio of the readings under the different conditions constitutes the experimental estimate of the effect produced by the condenser in question when in that position.

12. In order, however, to interpret the experimental results and compare them with theory, we need to trace the history of the wave-train reflected by the condenser until it is, practically, extinguished by its losses through repeated reflexion and attenuation. The effect at the electrometer when no condenser is on the line may be represented by a single time-integral which, without sensible error, owing to the damping of the wave-train, may be taken from the commencement of the time to infinity. The effect at the electrometer when a condenser is on the line requires, however, an infinite series of integrals.

13. To construct this series we must first determine the various elements of which it consists. We must accordingly treat the case of a condenser on the line, *not* at the end. This will be referred to as an *intermediate* condenser. We have here to determine the coefficients of reflexion and transmission and the phase-lags of both the reflected and transmitted portions. The case of a *terminal* condenser must also be treated, since the oscillator at the beginning of the line falls under this category. Here, only two constants are required, the amplitude and phase of the reflected portion. These points will now be dealt with in order, together with subsidiary matters that crop up by the way and are included for the sake of completeness.

14. We shall treat the circuit throughout as what Mr. Oliver Heaviside terms a "distortionless circuit," not that it is strictly distortionless, but because it is very nearly so; and this treatment, while sacrificing nothing of material accuracy,

considerably simplifies the analysis. Prof. W. B. Morton* has shown that for the circuit in question the errors thus involved are less than one in a thousand.

15. *Intermediate Condenser*.—At the place of attachment of the condenser let ϕ_1 denote the potential-difference of the two wires of the line due to the arrival there of the incident wave-train. Then ϕ_1 is a function of the time. Similarly let ϕ' and ϕ_2 correspond to the reflected and transmitted wave-trains respectively. Then by consideration of potential and Heaviside's theory of the distortionless circuit, assumed valid to the head of the wave-trains, we have

$$\left. \begin{aligned} \phi_1 + \phi' &= \phi_2 = Q/S', \\ i_1 &= \phi_1/Lv, \quad i' = -\phi'/Lv, \quad i_2 = \phi_2/Lv, \\ i_1 + i' - i_2 &= \frac{dQ}{dt}, \\ Q &= \phi_2 = 0 \text{ for } t = 0; \end{aligned} \right\} \quad \dots \quad (1)$$

where Q is the quantity of electricity at time t on the positively charged plate of the condenser of capacity S' (in electromagnetic units), and the i 's denote the currents in the positive wires due to the wave-trains denoted by the ϕ 's bearing the same subscript or dash; L denotes the inductance of the line per unit-length, and v is the speed of free æther radiation. These equations were used by Heaviside with reference primarily to waves of telephonic frequency, but are here adopted as a working approximation for the phenomena which obtain in the case of Hertzian waves.

On eliminating the i 's and Q , we have

$$\left(LvS' \frac{\partial}{\partial t} + 2 \right) \phi' = -LvS' \frac{\partial}{\partial t} \phi_1, \quad \dots \quad (2)$$

and

$$\left(LvS' \frac{\partial}{\partial t} + 2 \right) \phi_2 = 2\phi_1. \quad \dots \quad (3)$$

These expressions might have been written at once from Heaviside's operational results, the reflexion and transmission

* Phil. Mag., March 1899; Proc. Phys. Soc. vol. xvi. p. 281.

coefficients for an intermediate bridge being respectively

$$\frac{-Lv}{2Z + Lv} \quad \text{and} \quad \frac{2Z}{2Z + Lv}, \quad . \quad . \quad . \quad (4)$$

where the resistance operator Z is here equivalent to

$$\left(S' \frac{\partial}{\partial t}\right)^{-1}$$

since the bridge is a condenser merely.

16. Now if the incident wave-train is a damped sine function right up to its head, we may write without loss of generality,

$$\phi_1 = e^{-kpt} \cos(pt + \iota). \quad . \quad . \quad . \quad (5)$$

Substituting this value of ϕ_1 in equations (2) and (3), and satisfying the initial conditions expressed in (1), we obtain in the usual way the following solutions:—

$$\phi' = Ae^{-kpt} \cos(pt + \iota + \alpha) - B \cos(\iota + \beta) e^{-2pt/\chi'}, \quad . \quad (6)$$

$$\phi_2 = Be^{-kpt} \cos(pt + \iota + \beta) - B \cos(\iota + \beta) e^{-2pt/\chi'}; \quad . \quad (7)$$

where

$$\left. \begin{aligned} A \cos \alpha &= \frac{-\chi'^2 + 2k\chi' - k^2\chi'^2}{\Delta'} = a, \\ A \sin \alpha &= -2\chi'/\Delta' = a', \\ B \cos \beta &= \frac{4 - 2k\chi'}{\Delta'} = b = 1 + a, \\ B \sin \beta &= -2\chi'/\Delta' = b' = a', \\ \Delta' &= 4 + \chi'^2 - 4k\chi' + k^2\chi'^2, \\ \chi &= LvS'p. \end{aligned} \right\} \quad . \quad (8)$$

and

We have then finally

$$\left. \begin{aligned} A &= + \sqrt{a^2 + a'^2}, \quad \tan \alpha = a'/a, \\ B &= + \sqrt{b^2 + b'^2} \text{ and } \tan \beta = b'/b. \end{aligned} \right\} \quad . \quad (9)$$

The algebraic signs of a , a' , b , and b' being preserved show in which quadrants α and β should be taken.

17. We see from (6) and (7) that the reflected and transmitted wave-trains consist each of a damped periodic part and a purely exponential part. The exponential part we shall

have to omit from the first portion of the subsequent analysis, because we have no experimental proof of the exact shape of the head of the incident wave-train. For the same reason we shall at first omit the ι also. The greatest correction which this approximate theory can in any possible case require will be subsequently discussed and evaluated. (See arts. 60 and 61.)

18. Under these simplifications we may compactly represent the three wave-trains as follows :—

$$\left. \begin{array}{l} \text{Incident wave-train, real part of } e^{(i-k)pt}, \\ \text{Reflected wave-train, real part of } (a+a'i)e^{(i-k)pt}, \\ \text{Transmitted wave-train, real part of } (b+b'i)e^{(i-k)pt}; \end{array} \right\} (10)$$

where a , a' , b and b' have the values assigned to them in (8) and have been confirmed by Heaviside's operational methods.

19. It may be noted in passing that if the incident wave-train were not damped, the reflected and transmitted trains would differ in phase by $\frac{\pi}{2}$ exactly. For on putting $k=0$ in the expressions for $\tan \alpha$ and $\tan \beta$ they become $\frac{-2}{-\chi}$ and $\frac{-\chi'}{2}$ respectively, hence their product is -1 . This simple relation is seen to be slightly modified by the damping.

Again, if the damping were not present, we should have the simple result $A^2 + B^2 = 1$, as should be the case, showing that the sum of the energies of the reflected and transmitted wave-trains equals the energy of the incident one. To effect the corresponding check for the damped waves, one would need to take the time-integrals from zero to infinity of the square of the amplitude for the incident, reflected, and transmitted wave-trains, the exponential terms being no longer ignored.

As a further check it may be noted that the equation for potential-difference at the condenser is always satisfied. For the equation in question, viz. : $\phi_1 + \phi' = \phi_2$, reduces to $1 + a = b$ and $b' = a'$, and by (8) this is an identity for all values of k and χ' .

20. If the capacity of the condenser be imagined to increase from zero to infinity, while the circuit and the incident wave-train are maintained the same, we see by equations (6) and (8) that the reflexion-coefficient A grows from zero to unity, while the phase-difference α , but for the slight modification due to damping, changes from $\frac{-\pi}{2}$ to $-\pi$. Similarly, a reference to equations (7) and (8) shows that as the capacity of the condenser changes from zero to infinity the transmission-coefficient B changes from unity to zero, while the phase-difference, but for the damping, changes from zero to $\frac{-\pi}{2}$.

21. These features for several sizes of condenser are exhibited in Table I., and are graphically represented in figs. 2, 3, and 4, in which the dotted lines represent the values which the coefficients would assume if there were no damping, and the full lines the corresponding values for the damping actually present in the experimental case in question. The incident, reflected, and transmitted amplitudes are shown by OI , OR , and OT respectively. The curves in fig. 5 show at one glance the continuous changes in A , α , B , and β as the

Fig. 2.—Reflexion and Transmission at Small Condenser.

Fig. 3.—Reflexion and Transmission at Medium Condenser.

Fig. 4.—Reflexion and Transmission at Large Condenser.

Fig. 5.—Amplitudes and Phase Changes of Reflected and Transmitted Waves with Damping and without.

capacity changes without break from zero towards infinity. Here, again, the full lines show the values of the constants for the damping actually in use, and the broken lines the ideal values when damping is absent.

TABLE I.—Reflexion and Transmission Constants of Intermediate Condensers.

Condenser Constant $\chi' = L\omega S'p$.	Constants of Reflected Wave-Train.				Constants of Transmitted Wave-Train.				Damping Constant of Incident Wave-Train.
	a .	a' .	A.	α .	b .	b' .	B.	β .	
0	0	0	0	$-\frac{\pi}{2}$	+1	0	+1	0	Zero, <i>i. e.</i> Incident Wave-Train Undamped.
1.21	-.268	-.443	+.518	$-121^\circ 10'$	+.732	-.443	+.856	$-31^\circ 10'$	
3.47	-.751	-.433	+.867	$-150^\circ 5'$	+.249	-.433	+.500	$-60^\circ 5'$	
8.95	-.952	-.214	+.976	$-167^\circ 25'$	+.048	-.214	+.22	$-77^\circ 25'$	
∞	-1	0	+1	$-\pi$	0	0	0	$-\frac{\pi}{2}$	
0	0	0	0	$-84^\circ 36'$	+1	0	+1	0	Incident Wave-Train $is e^{-kpt} \cos pt$ where $k = 0.0955$ as in Experimental Case.
1.21	-.249	-.483	+.543	$-117^\circ 16'$	+.751	-.483	+.89	$-32^\circ 46'$	
3.47	-.775	-.468	+.906	$-148^\circ 52'$	+.225	-.468	+.52	$-64^\circ 19'$	
8.95	-.972	-.220	+.994	$-167^\circ 16'$	+.028	-.220	+.22	$-82^\circ 43'$	
∞	-1	0	+1	$-\pi$	0	0	0	$-95^\circ 24'$	

22. *Reflexion at Electrometer negligible.*—It is now necessary to inquire whether the electrometer reflects any appreciable part of the wave-train incident upon it. As previously mentioned, this is not the case.

Between the disks attached to the line at EE' (fig. 1) the ends of the needle are interposed, each disk with its adjacent needle-end forming a condenser whose plates are 1 centim. radius and 1 centim. apart. The electrometer as a whole constitutes a set of two such condensers in series. Hence the capacity interposed between E and E' is half that due to one disk and needle-end.

Taking the correction for the edges of the plates in a parallel disk condenser given in Kohlrausch's 'Practical Physics,' we thus obtain S' , the capacity between E and E', equals $\frac{1}{2} \times 10^{-21}$ in electromagnetic c.g.s. units. And for the present experimental case, $L\omega = 56 \times 10^{10}$ and $p = 2 \times 10^8$ in

the same units. Hence $\chi' = LvS'/p = \cdot 037$, $a = -\cdot 00035$, $a' = -\cdot 019$, and $A^2 = a^2 + a'^2 = \cdot 00036$.

Thus the energy reflected by the electrometer is of the order 0·04 per cent. of that incident upon it, and is therefore quite negligible.

23. *Intermediate Resistance*.—It seems desirable for the purpose of contrast with the reflexion from a condenser to consider now that of an intermediate resistance. In this case the phase-change introduced does not vary with the amplitude coefficient. But, however the resistance is varied, the phase-change introduced in the reflected wave-train remains constant and equal to a lag of π .

24. Thus, using the same notation as before, we have the following equations as the conditions connecting potential-difference and current where a bridge of resistance R' is placed on the line, the line being understood to extend beyond it.

$$\begin{aligned}\phi_1 + \phi' &= \phi_2, \\ i_1 &= \phi_1/Lv, \\ i' &= -\phi'/Lv, \\ i_2 &= \phi_2/Lv, \\ i_1 + i' - i_2 &= \phi_2/R'.$$

Whence, on elimination of the i 's, we obtain

$$\frac{\phi'}{\phi_1} = \frac{-Lv}{2R' + Lv} \quad \text{and} \quad \frac{\phi_2}{\phi_1} = \frac{2R'}{2R' + Lv}.$$

25. Or, we may say that if the incident wave-train be a cosine function of the time, damped or not, the reflexion-coefficient and phase-change are respectively

$$A' = \frac{+Lv}{2R' + Lv}, \quad \text{and} \quad \alpha' = -\pi. \quad . \quad . \quad (11)$$

Similarly, the amplitude-coefficient and phase-change for the transmitted wave-train are seen to be respectively

$$B' = \frac{+2R'}{2R' + Lv}, \quad \text{and} \quad \beta' = 0. \quad . \quad . \quad (12)$$

26. *Terminal Condenser*.—Let the capacity of the condenser at the end of the line be S_0 , the previous notation for

incident and reflected waves &c. being retained. We then have

$$\left. \begin{aligned} \phi_1 + \phi' &= Q/S_0, \\ i_1 &= \phi_1/Lv, \quad i' = -\phi'/Lv, \\ i_1 + i' &= \frac{\partial Q}{\partial t}, \\ Q &= \phi_1 + \phi' = 0, \quad \text{for } t=0. \end{aligned} \right\} \quad \dots (13)$$

On elimination of Q and the i 's these equations yield

$$\left(1 + LvS_0 \frac{\partial}{\partial t}\right) \phi' = \left(1 - LvS_0 \frac{\partial}{\partial t}\right) \phi_1; \quad \dots (14)$$

which is equivalent to Heaviside's expression for reflexion at a terminal condenser.

27. Writing for the incident wave-train

$$\phi_1 = e^{-kpt} \cos(pt + \iota),$$

the solution of (14) may be written

$$\left. \begin{aligned} \phi' &= Ce^{-kpt} \cos(pt + \iota + \gamma) - (\cos \iota + C \cos \iota + \gamma) e^{-pt/\chi_0}; \\ \text{where} \quad C \cos \gamma &= \frac{1 - \chi_0^2 - k^2 \chi_0^2}{\Delta_0} = c, \\ C \sin \gamma &= \frac{-2\chi_0}{\Delta_0} = c', \\ C &= +\sqrt{c^2 + c'^2}, \\ \tan \gamma &= \frac{c'}{c} = \frac{-2\chi_0}{1 - \chi_0^2 - k^2 \chi_0^2}, \\ \chi_0 &= LvS_0 p, \quad \text{and} \quad \Delta_0 = 1 + \chi_0^2 - 2k\chi_0 + k^2 \chi_0^2. \end{aligned} \right\} \quad (15)$$

Or, if we ignore, as before, the exponential term and let the incident wave-train be the real part of $e^{(i-k)pt}$, then the solution is

$$\phi' = (c + c'i)\phi_1, \quad \dots \dots \dots (16)$$

the c 's having the values given in equations (15).

In the case of a terminal condenser we see that if $k=0$, $C=1$ for all values of χ_0 . This simple relation is slightly disturbed by the damping when present.

28. This theory of the terminal condenser has been given for the sake of dealing with the reflexion which occurs at the

oscillator at the beginning of the line. It may be approximately treated as a condenser simply, because the interval elapsing between the launching of the wave-train along the line and its return after traversing about 300 metres is so small (say one-millionth of a second), that we may regard the spark-gap of the primary as hot enough to act as a short circuit to the secondary of the induction-coil, whose inductance is of the order 20 henries.

29. *Electrometer-throw for Single Passage of Wave-train.*—It might at first be supposed that the phase at the head of a wave-train would have no effect on the electrometer-throw, except when that wave-train was interfering with another, and so producing stationary waves in the neighbourhood of the electrometer. This, however, is not the case. On the contrary, the throw produced by the passage of a single wave-train of given amplitude, period, and damping depends to some extent on the initial phase. Thus we have

$$\int_0^{\infty} [e^{-kpt} \cos (pt + \iota)]^2 dt = \frac{1 - k \sin 2\iota + k^2(1 + \cos 2\iota)}{4kp(1 + k^2)}. \quad (17)$$

Hence, although we have hitherto been compelled, in one or two cases, to sacrifice rigour on minor points on account of insufficient data, the initial phase of a wave-train will hereafter always be taken account of when dealing with reflexions for which theory furnishes the value of the phase-change to be expected.

It may be seen from (17) that where $k=0.1$ the value of the integral varies with ι between limits having the ratio 11 to 9 approximately.

30. *Transmitted System.*—Let the condenser under examination be at a point on the line distant l cm. from its beginning at the oscillator, and let the electrometer be l_2 cm. beyond the condenser, *i. e.* $l+l_2$ from the beginning of the line. Take the origin of coordinates at the condenser, and let the potential-difference of the wires due to the incident wave-train be represented by ϕ_1 , where

$$\phi_1 = e^{-kp(t-x/v) - \sigma x} \cos p(t-x/v). \quad (18)$$

The σ here provides for the attenuation of the waves in

their passage along the line. Its value has been previously determined experimentally.

The train incident at the condenser is accordingly

$$\phi_1 = e^{-kpt} \cos pt. \quad . \quad . \quad . \quad . \quad (19)$$

31. Now, owing to the irregularity of the sparks at the primary oscillator, it is necessary in the experimental examination of any condenser to alternate electrometer-readings with and without the condenser on the line; the bridge which absorbs all waves incident upon it being throughout in place at the end of the line.

We have therefore to determine theoretically the ratio y , of the mean reading d' with the condenser on the line, to the mean reading d with no condenser on. Now the electrometer-throw, as previously explained, is proportional to the time-integral of the square of the amplitude of the wave-train passing it. Hence, denoting by E the electrometer-constant, we have

$$\begin{aligned} Ed &= e^{-2\sigma l_2} \int_{l_2/v}^{\infty} e^{-2kp(t-l_2/v)} \cos^2 p(t-l_2/v) dt \\ &= e^{-2\sigma l_2} \int_0^{\infty} e^{-2kpt} \cos^2 pt \, dt, \end{aligned}$$

or

$$4kp(1+k^2)e^{2\sigma l_2}Ed = 1+2k^2. \quad . \quad . \quad . \quad (20)$$

32. Consider now the throw d' obtained with the condenser on the line. Retaining the previous notation for the transmission and reflexion operators of the condensers we have for the wave-trains successively transmitted at the condenser the following series:—

$$\left. \begin{array}{ll} \text{First wave-train:} & B e^{-kpt} \cos(pt + \beta), \\ \text{Second do.} & CAB e^{-kpt-2\sigma} \cos(pt + \gamma + \alpha + \beta), \\ \text{Third do.} & C^2 A^2 B e^{-kpt-4\sigma} \cos(pt + 2\gamma + 2\alpha + \beta). \\ & \dots\dots\dots \end{array} \right\} (21)$$

The formation of this series is easily traced by noting that each time a wave-train is incident upon the condenser, not only is a part transmitted but also a part is reflected, suffers attenuation along the line in its passage to the oscillator, is

there reflected, and again suffers attenuation along the line before reaching the condenser a second time. These effects are all provided for in the above series, in which also the simplification is adopted of taking $t=0$ for each new arrival of a wave-train at the condenser.

33. We thus have, on taking the time-integrals of the squares of the successive trains, the following series:—

$$\begin{aligned} \text{Ed}'e^{2\sigma l_2} &= \text{B}^2 \int_0^\infty e^{-2kpt} \cos^2(pt + \beta) dt \\ &+ r^2 \text{B}^2 \int_0^\infty e^{-2kpt} \cos^2(pt + \beta + \kappa) dt \\ &+ r^4 \text{B}^2 \int_0^\infty e^{-2kpt} \cos^2(pt + \beta + 2\kappa) dt \\ &+ \dots + \dots + \dots, \dots \quad (22) \end{aligned}$$

where $r = \text{CA}e^{-2\sigma}$ and $\kappa = \alpha + \gamma$.

Now all the above integrals fall under the type

$$\int_0^\infty e^{-2kpt} \cos^2(pt + \delta) dt = \frac{1 - k \sin 2\delta + k^2(1 + \cos 2\delta)}{4kp(1 + k^2)}. \quad (23)$$

Further, the series, to which (22) when integrated gives rise, are represented by

$$\begin{aligned} \sin \phi + r^2 \sin \overline{\phi + \psi} + r^4 \sin \overline{\phi + 2\psi} + \dots \\ = \frac{\sin \phi - r^2 \sin \overline{\phi - \psi}}{1 - 2r^2 \cos \psi + r^4}. \quad (24) \end{aligned}$$

Hence (22) is easily transformed to

$$\begin{aligned} 4kp(1 + k^2) \text{Ed}'e^{2\sigma l_2} \\ = \text{B}^2 \left\{ \frac{1}{1 - r^2} - \frac{k}{D} (\sin 2\beta - r^2 \sin \overline{2\beta - 2\kappa}) \right. \\ \left. + \frac{k^2}{1 - r^2} + \frac{k^2}{D} (\cos 2\beta - r^2 \cos \overline{2\beta - 2\kappa}) \right\}; \quad (25) \end{aligned}$$

where $D = 1 - 2r^2 \cos 2\kappa + r^4$.

34. Now in the experimental case under consideration k is less than 0.1; we may therefore, for the order of accuracy aimed at, simplify by neglect of k^2 . Then dividing (25) by

(20) gives

$$y = \frac{d'}{d} = B^2 \left\{ \frac{1}{1-r^2} - \frac{k}{D} (\sin 2\beta - r^2 \sin \overline{2\beta - 2\kappa}) \right\}; \quad (26)$$

which is the working formula for comparison with the experimental results.

35. Regarding this expression physically, we may note that its essence is the factor B^2 , the square of the amplitude transmitted. The denominator $1-r^2$ of the first term in the brackets is due to the repeated coursings of the waves to and fro between the condenser and oscillator. Finally, the second term, which vanishes with k , or with β and κ , is a small correction whose existence is due to the fact that with a damped wave-train the electrometer-throw varies slightly with the phase at the head, as shown in Art. 29.

36. *Stationary Waves at Electrometer.*—Now suppose the electrometer to be at a point on the line distant l_1 cm. from its beginning at the oscillator, and let the condenser be at $l_1 + l_2 = l$ cm. from the beginning, *i. e.* l_2 cm. beyond the electrometer. This disposition is exhibited in fig. 1. Let the absorbing bridge be at the end of the line as before. Take the origin of coordinates at the electrometer, and let the incident wave-train be represented by

$$\phi_1 = e^{-kp(t-x/v) - \sigma x} \cos p(t-x/v), \quad (27)$$

which accordingly becomes at the electrometer

$$\phi_1 = e^{-kpt} \cos pt. \quad (28)$$

The notation previously used for the various wave-trains and for the coefficients of reflexion and transmission will be retained, and the following abbreviations used:—

$$\begin{aligned} l_2/v &= t_2, & 2pl_2/v &= 2pt_2 = \theta, \\ Ae^{-2l_2\sigma} &= s, & \alpha + \gamma &= \kappa, \\ CAe^{-2l\sigma} &= r, & CAe^{-2l_1\sigma} &= r_1, \\ 1 - 2r^2 \cos 2\kappa + r^4 &= D, & 1 - 2r_1^2 \cos 2\kappa + r_1^4 &= D_1. \end{aligned}$$

37. Then we have as before, if d be the electrometer-throw with no condenser on the line,

$$Ed = \int_0^\infty [e^{-kpt} \cos pt]^2 dt,$$

whence

$$4kp(1+k^2)Ed=1+2k^2, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

or, neglecting k^2 ,

$$4kpEd=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

38. Now suppose the condenser placed on the line and consider the effect at the electrometer of a single interference of the initial wave-train incident on the condenser with that reflected from the condenser. This is obviously represented by the following integrals:—

$$\begin{aligned} & \int_0^{2t_2} \phi_1^2 dt + \int_{2t_2}^{\infty} (\phi_1 + \phi')^2 dt \\ &= \int_0^{\infty} \phi_1^2 dt + \int_{2t_2}^{\infty} \phi'^2 dt + 2 \int_{2t_2}^{\infty} \phi_1 \phi' dt \\ &= \frac{F_1 + G_1 + H_1}{4kp(1+k^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31) \end{aligned}$$

where $\frac{F_1}{4kp(1+k^2)}$ represents the integral under which it is placed; and the same remark applies to G_1 and H_1 .

39. But here again, as in the transmitted system already dealt with, the wave-train first reflected at the condenser returns along the wires, suffering attenuation during its transit, is reflected at the oscillator, and is again attenuated in passing along the line to the electrometer. This second wave-train produces a second interference phenomenon, like that due to the initial wave already treated; but phase changes have been introduced and the amplitude diminished. Evidently, then, to complete the account of what happens at the electrometer, and frame a theory of the throw d' to be expected, we need an infinite decreasing series of integrals of each of the three types given in (31) as due to the first interference effect.

40. All these integrals, when evaluated, have the same denominator; hence bringing that to the left side of the

equation, we may write

$$\left. \begin{aligned} 4kp(1+k^2)Ed' &= F_1 + G_1 + H_1 \\ &+ F_2 + G_2 + H_2 \\ &+ F_3 + G_3 + H_3 \\ &+ \dots \dots \dots \\ &+ \dots \dots \dots \\ &+ \dots \dots \dots \\ &= F + G + H; \end{aligned} \right\} \dots \dots (32)$$

where $F = F_1 + F_2 + F_3 + \dots$;

and the same applies to G and H .

41. In constructing the integrals for the first line, we have to note that the reflexion at the condenser and the return of the waves to the electrometer introduce the space coordinate $2l_2$ in two exponential factors and in the argument of the cosine, besides introducing the amplitude factor A and the phase change α .

Again, in passing down any one of the columns in (32) we have to affect each succeeding line by the factor r^2 , and introduce κ in the angles whose cosines are taken.

We thus obtain the following scheme of integrals:—

$$\left. \begin{aligned} Ed' &= \int_0^\infty e^{-2kpt} \cos^2 pt \, dt + s^2 \int_{2t_2}^\infty e^{-2kp(t-2t_2)} \cos^2 (p \cdot \overline{t-2t_2} + \alpha) \, dt \\ &+ 2s \int_{2t_2}^\infty e^{-2kp(t-t_2)} \cos pt \cos (p \cdot \overline{t-2t_2} + \alpha) \, dt \\ &\int_0^\infty e^{-2kpt} \cos^2 (pt + \kappa) \, dt + r^2 s^2 \int_{2t_2}^\infty e^{-2kp(t-2t_2)} \cos^2 (p \cdot \overline{t-2t_2} + \alpha + \kappa) \, dt \\ &+ 2r^2 s \int_{2t_2}^\infty e^{-2kp(t-t_2)} \cos (pt + \kappa) \cos (p \cdot \overline{t-2t_2} + \alpha + \kappa) \, dt \\ &+ \dots \dots \dots + \dots \dots \dots + \dots \dots \dots \\ &+ \dots \dots \dots + \dots \dots \dots + \dots \dots \dots \end{aligned} \right\} (33)$$

to an infinite number of lines.

42. All the above integrals are included in the following typical form:—

$$\left. \begin{aligned} & \int_{2t_2}^{\infty} e^{-2kp(t-t_2)} \cos(pt + \delta) \cos(\overline{pt - 2t_2} + \delta + \epsilon) dt \\ &= \frac{e^{-k\theta}}{4kp(1+k^2)} \left\{ \begin{aligned} & \cos(\theta - \epsilon) - k \sin(\theta + 2\delta + \epsilon) \\ & - k^2(\cos \overline{\theta - \epsilon} + \cos \overline{\theta + 2\delta + \epsilon}) \end{aligned} \right\} \end{aligned} \right\} \quad (34)$$

And the series to which the columns in (32) and (33) when evaluated give rise are of the type given in equation (24). The evaluation of (33) then yields

$$4kp(1+k^2)Ed' = F + G + H, \quad . \quad . \quad . \quad (35)$$

where

$$F = \frac{1}{1-r^2} - \frac{k}{D} r^2 \sin 2\kappa + \frac{k^2}{1-r^2} + \frac{k^2}{D} (1-r^2 \cos 2\kappa), \quad . \quad (36)$$

$$\begin{aligned} G = s^2 \left\{ \frac{1}{1-r^2} - \frac{k}{D} (\sin 2\alpha + r^2 \sin \overline{2\kappa - 2\alpha}) \right. \\ \left. + \frac{k^2}{1-r^2} + \frac{k^2}{D} (\cos 2\alpha - r^2 \cos \overline{2\kappa - 2\alpha}) \right\}, \quad . \quad . \quad (37) \end{aligned}$$

$$\begin{aligned} H = 2se^{-k\theta} \left\{ \frac{\cos \theta - \alpha}{1-r^2} - \frac{k}{D} (\sin \overline{\theta + \alpha} - r^2 \sin \overline{\theta + \alpha - 2\kappa}) \right. \\ \left. + \frac{k^2 \cos \overline{\theta - \alpha}}{1-r^2} + \frac{k^2}{D} (\cos \overline{\theta - \alpha} - r^2 \cos \overline{\theta + \alpha - 2\kappa}) \right\}. \quad (38) \end{aligned}$$

43. *Discussion of Interference Phenomena.*—Let us now consider the various features of this result and their physical interpretation. In equations (36) to (38) the values of F, G, and H are arranged in terms commencing with one of the order unity and proceeding by ascending powers of k . The terms involving k are all small as their coefficients are of the order unity, and k is less than 0.1 in the experimental case under examination. Those involving k^2 will be omitted finally as their coefficients are likewise of the order unity.

The presence of the terms in k is due to the fact, already discussed in art. 29, that the electrometer-throw is slightly affected by the phase at the head of a wave-train of given amplitude, period, and damping. They are therefore but

small corrections to the main features of the case which are represented by the first terms only of each of the expressions in (36) to (38) for F, G, and H. The denominators of these terms supply the correction for the successive arrivals at the electrometer of wave-trains which have suffered reflexion at the condenser and oscillator and passed to and fro along the line.

44. Let curves be imagined with l_2 , the distance between electrometer and condenser, as abscissæ, and F, G, and H as ordinates. It is then seen by inspection of (36) that the curve for F is practically a straight line parallel to the axis of abscissæ. This was to be expected on physical grounds, for it represents the effect at the electrometer due to all the positively-travelling waves. And they are almost independent of any change in the value of l_2 when that change is small compared with l_1 .

45. Similarly, on inspection of (37), we see that, apart from the small terms in k and the slight change of r with l_2 , the curve for G is a logarithmically drooping one. But the droop is very slight for $s^2 = A^2 e^{-4l_2\sigma}$. This again is what might be expected from physical considerations, since this term, G, is due to the return of the wave-train reflected at the condenser apart from interference effects.

46. Turning finally to equation (38), we see that the main features of H are those represented by the first term. This term consists essentially of two factors, one exponential and the other trigonometrical. The exponential factor here gives a much quicker droop than that in G, for we now have $se^{-k\theta}$ instead of s^2 simply as before.

47. The essential nature of H will be made clearer if we write the chief parts of its first term in full. Thus

$$2Ae^{-2kpl_2/v} \cos\left(\frac{2pl_2}{v} - \alpha\right) = h \text{ (say)}. \quad (39)$$

Let now a curve be plotted with h as ordinates and l_2 as abscissæ, and call this the *interference curve*, since it represents the essential features of the electrometer-throws due to interference of waves reflected by the condenser with those incident upon it. Then by inspection of (39) and comparison

with equation (27), representing the incident wave-train, we have the following results :—

- (1) The interference curve is of *damped wave-form*.
- (2) The *wave-length* of the interference curve is *half* that of the incident wave-train.
- (3) The *logarithmic decrement per wave* of the interference curve is *equal* to that of the incident wave-train.
- (4) The *amplitude* of the interference curve is *double* the amplitude factor introduced by the condenser on reflexion.
- (5) The *phase change* exhibited in the interference curve compared with the incident wave-train is *equal in magnitude but opposite in sign* to that introduced by the condenser on reflexion.

48. It may be noted here also in corroboration of these results that, if we simplify to the case of a resistance-bridge of resistance infinity or zero and omit all small terms and corrections, we obtain Bjerknes' expression for interference*.

Thus taking only the first terms of F, G, and H, omitting their denominators and the factor $e^{-2\frac{1}{2}\sigma}$, and writing $A=1$, and $\alpha=0$ or π , we have

$$F + G + H = 2(1 \pm e^{-k\theta} \cos \theta), \quad . \quad . \quad (40)$$

which is equivalent to Bjerknes' equation just quoted.

49. If we now simplify equations (35) to (38) by neglect of k^2 and divide by the corresponding equation (30), we obtain

$$y = \frac{d'}{d} = F + G + H,$$

where

$$F = \frac{1}{1-r^2} - \frac{k}{D} r^2 \sin 2\kappa,$$

$$G = s^2 \left\{ \frac{1}{1-r^2} - \frac{k}{D} (\sin 2\alpha + r^2 \sin 2\gamma) \right\},$$

$$H = 2se^{-k\theta} \left\{ \frac{\cos \overline{\theta - \alpha}}{1-r^2} - \frac{k}{D} (\sin \overline{\theta + \alpha} - r^2 \sin \overline{\theta + \alpha - 2\kappa}) \right\}.$$

(41)

* Wied. Ann. xlv. pp. 513-526 (1891). Equation near top of page 517.

This is the general formula to the desired degree of accuracy.

50. *Reflected System*.—In order to realize an experimental arrangement by which the value of A , the amplitude factor on reflexion at the condenser, may be inferred as simply as possible, let l_2 be taken very large. The term H is then practically extinguished by its very small exponential factor. But, lest its value should still be appreciable, let two values of l_2 be used, viz.: $l_2 + \lambda/8$ and $l_2 - \lambda/8$, where λ is the wave-length of the incident waves. These values of l_2 will give to H opposite algebraical signs, and since H is here very small the difference between its values for each position will be quite negligible. Moreover the values of F and G are almost the same for $l_2 + \lambda/8$ and $l_2 - \lambda/8$. We may, accordingly, write as the working formula for the reflected system,

$$\text{mean } y = F + G, \quad . \quad . \quad . \quad . \quad . \quad (42)$$

the 'mean y ' in the experimental case being the arithmetic mean of the ratios $\frac{d'}{d}$ obtained with $l_2 + \lambda/8$ and $l_2 - \lambda/8$, and F and G may be calculated from (41) as for l_2 simply.

51. *Detection of Phase Change of Reflected Waves*.—We have just noticed a method of practically extinguishing H to simplify the determination of A . Now to determine the phase-change, α , we must take exactly the opposite course and make H paramount, since this term represents the interference phenomena which alone involve α except in a subordinate degree.

We therefore choose a number of values of l_2 differing by small amounts and ranging from zero to about two wave-lengths of the interference-curve.

Since in these cases l_2 is very small compared with l_1 (say 0 to 10 metres and 100 metres respectively), the variable $l = l_1 + l_2$ may be put equal to the constant l_1 without material loss of accuracy. Consequently r is replaced by r_1 and D becomes D_1 .

52. We thus obtain from (41), under these simplifications, our interference formula, viz.:—

$$y = \frac{d'}{d} = F + G + H,$$

where

$$F = \frac{1}{1-r_1^2} - \frac{k}{D_1} r_1^2 \sin 2\kappa,$$

$$G = s^2 \left\{ \frac{1}{1-r_1^2} - \frac{k}{D_1} (\sin 2\alpha + r_1^2 \sin 2\gamma) \right\},$$

and

$$H = 2se^{-k\theta} \left\{ \frac{\cos \overline{\theta-\alpha}}{1-r_1^2} - \frac{k}{D_1} (\sin \overline{\theta+\alpha} - r_1^2 \sin \overline{\theta+\alpha-2\kappa}) \right\}.$$

53. A further simplification is made in the actual calculation and plotting of the theoretical curve, by writing

$$H = \frac{2se^{-k\theta}}{1-r_1^2} (1+n) \cos (\theta-\alpha+\nu). \quad (44)$$

The small quantities n and ν are defined as functions of k , r_1 , and κ by the identity of these two expressions for H . Both n and ν vanish for $k=0$, and assume a simplified value for $\kappa=0$. The actual values of n and ν in any numerical case can be readily found with sufficient accuracy by the graphical composition of sine and cosine functions, the amplitudes being represented by the lengths of lines whose inclination denotes the relative phase of the respective vibrations.

54. *Interference by Reflexion from a Resistance.*—To obtain the interference effect produced by an intermediate resistance-bridge placed on the line beyond the electrometer we have simply to introduce its constants from equation (11) into equation (43). This substitution gives

$$y = \frac{d'}{d} = F + G + H,$$

where

$$F = \frac{1}{1-r_1^2} - \frac{k}{D_1} r_1^2 \sin 2\gamma,$$

$$G = s^2 \left\{ \frac{1}{1-r_1^2} - \frac{k}{D_1} r_1^2 \sin 2\gamma \right\},$$

and

$$H = 2se^{-k\theta} \left\{ \frac{\cos \overline{\theta+\pi}}{1-r_1^2} + \frac{k}{D_1} (\sin \theta - r_1^2 \sin \overline{\theta-2\gamma}) \right\}$$

$$= \frac{2se^{-k\theta}}{1-r_1^2} (1+n) \cos (\theta+\pi+\nu).$$

For a no-resistance bridge this is further simplified by the amplitude factor being unity.

EXPERIMENTS.

55. *Positions of Condensers.*—For the transmitted system the condensers were placed at FF' (fig. 1, p. 275), the distance SF being 59 metres $\pm \lambda/8$. These two distances were chosen to eliminate any error which would arise if interference occurred on the oscillator side of the condenser. This might conceivably take place between the head of a wave-train arriving at the condenser simultaneously with the departure of the tail of a previous train reflected there. To reduce this anticipated disturbance to a minimum, the distance SF was chosen of the order 60 m. in preference to any smaller distance. It is easily seen that if this interference was additive in one of the positions chosen, then it would be subtractive in the other, and both being small the arithmetic mean could be taken. From the tables of readings it may be noted that neither of these precautions was unnecessary.

56. In the reflected system the distance ED of the condenser beyond the electrometer, denoted by l_2 in the theory, was made 48 m. $\pm \lambda/8$; the two positions being again adopted to obviate any error due to interference. Here again the precautions were not in vain. The wave-length λ of the incident waves was determined by Bjerknes' method to be 8.5 m.

For the interference phenomena, observed in order to detect the phase change on reflexion, the distance ED was varied by half a metre from 0.75 m. to about 8.0 m. so as to furnish nearly two wave-lengths of the interference curve.

The methods of taking the readings and their results will be sufficiently understood from the following tables and accompanying curves. The three condensers used were chosen by reference to the curve in fig. 5 as likely to yield typical results.

57. *Large Condenser.*—Disks 15 centim. radius and placed 1 centim. apart.

(i.) *Transmitted System.* On taking seven readings with-

out condenser on alternated with six readings with condenser on, the values were respectively of the order

$$\left. \begin{array}{l} d=26\cdot8 \text{ and } d'=2\cdot6 \\ y=\frac{d'}{d}=0\cdot097 \text{ nearly} \end{array} \right\} \dots \dots (46)$$

whence

With such small readings as those here obtained for d' it cannot be professed that they are free from errors due to small variations of the zero, which is never quite steady and

TABLE II.—Reflected System for Large Condenser.

Electrometer-Throws, d , without Condenser on.		Electrometer-Throws, d' , with Condenser at l_2 beyond Electrometer.		Ratio of Throws, $\frac{d'}{d}$, for each position, viz.:	
Actual Throws.	Interpolated Means.	$l_2 =$ $48^m - \lambda/8$.	$l_2 =$ $48^m + \lambda/8$.	$l_2 =$ $48^m - \lambda/8$.	$l_2 =$ $48^m + \lambda/8$.
23	23	45		1·957	
23	22·5		42		1·867
22	22	44	42	2·000	1·909
22	22				
22	25	50		2·000	
28	25·5		50		1·961
23	24	49		2·042	
25	25		54		2·160
25	27·5	56		2·036	
30	29·5		58		1·966
29	29·5	60		2·034	
30	29		57		1·966
28	28	59		2·107	
28	28		55		1·964
28	26	54		2·077	
24	26		54		2·077
28	29	50		1·724	
30	28·5		49		1·719
27	24·5	50		2·041	
22	20·5		43		2·098
19					
Mean Values of Ratios				2·002	1·969
Whence Final Value or Mean of Means = 1·986.					

Note.—This set of observations occupied 2 hours 45 minutes, during which time the zero crept 22 scale-divisions in the positive direction, viz., from -32 to -10 .

is always read before each throw is taken throughout all the experimental work.

(ii.) *Reflected System.* Two sets of 41 readings each were taken for this arrangement. The first gave

$$\text{mean } y = \frac{\text{mean } d'}{d} = 1.862. \quad . \quad . \quad . \quad . \quad (47)$$

The second, in which the behaviour of sparks and electrometer was more regular, gave

$$\text{mean } y = \frac{\text{mean } d'}{d} = 1.986. \quad . \quad . \quad . \quad . \quad (48)$$

This second set of readings is given in Table II. (p. 298).

(iii.) *Interference System.* Four sets of readings were taken with this arrangement of the apparatus. The last of these is given in Table III., and the points corresponding to all of them are plotted in fig. 6, a curve is then drawn through the mean position to represent as fairly as may be the result of all the readings.

TABLE III.—Interference System for Large Condenser.

Electrometer-Throws, d , without Condenser on.		Condenser on Line.		Ratios, $\frac{d'}{d} = y$, of Throws with and without Condenser.
Actual Readings.	Interpolated Means.	Electrometer Throws, d' .	Distance, l_2 , of Condenser beyond Electrometer.	
33	30	29	0.75 m.	0.97
27	29.5	53	1.0	1.80
32	28.5	72	1.5	2.53
25	25.5	112	2.0	4.39
26	16 *	67	2.5	4.19
16	17	64	3.0	3.76
18	18	48	3.5	2.67
18	19.5	32	4.0	1.64
21	22	29	4.5	1.32
23	25	82	6.25	3.28
27	27	86	7.0	3.19
27	26	64	7.5	2.46
25	22.5	48	8.0	2.13
20				

* A pause was made here to adjust balls of spark-gap in oscillator; hence the 26 is disregarded, and 16 taken as true reading.

Since the amplitude factor of the wave-train reflected from this condenser is almost unity, a bridge of no-resistance, whose reflexion factor is exactly unity, was used as a comparison and to show the phase-difference between the two curves. Three sets of readings were taken with this bridge, the points and mean curve for which are plotted on fig. 7.

Fig. 6.—Experimental Results:—Interference System for Large Condensers. First three sets of readings indicated by ., o, and † respectively. Fourth Set (Table III.) by ×.

Fig. 7.—Experimental Results:—Interference System for Bridge of Negligible Resistance. Three sets of readings indicated by ., o, and × respectively.

58. *Medium Condenser*.—Disks 9 centim. radius and placed 1 centim. apart.

(i.) *Transmitted System*. The readings for this arrangement result in the ratio

$$\text{mean } y = \frac{\text{mean } d'}{d} = 0.349. \quad . \quad . \quad . \quad (49)$$

The readings are shown in Table IV.

TABLE IV.

Transmitted System for Medium Condenser.

Electrometer-Throws, d , without Condenser on.		Electrometer Throws, d' , with Condenser at l beyond Oscillator, where		Ratios of Throws, $\frac{d'}{d}$, for each position, viz. :—	
Actual Throws.	Interpolated Means.	$l =$ $59^m - \lambda/8$.	$l =$ $59^m + \lambda/8$.	$l =$ $59^m - \lambda/8$.	$l =$ $59^m + \lambda/8$.
28	27.5	10		0.364	
27	30		10		0.333
33	31	11		.355	
29	29.5		10		.339
30	30	11		.367	
30	29.5		10		.339
29	28	9		.321	
27	25		7		.280
23	24.5	9		.367	
26	26		8		.308
26	25	9		.360	
24	23.5		8		.340
23	23	8		.348	
23	22.5		8		.356
22	22	8		.364	
22	22		7		.318
22	21	8		.381	
20	20		8		.400
20	20.5	7		.341	
21	20		8		.400
19					
Arithmetic Means.....				0.357	0.341
Whence Mean of Means = 0.349.					

(ii.) *Reflected System.* For this arrangement two sets each of 41 readings were taken. The first set gave

$$\text{mean } y = \frac{\text{mean } d'}{d} = 1.523; \quad (50)$$

the second set gave

$$\text{mean } y = \frac{\text{mean } d'}{d} = 1.626; \quad (51)$$

the readings being contained in Table V.

TABLE V.
Reflected System for Medium Condenser.

Electrometer-Throws, d , without Condenser on.		Electrometer-Throws, d' , with Condenser at l_2 beyond Electrometer.		Ratio of Throws, $\frac{d'}{d}$, for each position.	
Actual Throws.	Interpolated Means.	$l_2 =$ $48^m - \lambda/8$.	$l_2 =$ $48^m + \lambda/8$.	$l_2 =$ $48^m - \lambda/8$.	$l_2 =$ $48^m + \lambda/8$.
25	25	42		1.680	
25	26		42		1.615
27	26.5	46		1.736	
26	28.5		53		1.860
31	29.5	55		1.864	
28	29		50		1.724
30	31	54		1.742	
32	33		54		1.636
34	33.5	52		1.552	
33	33		50		1.515
33	32	51		1.594	
31	30.5		47		1.541
30	30	49		1.633	
30	30		47		1.567
30	30	50		1.667	
30	29		44		1.517
28	28.5	42		1.474	
29	28		44		1.571
27	26.5	42		1.585	
26	23		33		1.435
20					
Arithmetic Means.....				1.653	1.598
Whence Mean of Means = 1.626.					

(iii.) *Interference System.* For this disposition of the apparatus four sets of readings were taken. One of these is given in Table VI. On fig. 8 are plotted the points from all the readings, and the curve through their mean position is shown also.

TABLE VI.—Interference System for Medium Condenser.

Electrometer-Throws, d , without Condenser on.		Condenser on Line.		Ratios, $\frac{d'}{d} = y$, of Throws with and without Condenser.
Actual Throws.	Interpolated Means.	Electrometer Throws, d' .	Distances, l_2 , of Condenser beyond Electrometer	
			m.	
37	36.5	22	0.75	0.60
36	33	36	1.0	1.09
30	32	73	1.5	2.28
34	33.5	105	2.0	3.13
33	32.5	115	2.5	3.54
32	32	113	3.0	3.53
32	34	75	3.5	2.21
36	34.5	51	4.0	1.48
33	30.5	35	4.5	1.15
28	30.5	32	5.0	1.05
33	34.5	82	6.25	2.38
36	34.5	85	7.0	2.46
33	32.5	70	7.5	2.15
32	33.5	54	8.0	1.61
35				

Fig. 8.—Experimental Results. Interference System for Medium Condenser. The several sets of readings are respectively indicated by *, o, †, and ×.

59. *Small Condenser.*—Disks 5 centim. radius and placed 1 centim. apart.

(i.) *Transmitted System.* For this arrangement the set of 41 readings shown in Table VII. was taken. These readings yield the result

$$y = \frac{\bar{d}'}{\bar{d}} = 0.868. \quad . \quad . \quad . \quad . \quad . \quad (52)$$

TABLE VII.—Transmitted System for Small Condenser.

Electrometer-Throws, d , without Condenser on.		Electrometer Throws, d' , with Condenser at l beyond Oscillator, where		Ratio of Throws, $\frac{d'}{d}$, for each position, viz.:	
Actual Throws.	Interpolated Means.	$l=$ $59^m - \lambda/8$.	$l=$ $59^m + \lambda/8$.	$l=$ $59^m - \lambda/8$.	$l=$ $59^m + \lambda/8$.
20	21	15		0.714	
22	22.5		15		0.667
23	24	20		.833	
25	23		18		.783
21	23	19		.826	
25	23.5		21		.894
22	22	19		.864	
22	22.5	19	19		.844
23	20.5	19		.927	
18	16		14		.875
14	16	14		.875	
18	18.5		14		.757
19	18	18		1.000	
17	17.5		16		.914
18	18.5	17		0.919	
19	17		17		1.000
15	16	14		.875	
17	17.5		14		0.800
18	15	16		1.067	
12	12		11		.917
12					
Arithmetic Means				0.890	0.845
Whence Mean of Means = 0.868					

(ii.) *Reflected System.* For this arrangement two sets each of 41 readings were taken. The first set gave

$$\text{mean } y = \frac{\text{mean } d'}{d} = 1.168. \quad . \quad . \quad . \quad (53)$$

The second set of readings is given in Table VIII. and yields the result

$$\text{mean } y = \frac{\text{mean } d'}{d} = 1.188. \quad . \quad . \quad . \quad (54)$$

TABLE VIII.—Reflected System for Small Condenser.

Electrometer-Throws, d , without Condenser on.		Electrometer-Throws, d' , with Condenser at l_2 beyond Electrometer.		Ratios of Throws, $\frac{d'}{d}$, for each position.	
Actual Throws.	Interpolated Means	$l_2 =$ $48^m - \lambda/8.$	$l_2 =$ $48^m + \lambda/8.$	$l_2 =$ $48^m - \lambda/8.$	$l_2 =$ $48^m + \lambda/8.$
37	32.5	37		1.138	
28	24.5		29		1.184
21	14.5	19		1.310	
* { 8					
32	33.5		35		1.045
35	34	41		1.206	
33	31.5		37		1.175
30	32	39		1.219	
34	32		36		1.125
30	31.5	40		1.270	
33	33.5		38		1.134
34	31.5	39		1.238	
29	31		37		1.194
33	32.5	40		1.231	
32	33		37		1.121
34	32.5	39		1.200	
31	32.5		38		1.169
34	33.5	39		1.164	
33	33		37		1.121
33	32.5	41		1.262	
32	32		40		1.250
32					
Arithmetic Means				1.224	1.152
Whence Mean of Means is 1.188.					

* The balls of the spark-gap of the primary oscillator were adjusted between these two readings.

(iii.) *Interference System.* With the apparatus arranged for observing the interference phenomena with this condenser, three sets of readings were taken. One of these is shown in Table IX. It will be seen to include readings for a *resistance-bridge* also. This bridge consisted of graphite markings upon a half disk of ground glass 3 inches diameter, and was made of a resistance of 235 ohms, the value calculated to give the same amplitude factor on reflexion as the condenser

TABLE IX.—Interference System for Small Condenser and Corresponding Resistance Bridge.

Electrometer-Throws, l , without either Con- denser or Resistance Bridge on.		Electrometer-Throws, d' , with apparatus at l_2 beyond Electrometer viz. :—		Values of the Distance l_2 beyond Elec- trometer at which Con- denser or Resistance Bridge is placed.	Ratios of Throws, $\frac{d'}{d}$, obtained with	
Actual Throws.	Inter- polated Means.	Condenser on Line.	Resistance Bridge on Line.		Con- denser.	Resistance Bridge.
27			27	m.		
25			39	8.0	1.31	1.04
27	25	42	43	7.5	1.62	1.50
23	24.5	31	48	7.0	1.68	1.72
26	26	19	27	6.25	1.27	1.96
26	25.5	22	19	5.0	0.73	1.04
25	24	30	20	4.5	0.86	0.75
23	23.5	40	25	4.0	1.25	0.83
24	23	43	41	3.5	1.70	1.06
22	22	39	45	3.0	1.87	1.78
22			43	2.5	1.77	2.05
22	19	18	35	2.0	1.36	1.95
16	14	10	14	1.5	0.95	1.84
12	13.5	8	12	1.0	0.71	1.00
15				0.75	0.59	0.89

now in use. This resistance was tested, and if necessary adjusted to 235 ohms, immediately before each time it was used.

Table X. gives a summary of the results, for both condenser and resistance-bridge, of all three sets of readings.

Fig. 9.—Interference Systems for Small Condenser and for Bridge of 235 ohms Resistance. Experimental results in full lines, theoretical curves in broken lines. C and c denote curves for Condenser. B and b denote those for Bridge.

From the mean values thus obtained the experimental curves for condenser and resistance-bridge are plotted in fig. 9.

TABLE X.—Summary of Results for Interference System with Small Condenser and Corresponding Resistance Bridge.

Ratios, $\frac{d'}{d}$, with the Condenser on the Line.				Values of the Distances l_2 beyond Electrometer at which Condenser or Resistance Bridge is placed.	Ratios $\frac{d'}{d}$, with the Resistance Bridge on the Line.			
Separate Values.			Means.		Means.	Separate Values.		
(A)	(B)	(C)		m.		(C)	(B)	(A)
0.526	0.39	0.59	0.50	0.75	0.90	0.89	0.89	0.912
.604	.48	.71	.60	1.0	1.11	1.00	1.13	1.208
.898	.95	.95	.93	1.5	1.76	1.84	1.87	1.551
1.333	1.57	1.36	1.42	2.0	2.05	1.95	2.03	2.157
1.835	1.56	1.77	1.74	2.5	2.15	2.05	2.10	2.308
1.738	1.79	1.87	1.75	3.0	1.45	1.78	1.36	1.639
	1.59						1.02	
1.486	1.20	1.70	1.46	3.5	1.19	1.06	1.13	1.371
1.206	1.50	1.25	1.38	4.0	0.97	0.83	1.11	0.971
0.812	0.93	0.86	0.87	4.5	0.81	0.75	0.96	0.719
.926	.68	.73	.78	5.0	1.05	1.04	1.03	1.074
1.309	1.24	1.27	1.27	6.25	2.02	1.96	1.97	2.145
1.569	1.66	1.68	1.64	7.0	1.75	1.72	1.87	1.662
1.636	1.73	1.62	1.66	7.5	1.45	1.50	1.48	1.364
1.118	1.45	1.31	1.29	8.0	1.15	1.04	1.18	1.235

The letters (A), (B), and (C) at the heads of the columns of separate values show corresponding sets of readings for Condenser and Resistance-bridge.

COMPARISON OF THEORY AND EXPERIMENT.

60. *Transmitted and Reflected Systems.*—The comparison of theoretical and experimental results for the transmitted and reflected systems of all three condensers is exhibited in Table XI.

The data for the theoretical values of the ratios are as follows.

The Reflexion and Transmission Constants for the three condensers are those given in fig. 5 and in the lower part of Table I.

The Amplitude Factor, C , for the oscillator is taken as 0·716 from the experiments * made in 1898 on the very line now in use.

The Attenuation Constant, σ , was also taken, from the experimental determinations of the paper just quoted, as equal to 0·0000130.

TABLE XI.

Comparison of Theory and Experiment for Transmitted and Reflected Systems of all three Condensers.

Condenser and System of Waves observed.		Values of Ratios, $\frac{d'}{d}$, of Electrometer-Throws with and without Condenser.		
		Calculated by Approximate Theory from equations (26) and (42).	Experimentally determined Values.	Calculated by Approximate Theory with corrections from equations (56) and (57).
Large.	Transmitted.	0·079	0·097 (?)	0·108
	Reflected.	2·236	1·924	2·095
Medium.	Transmitted.	0·420	0·349	0·367
	Reflected.	1·924	1·574	1·754
Small.	Transmitted.	0·966	0·868	0·706
	Reflected.	1·330	1·178	1·178

* "Attenuation of Electric Waves along a Line of Negligible Leakage," Phil. Mag. Sept. 1898. See in Table II. (p. 301) the value of ρ , which is the C of the present notation.

The *Phase-change*, γ , introduced by the oscillator is calculated theoretically as equal to $-23^\circ 28'$.

Other data are:—

$Lv = 56 \times 10^{10}$ C.G.S. E.M. units (calculated and verified experimentally).

$k = 0.0955$ as determined experimentally.

$p = 2\pi v/\lambda = 1.96 \times 10^8$ radians per second as determined experimentally.

61. It is seen on reference to Table XI. that there is a considerable discrepancy between the experimentally-determined values of the ratios of electrometer-throws and those calculated from the approximate theory. This theory, used as a first working approximation, takes $\iota = 0$ in the original incident wave-train $e^{-kpt} \cos(pt + \iota)$, and omits the exponential term introduced on reflexion at the condenser. This was done because the value of ι is unknown. Let us now determine the *maximum* correction needed to the above theory, the results of which are placed in the last column of Table XI. Assume for ι such values as to make for each condenser $\iota + \beta = 0$. This makes the coefficient of the exponential term a maximum. Further, let us suppose (what is not possible) that the coefficient of the exponential term retains its maximum value all through; *i. e.*, put $\iota + \beta + 2\kappa = 0$ or π , $\iota + \beta + 4\kappa = 0$ or 4π &c. for the successive reflexions and transmissions. In order to convert the time-integral of the square of the amplitude, on the approximate theory, to its true value when the exponential term is retained, let A^2 and B^2 be now replaced by ρA^2 and τB^2 respectively.

Then we shall have

$$\left. \begin{aligned} \rho &= \int_0^\infty [Ae^{-kpt} \cos(pt + \iota + \alpha) - B \cos(\iota + \beta) e^{-2pt/\chi'}]^2 dt \\ &\quad \div \int_0^\infty [Ae^{-kpt} \cos(pt + \iota + \alpha)]^2 dt, \\ \text{and} \\ \tau &= \int_0^\infty [Be^{-kpt} \cos(pt + \iota + \beta) - B \cos(\iota + \beta) e^{-2pt/\chi'}]^2 dt \\ &\quad \div \int_0^\infty [Be^{-kpt} \cos(pt + \iota + \beta)]^2 dt \end{aligned} \right\} \quad (55)$$

where $\iota + \beta$ is to be written equal to zero.

Further, since ι has now a finite value, and different for each condenser, equations (26) and (42) need modifying by the introduction of ι . We thus obtain the following corrected equations:—

For the transmitted system:

$$y = \tau B^2 \left\{ \frac{1}{1-r^2} - \frac{k}{D} r^2 \sin 2\kappa \right\}; \quad . \quad . \quad . \quad (56)$$

and for the reflected system:

$$\text{mean } y = F + \rho G;$$

where

$$\left. \begin{aligned} F &= \frac{1}{1-r^2} - \frac{k}{D} (\sin 2\iota - r^2 \sin 2\iota - 2\kappa), \\ \text{and} \\ G &= s^2 \left\{ \frac{1}{1-r^2} - \frac{k}{D} (\sin 2\iota + 2\alpha - r^2 \sin 2\iota + 2\alpha - 2\kappa), \right\} \end{aligned} \right\} (57)$$

ρ and τ being defined by (55).

62. It is to be noted that these formulæ in *two* respects *over-correct* the original approximate expressions. *First*, they assume that the trigonometrical coefficient of the exponential term is a maximum for *every* successive incidence; and *second*, they assume that for each condenser $\iota + \beta = 0$, which necessitates a different ι for each case, whereas the wave-train launched along the line must have the same shaped head in each case. Thus the truth on the long-wave theory may be expected to lie between the values given in the first and third columns of Table XI.

63. In comparing the experimental values with the theory, the following points may be noted:—

- (1) In one case (in Table XI.) experiment and theory give identical values (last line).
- (2) In two cases the theory seems over-corrected (lines 1 & 5).
- (3) In three cases the correction to the theory fails to bring the values down to those experimentally determined.
- (4) Hence the outstanding discrepancy is on the side of slightly less reflected and less transmitted than theory

predicts, though further experiments under a greater variety of conditions would be needed to establish this view of the case.

- (5) Turning now from the question of the amplitudes of the reflected and transmitted waves to that of the phase-change, α , introduced on reflexion, we see from figs. 9, 10, & 11 that the theoretical and experimental results are in fair agreement.

Fig. 10.—Interference Systems for Large Condenser and for Bridge of Negligible Resistance. Experimental results in full lines, theoretical curves in broken lines. C and c denote curves for Condenser; B and b denote curves for Bridge.

Fig. 11.—Interference System for Medium Condenser. The full line is the Experimental Curve, and the broken line is the Theoretical Curve.

Our indebtedness to the works of Oliver Heaviside, obviously great, has been increased by the receipt in MS. of supplementary elucidations with which Mr. Heaviside has favoured us. This invaluable help we here gratefully acknowledge.

Univ. Coll. Nottingham,
August 1899.

XX. *Some Lecture Experiments illustrating Syntony.*

By PHILIP E. SHAW, B.A., B.Sc.*

IN the Journal of the Institution of Electrical Engineers (Feb. 1899) will be found an account by Prof. Oliver Lodge of his researches in Magnetic Space Telegraphy. The present writer then set up some apparatus to show these methods in a popular lecture. Considerable difficulties soon presented themselves, but success was ultimately attained. Those who are conversant with the above paper will admit that the theory involved in it is of importance, and that any new demonstration of it would be valuable.

Besides the interest attaching to the theory of syntony, this method of magnetic space-telegraphy has promise of practical and commercial value, and has been shown already by Prof. Lodge to work well over considerable distances.

The following brief account contains a notice of the general method, of special details introduced by the author, and of precautions to be adopted.

Theory.

A summary will be given of the more essential points.

I. The frequency (N) of a circuit of capacity S and self-induction L is

$$N = \frac{160}{\sqrt{L \cdot S}}$$

where L is reckoned in henries and S in microfarads.

* Read March 23, 1900.