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Calculation of the Hyperbolic Logarithm of π.

By J. W. L. Glaisher, M.A., F.R.S.

[*Read March 8th*, 1883.]

1. In the Introduction to his *Table des Diviseurs pour tous les nombres du premier Million* (Paris, 1817), Burckhardt explains the application of the table to the calculation of logarithms, taking as his example the hyperbolic logarithm of π.

By means of the table, he finds that $3141593 = 7 \times 13 \times 19 \times 23 \times 79$, but he remarks that a better approximation to π is afforded by 31415925, which $= 3 \times 5^2 \times 373 \times 1123$. As an approximation which is closer still, he gives $314159265 = 3^2 \times 5 \times 7 \times 127 \times 7853$.

Adding the logarithms of the factors forming this last product, and subtracting $8 \times \log_e 10$, he finds

$$\log_e 3{\cdot}14159265 = 1{\cdot}144\,729\,884\,706\,733\,496\,332\,198\,442\,742\,4,$$

which differs from $\log_e \pi$ only in the ninth decimal place. Putting

$$\frac{\pi}{314159265} = 1 + a,$$

he finds $\quad a = 11{\cdot}426\,666\,784\,640\,724\,771\,493 \times 10^{-31}$,

giving $\quad \log_e \pi = 1{\cdot}144\,729\,885\,849\,400\,174\,143\,427$.

Burckhardt also mentions that we may obtain even better approximations by seeking for a number which shall be nearly equal to π, and also be the difference of two squares. Thus, for example,

$$(1772454)^2 = 314\,159\,318\,211\,6, \quad \pi = 314\,159\,265\,359\,0,$$

the difference 528526 = $(727)^2$ nearly, whence

$$\pi = (1772454)^2-(727)^2 = (1772454+727)(1772454-727)$$
$$= 1773181 \times 1771727.$$

The first factor being a prime, this formula, he remarks, is not available; but, as the product is not changed sensibly by increasing one of the factors by a small quantity and diminishing the other by the same quantity, he adds $\frac{2}{10}$ to one factor, and subtracts it from the other, and thus obtains the formula

$$17731808 \times 17717272 = 2^8 \times 19 \times 173 \times 229 \times 509 \times 3203$$
$$= 314\,159\,265\,387\,776.$$

2. Wolfram's table gives the hyperbolic logarithms of all numbers up to 2,200, and of all primes (and of many composite numbers also) up to 10,009, to 48 places of decimals. It first appeared in Schulze's *Sammlung logarithmischer... Tafeln*, pp. 190—259 (Berlin, 1778), and was reprinted in Vega's *Thesaurus logarithmorum completus*, pp. 642—684 (Leipzig, 1794), with the addition of six logarithms which had been omitted owing to Wolfram's death. By means of this table, therefore, we can obtain at once, by simple addition, the hyperbolic logarithm of any number which can be resolved into factors none of which exceeds 10,009. The method of calculating a logarithm to which Burckhardt draws attention is a very convenient one, and I was tempted by my father's calculation of the factor tables of the fourth, fifth, and sixth millions (which, in conjunction with Burckhardt's and Dase's tables, afford the means of resolving into factors at once any number not exceeding 9,000,000) to seek for a number very nearly equal to π and having no prime factor exceeding 10,009, and in this manner to calculate the hyperbolic logarithm of π to 48 places of decimals.

3. The two nearest approximations to π, satisfying the required conditions, which I was successful in finding were:—

(i.) 3·141 592 653 584 947 2,

which differs from π by ·000 000 000 004 846 0 ...,

and (ii) 3·141 592 653 535,

which differs from π by ·000 000 000 054 ... ;

the resolutions into factors being as follows:—

(i.) $314\,159\,265\,358\,494\,72 = 2^? \times 3 \times 19 \times 269 \times 307 \times 3433 \times 3797$;

(ii.) $314\,159\,265\,353\,5 = 5 \times 7 \times 17^2 \times 19 \times 37 \times 107 \times 4129.$

4. Taking the first resolution (i.), we have

$$\pi = x + h = 3{\cdot}141\ 592\ 653\ 589\ 793\ 238\ \ldots,$$

$$x = 3{\cdot}141\ 592\ 653\ 584\ 947\ 2,$$

whence $$h = {\cdot}(11)\ 4\ 846\ 038\ 462\ 643\ 383\ 279\ 502\ 884\ 197\ 169\ 399\ 375\ 105\ 820\ 974\ 945\ \ldots,$$

where the notation (11) indicates that there are eleven ciphers between the decimal point and the first significant figure.

Dividing the value of h by x, and retaining 37 figures of the quotient, we find, true to 48 decimal places,

$$\frac{h}{x} = {\cdot}(11)\ 1\ 542\ 541\ 951\ 488\ 666\ 689\ 372\ 045\ 964\ 447\ 941\ 218.$$

Multiplying this value of $\frac{h}{x}$ by itself, and dividing by 2, we deduce

$$\tfrac{1}{2}\frac{h^2}{x^2} = {\cdot}(23)\ 1\ 189\ 717\ 836\ 051\ 232\ 069\ 032\ 064;$$

multiplying the value of $\frac{h^2}{x^2}$ by that of $\frac{h}{x}$, and dividing by 3,

$$\tfrac{1}{3}\frac{h^3}{x^3} = {\cdot}(35)\ 1\ 223\ 459\ 781\ 696;$$

and the next term $$\tfrac{1}{4}\frac{h^4}{x^4} = {\cdot}(47)\ 1.$$

Taking the logarithms from Wolfram's table, it is found that

$$\log_e (x \times 10^{16}) = 37{\cdot}\ 986\ 091\ 373\ 752\ 588\ 576\ 479\ 803\ 149\ 331\ 350\ 037\ 309\ 079\ 940\ 128,$$

whence, subtracting $16 \times \log_e 10$,

$$\log_e x = 1{\cdot}\ 144\ 729\ 885\ 847\ 857\ 632\ 191\ 939\ 874\ 381\ 522\ 715\ 691\ 456\ 122\ 064.$$

Substituting for $\log_e x$, $\frac{h}{x}$, &c., their values in the formula

$$\log_e \pi = \log_e x + \frac{h}{x} - \tfrac{1}{2}\frac{h^2}{x^2} + \tfrac{1}{3}\frac{h^3}{x^3} - \tfrac{1}{4}\frac{h^4}{x^4} + \&c.,$$

we finally obtain the result

$$\log_e \pi = 1{\cdot}\ 144\ 729\ 885\ 849\ 400\ 174\ 143\ 427\ 351\ 353\ 058\ 711\ 647\ 294\ 812\ 913.$$

5. Similarly, in the case of the second resolution (ii.), the values of the same expressions, obtained in the same manner, are :—

$$\pi = x + h = 3{\cdot}\ 141\ 592\ 653\ 589\ 793\ \ldots,$$

$$x = 3{\cdot}\ 141\ 592\ 653\ 535,$$

$$h = \cdot(10)\ 54\ 793\ 238\ 462\ 643\ 383\ 279\ 502\ 884$$
$$197\ 169\ 399\ 375\ 105\ 820\ 974\ 944\ \ldots,$$

$$\frac{h}{x} = \cdot(10)\ 17\ 441\ 229\ 498\ 989\ 513\ 169\ 562\ 245\ 806\ 364\ 125\ 677,$$

$$\tfrac{1}{2}\frac{h^2}{x^2} = \cdot(21)\ 152\ 098\ 243\ 218\ 210\ 992\ 284\ 118\ 085,$$

$$\tfrac{1}{3}\frac{h^3}{x^3} = \cdot(32)\ 1\ 768\ 520\ 244\ 241\ 295,$$

$$\tfrac{1}{4}\frac{h^4}{x^4} = \cdot(43)\ 23\ 134,$$

$$\log_e x = 1\cdot 144\ 729\ 885\ 831\ 958\ 944\ 644\ 589$$
$$936\ 426\ 712\ 908\ 312\ 970\ 587\ 159,$$

giving
$$\log_e \pi = 1\cdot 144\ 729\ 885\ 849\ 400\ 174\ 143\ 427$$
$$351\ 353\ 058\ 711\ 647\ 294\ 812\ 912.$$

6. The two results differ only by a unit in the forty-eighth place, and, since all the portions of the two calculations are entirely distinct, we may therefore conclude with certainty that to forty-seven places of decimals the value of the hyperbolic logarithm of π is

$$1\cdot 144\ 729\ 885\ 849\ 400\ 174\ 143\ 427$$
$$351\ 353\ 058\ 711\ 647\ 294\ 812\ 91.$$

7. The resolution (ii.) was found without the aid of any definite process; but the resolution (i.) was the result of a systematic series of trials by Burckhardt's method. The first seventeen figures of π were subtracted from $(177245386)^2$, $(177245387)^2$, ... $(177245400)^2$ in succession, and the square nearest to the difference in each case was taken out from Kulik's *Tafeln der Quadrat- und Kubik-Zahlen*, which gives squares and cubes of numbers up to 100,000. The differences of squares thus obtained give resolutions into pairs of factors, each containing nine figures, and these were resolved into their prime factors as far as was found practicable or necessary. In only one case were the factors thus found all included within the range of Wolfram's table; the resolution (i.) was thus obtained, the process being as follows:—The difference between $(177245396)^2$ and the number whose digits are the first seventeen figures of π is 38672988884, the nearest square to which is $3867347344 = (62188)^2$,

and
$$(177245396)^2 - (62188)^2 = 177307584 \times 177183208$$
$$= (2^6 \times 3 \times 269 \times 3433) \times (2^3 \times 19 \times 307 \times 3797),$$
which gives (i.).

The following resolution derived from this system may also be noticed:— $(177245387)^2 - (26017)^2 = 177271404 \times 177219370$
$$= (2^2 \times 3 \times 47 \times 257 \times 1223) \times (2 \times 5 \times 23 \times 770519);$$

but the last factor 770519 is prime. The product in this case is 314 159 265 358 954 80, which differs from π in the fourteenth figure.

8. I made some attempts in which the larger of the two numbers that were squared contained ten figures; the one which came nearest to success was

$$(1772453853)^2-(86167)^2 = 1772540020 \times 1772367686.$$

The first factor $= 2^2 \times 5 \times 17 \times 19 \times 89 \times 3083$,

and the second $= 2 \times 79 \times 11217517$;

but I found no factor of 11217517. The product in this case differs from π only in the seventeenth figure.

It may also be remarked that

$$(17724539)^2-(4171)^2 = 17728710 \times 17720368$$
$$= (2 \times 3 \times 5 \times 19^2 \times 1637) \times (2^4 \times 1107523).$$

The product differs from π in the twelfth figure.

9. The different resolutions which seem worthy of notice are collected together in the following list. The number in [ ] denotes the number of figures which agree with those of π; and the two figures following the [ ] are the first two figures of the difference between the given number and the figures of π.

(i.) 314 159 265 358 494 72 $= 2^9 \times 3 \times 19 \times 269 \times 307 \times 3433 \times 3797$,
Difference = [12] 48.

(ii.) 314 159 265 353 5 $= 5 \times 7 \times 17^2 \times 19 \times 37 \times 107 \times 4129$,
Difference = [11] 54;

these are the resolutions by means of which $\log_e \pi$ was calculated in §§ 4 and 5.

(iii.) 314 159 265 387 776 $= 2^6 \times 19 \times 173 \times 229 \times 509 \times 3203$,
Difference = [10] 28;

this is Burckhardt's resolution (§ 1).

(iv.) 314 159 265 42 $= 2 \times 3 \times 7^2 \times 11 \times 13 \times 19 \times 67 \times 587$,
Difference = [10] 61.

(v.) 314 159 265 358 954 80 $= 2^3 \times 3 \times 5 \times 23 \times 47 \times 257 \times 1223 \times 770519$,
Difference = [13] 24, (§ 7).

(vi.) 314 159 265 365 280 $= 2^5 \times 3 \times 5 \times 19^2 \times 1637 \times 1107523$,
Difference = [11] 63, (§ 8).

(vii.) 314 159 265 358 979 372 0 $= 2^3 \times 5 \times 17 \times 19 \times 79 \times 89 \times 3083 \times 11217517$,
Difference = [16] 48, (§ 8).

10. In applying the method of calculating logarithms made use of in this note, I have invariably experienced but little difficulty in find-

ing a number having nine or ten figures identical with those of a given number, and admitting of resolution into prime factors less than 10,000; but, judging from the preceding resolutions, it appears that the difficulty is very greatly increased when the number of figures exceeds ten.

On Monge's "Mémoire sur la Théorie des Déblais et des Remblais."

By Prof. CAYLEY, F.R.S.

[*Read March 8th*, 1883.]

The Memoir referred to, published in the *Mémoires de l'Académie*, 1781, pp. 666 to 704, is a very remarkable one, as well for the problem of earthwork there considered as because the author was led by it to his capital discovery of the curves of curvature of a surface. The problem is, from a given area, called technically the Déblai, to transport the earth to a given equal area, called the Remblai, with the least amount of carriage. Taking the earth to be of uniform infinitesimal thickness over the whole of each area (and therefore of the same thickness for both areas), the problem is a plane one: viz., stating it in a purely geometrical form, the problem is, Given two equal areas, to transfer the elements of the first area to the second area in such wise that the sum of the products of each element into the traversed distance may be a minimum; the route of each element is, of course, a straight line. And we have the corresponding solid problem: Given two equal volumes, to transfer the elements of the first volume to the second volume in such wise that the sum of the products of each element into the traversed distance may be a minimum; the route of each element is, of course, a straight line. The Memoir is divided into two parts: the first relating to the plane problem (and to some variations of it): the second part contains a theorem as to congruences, the general theory of the curvature of surfaces, and finally a solution of the solid problem; in regard to this, I find a difficulty which will be referred to further on.

I have said that Monge gives a theorem as to congruences. This is not stated quite in the best form,—viz., instead of speaking of a singly infinite system of lines, or even of the lines drawn according to a given law from the several points of a *surface*, he speaks of the lines