We have, therefore, the required theorem in the form

$$\begin{aligned} & \sum \int_{p_i}^{q_i} dw + \sum_j m_j B_j - \sum_j n_j A_j \\ & + \sum_k g \{ G_0 E_{-g} + G_1 E_{-g-1} + \dots G_{-p..g} E_p \} \\ & + \sum g \{ G_0 E_g + G_1 E_{g-1} + \dots G_{g..} E'_g \} = 0, \end{aligned}$$

$2\pi i$ being a common factor, and so disappearing.

On some Definite Integrals expressible in terms of the First Complete Elliptic Integral and of Gamma Functions. By J. W. L. GLAISHER, M.A., F.R.S.

[Read February 10th, 1881.]

1. *Theorem.*—If the value of the integral

$$\int_0^\infty \phi(t^2) dt$$

be denoted by A , then

$$\int_0^\infty \int_0^\infty \phi(x^2 + 2x^2 y^2 \cos 2\gamma + y^4) dx dy = \frac{1}{2} A \cdot F^1(\sin \gamma);$$

where $F^1(\sin \alpha)$ denotes the complete elliptic integral of the first kind whose modulus is $\sin \gamma$.

To prove this, substitute $\sqrt{c} \cdot t$ for t in the equation

$$A = \int_0^\infty \phi(t^2) dt,$$

and we have $\frac{A}{\sqrt{c}} = \int_0^\infty \phi(ct^2) dt;$

whence, putting $c = 1 - k^2 \sin^2 \theta$,

$$\frac{A}{\sqrt{(1 - k^2 \sin^2 \theta)}} = \int_0^\infty \phi(t^2 - k^2 t^2 \sin^2 \theta) dt;$$

and therefore, integrating with regard to θ between the limits 0 and π ,

$$\begin{aligned} A \int_0^\pi \frac{d\theta}{\sqrt{(1 - k^2 \sin^2 \theta)}} &= \int_0^\pi \int_0^\infty \phi(t^2 - k^2 t^2 \sin^2 \theta) dt d\theta \\ &= \int_0^\pi \int_0^\infty \phi(t^2 - 4k^2 t^2 \sin^2 \frac{1}{2}\theta \cos^2 \frac{1}{2}\theta) dt d\theta. \end{aligned}$$

Transforming the double integral by putting $r^2 = t$ and replacing $\frac{1}{2}\theta$ by

$$\theta, \text{ it becomes } = 4 \int_0^\pi \int_0^{1^*} \phi (r^4 - 4k^2 r^4 \sin^2 \theta \cos^2 \theta) r dr d\theta ;$$

which, transforming from polar to rectangular coordinates,

$$= 4 \int_0^\pi \int_0^\pi \phi (x^4 + 2x^2 y^2 + y^4 - 4k^2 x^2 y^2) dx dy.$$

$$\text{Now} \quad \int_0^\pi \frac{d\theta}{\sqrt{(1-k^2 \sin^2 \theta)}} = 2F^1(k),$$

$$\text{and if } k = \sin \gamma, \quad 1 - 2k^2 = \cos 2\gamma,$$

so that it has been shown that

$$A \cdot 2F^1(\sin \gamma) = 4 \int_0^\pi \int_0^\pi \phi (x^4 + 2x^2 y^2 \cos 2\gamma + y^4) dx dy,$$

which is the result enunciated above.

2. As particular cases, since

$$\int_0^\pi e^{-t^2} dt = \frac{1}{2} \sqrt{\pi},$$

$$\int_0^\pi e^{-\sqrt{t^2}} dt = 1,$$

$$\int_{-\infty}^\infty \frac{dt}{1+t^2} = \frac{1}{2} \pi,$$

$$\text{we have} \quad \int_0^\pi \int_0^\pi e^{-(x^4 + 2x^2 y^2 \cos 2\gamma + y^4)} dx dy = \frac{1}{2} \sqrt{\pi} \cdot F^1(\sin \gamma),$$

$$\int_0^\pi \int_0^\pi e^{-\sqrt{(x^4 + 2x^2 y^2 \cos 2\gamma + y^4)}} dx dy = \frac{1}{2} \cdot F^1(\sin \gamma),$$

$$\int_0^\pi \int_0^\pi \frac{dx dy}{1 + x^4 + 2x^2 y^2 \cos 2\gamma + y^4} = \frac{1}{2} \pi \cdot F^1(\sin \gamma).$$

The first of these evaluations is given among the examples at the end of Chapter xiii. of Todhunter's "Integral Calculus," and it is this result which suggested the present paper.

3. It may also be shown that if, as before,

$$\int_0^\pi \phi(t^2) dt = A,$$

then

$$\int_0^\pi \int_0^\pi \phi \{ (a^2 x^2 + b^2 y^2)(\alpha^2 x^2 + \beta^2 y^2) \} dx dy = \frac{1}{4} A \int_0^{1^*} \frac{d\theta}{\sqrt{(a^2 \beta^2 \cos^2 \theta + \alpha^2 b^2 \sin^2 \theta)}}.$$

$$\text{For} \quad \frac{A}{\sqrt{(p^2 \cos^2 \theta + q^2 \sin^2 \theta)}} = \int_0^\pi \phi(p^2 \cos^2 \theta + q^2 \sin^2 \theta) dt,$$

and therefore

$$\begin{aligned} A \int_0^{1^*} \frac{d\theta}{\sqrt{(p^2 \cos^2 \theta + q^2 \sin^2 \theta)}} &= 2 \int_0^\infty \int_0^{1^*} \phi(p^2 r^4 \cos^2 \theta + q^2 r^4 \sin^2 \theta) r dr d\theta \\ &= 2 \int_0^\infty \int_0^\infty \phi\{(x^2 + y^2)(p^2 x^2 + q^2 y^2)\} dx dy. \end{aligned}$$

Transforming the double integral by substituting $\alpha x, \beta y$ for x, y respectively, it becomes

$$= \alpha \beta \int_0^\infty \int_0^\infty \phi\{(\alpha^2 x^2 + \beta^2 y^2)(p^2 \alpha^2 x^2 + q^2 \beta^2 y^2)\} dx dy;$$

and therefore, putting $a = p\alpha, b = q\beta$, we have

$$A \int_0^{1^*} \frac{d\theta}{\sqrt{(a^2 \beta^2 \cos^2 \theta + \alpha^2 b^2 \sin^2 \theta)}} = 2 \int_0^\infty \int_0^\infty \phi\{(a^2 x^2 + b^2 y^2)(\alpha^2 x^2 + \beta^2 y^2)\} dx dy.$$

4. If
$$\int_0^{1^*} \frac{d\theta}{\sqrt{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)}}$$

be denoted by $F(a, b)$, so that

$$F(a, b) = \frac{1}{a} F^1\left(\frac{\sqrt{(a^2 - b^2)}}{a}\right) = \frac{1}{b} F^1\left(\frac{\sqrt{(b^2 - a^2)}}{b}\right),$$

then it follows from the result in the last section that

$$\begin{aligned} \frac{1}{4} A \cdot F(a, b) &= \int_0^\infty \int_0^\infty \phi\{(ax^2 + by^2)(bx^2 + ay^2)\} dx dy \\ &= \int_0^\infty \int_0^\infty \phi\{(x^2 + a^2 y^2)(x^2 + b^2 y^2)\} dx dy. \end{aligned}$$

5. In the process in § 1, the limits of the integration with regard to θ were 0 and π , and $\frac{1}{2}\theta$ was then replaced by θ , so that x, y were equivalent to $r \cos \frac{1}{2}\theta, r \sin \frac{1}{2}\theta$ of the original variables; but in § 3 the limits of integration were 0 and $\frac{1}{2}\pi$, and x, y represented $r \cos \theta, r \sin \theta$.

In order to connect the results derived from the two processes, transform that given in § 1 by putting $\sqrt{(aa)}.x, \sqrt{(bb)}.y$ for x, y ; we thus have, from § 1,

$$\int_0^\infty \int_0^\infty \phi\{(a^2 \alpha^2 x^2 + 2aa b \beta x^2 y^2 \cos 2\gamma + b^2 \beta^2 y^4)\} dx dy = \frac{\frac{1}{2}A}{\sqrt{(aab\beta)}} F^1(\sin \gamma),$$

and from § 3,

$$\int_0^\infty \int_0^\infty \phi\{(a^2 x^2 + b^2 y^2)(\alpha^2 x^2 + \beta^2 y^2)\} dx dy = \frac{1}{4} A \cdot F(a\beta, a\alpha).$$

If the expressions subject to the sign ϕ in these two formulæ are the same,

$$2aa\beta\cos 2\gamma = a^3\beta^2 + a^2b^2,$$

whence

$$\cos 2\gamma = \frac{1}{2} \left(\frac{a\beta}{ab} + \frac{ab}{a\beta} \right),$$

and therefore

$$\sin \gamma = i \frac{\frac{1}{2}(a\beta - ab)}{\sqrt{(aa\beta\beta)}}.$$

Thus the two formulæ are in agreement, if

$$\frac{1}{\sqrt{(aa\beta\beta)}} F^1 \left(i \frac{\frac{1}{2}(a\beta - ab)}{\sqrt{(aa\beta\beta)}} \right) = F(a\beta, ab).$$

Now the left-hand member of this equation

$$\begin{aligned} &= \int_0^{1\pi} \frac{d\theta}{\sqrt{\{aa\beta\beta + \{\frac{1}{2}(a\beta - ab)\}^2 \sin^2 \theta\}}} \\ &= \int_0^{1\pi} \frac{d\theta}{\sqrt{\{aa\beta\beta \cos^2 \theta + \{\frac{1}{2}(a\beta + ab)\}^2 \sin^2 \theta\}}} \\ &= F\left\{\sqrt{(aa\beta\beta)}, \frac{1}{2}(a\beta + ab)\right\}, \end{aligned}$$

and this $= F(a\beta, ab)$ in virtue of Gauss's Theorem of the Arithmetico-geometric Mean.

6. It is thus evident that the transformations in § 1 and § 3, afford a proof of this theorem of Gauss's, viz., that

$$F(a, b) = F\left\{\sqrt{(ab)}, \frac{1}{2}(a+b)\right\}.$$

The proof may be exhibited as follows,

$$\begin{aligned} A \int_0^{1\pi} \frac{d\theta}{\sqrt{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)}} &= \int_0^\infty \int_0^{1\pi} \phi(a^2 t^2 \cos^2 \theta + b^2 t^2 \sin^2 \theta) dt d\theta, \\ &= 2 \int_0^\infty \int_0^\infty \phi\{(x^2 + y^2)(a^2 x^2 + b^2 y^2)\} dx dy, \\ &= 2 \int_0^\infty \int_0^\infty \phi\{(ax^2 + by^2)(bx^2 + ay^2)\} dx dy \dots (1), \end{aligned}$$

$$\begin{aligned} \text{and also } 2A \int_0^{1\pi} \frac{d\theta}{\sqrt{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)}} \\ &= \int_0^\infty \int_0^\infty \phi\{a^2 t^2 + 4(b^2 - a^2) t^2 \sin^2 \frac{1}{2}\theta \cos^2 \frac{1}{2}\theta\} dt d\theta \\ &= 4 \int_0^\infty \int_0^\infty \phi\{x^4 (x^4 + y^4) + 2(2b^2 - a^2) x^2 y^2\} dx dy \dots (2). \end{aligned}$$

Now, if we put $a^2 = a\beta, \quad b = \frac{1}{2}(a + \beta),$

the expression subject to the sign ϕ in (2)

$$\begin{aligned} &= \alpha\beta (x^4 + y^4) + (\alpha^2 + \beta^2) x^2 y^2, \\ &= (\alpha x^2 + \beta y^2) (\beta x^2 + \alpha y^2), \end{aligned}$$

and therefore, from (2),

$$\begin{aligned} A \int_0^{1\pi} \frac{d\theta}{\sqrt{(\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta)}} &= 2 \int_0^\infty \int_0^\infty \phi \{ (\alpha x^2 + \beta y^2) (\beta x^2 + \alpha y^2) \} dx dy \\ &= A \int_0^{1\pi} \frac{d\theta}{\sqrt{(\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta)}}, \text{ from (1),} \end{aligned}$$

that is,

$$F\{\alpha\beta, \tfrac{1}{2}(\alpha + \beta)\} = F(\alpha, \beta).$$

$$7. \text{ If } \int_0^\infty \phi(t^2) dt = B,$$

then

$$\frac{B}{\sqrt[3]{c}} = \int_0^\infty \phi(ct^2) dt,$$

and therefore, putting as before $c = 1 - k^2 \sin^2 \theta$,

$$\frac{B}{\sqrt[3]{(1 - k^2 \sin^2 \theta)}} = 2 \int_0^\infty \phi(r^2 - k^2 r^2 \sin^2 \theta) r dr.$$

Integrating with regard to θ between the limits 0 and π ,

$$2B \int_0^{1\pi} \frac{d\theta}{\sqrt[3]{(1 - k^2 \sin^2 \theta)}} = 4 \int_0^\infty \int_0^\infty \phi(r^2 - 4k^2 r^2 x^2 y^2) dx dy.$$

The quantity $\sqrt[3]{(1 - k^2 \sin^2 \theta)}$ is usually denoted by $\Delta\theta$, so that the integral on the left-hand side of this equation

$$= \int_0^{1\pi} \frac{d\theta}{(\Delta\theta)^{\frac{1}{3}}} = \int_0^K \frac{dn u}{(\operatorname{dn} u)^{\frac{1}{3}}} du,$$

on putting $\theta = \operatorname{am} u$,

$$= \int_0^K \sqrt[3]{\operatorname{dn} u} du,$$

and therefore

$$\begin{aligned} \tfrac{1}{2} B \int_0^K \sqrt[3]{\operatorname{dn} u} du &= \int_0^\infty \int_0^\infty \phi \{ x^6 + y^6 + (3 - 4k^2) x^2 y^2 (x^2 + y^2)^{\frac{1}{2}} \} dx dy, \\ &= \int_0^\infty \int_0^\infty \phi \left\{ x^6 + \frac{\sin 3\gamma}{\sin \gamma} x^2 y^2 (x^2 + y^2)^{\frac{1}{2}} + y^6 \right\} dx dy, \end{aligned}$$

where, as before, $k = \sin \gamma$.

If, instead of integrating from 0 to π , we integrate from 0 to $\frac{1}{2}\pi$, or from 0 to $\frac{3}{2}\pi$, and substitute θ for $\frac{1}{2}\theta$, we find that, in the last equation, we may replace the expression subject to the sign ϕ by

$$x^6 + (3 - k^2) x^4 y^2 + (3 - 2k^2) x^2 y^4 + (1 - k^2) y^6,$$

or by

$$x^6 + 3(1 - 3k^2) x^4 y^2 + 3(1 + 2k^2) x^2 y^4 + (1 - k^2) y^6.$$

8. Similarly, if

$$C = \int_0^{\pi} \phi(t) dt,$$

then, by integrating from 0 to π , and from 0 to 2π , and substituting θ for $\frac{1}{2}\theta$ and $\frac{1}{4}\theta$ respectively, we find that

$$\begin{aligned} \frac{1}{2}C \int_0^{\pi} \frac{d\theta}{\sqrt{(\Delta\theta)}} &= \frac{1}{4}C \int_0^{\pi} \sqrt{(\operatorname{dn} u)} du \\ &= \int_0^{\pi} \int_0^{\pi} \phi \{x^4 + 4(1-k^2)x^2y^2(x^4+y^4) + 2(3-4k^2)x^4y^4+y^8\} dx dy \\ &= \int_0^{\pi} \int_0^{\pi} \phi \{x^8 + 4(1-4k^2)x^2y^2(x^4+y^4) + 2(3+16k^2)x^4y^4+y^8\} dx dy, \end{aligned}$$

and so on.

9. As particular cases of the foregoing results, we have

$$\begin{aligned} \int_0^{\pi} \int_0^{\pi} \phi(x^4+y^4) dx dy &= \frac{1}{4}A \cdot F^1\left(\frac{1}{\sqrt{2}}\right), \\ \int_0^{\pi} \int_0^{\pi} \phi(x^8+y^8) dx dy &= \frac{1}{2}B \int_0^{\pi} \sqrt{(\operatorname{dn} u)} du \quad \left(k = \frac{\sqrt{3}}{2}\right), \\ \int_0^{\pi} \int_0^{\pi} \phi(x^8-2x^4y^4+y^8) dx dy &= \frac{1}{2}C \int_0^{\pi} \sqrt{(\operatorname{dn} u)} du \quad (k=1), \\ \int_0^{\pi} \int_0^{\pi} \phi(x^8+x^6y^2+x^2y^6+y^8) dx dy &= \frac{1}{2}C \int_0^{\pi} \sqrt{(\operatorname{dn} u)} du \quad \left(k = \frac{\sqrt{3}}{2}\right), \\ \int_0^{\pi} \int_0^{\pi} \phi(x^8+14x^4y^4+y^8) dx dy &= \frac{1}{2}C \int_0^{\pi} \sqrt{(\operatorname{dn} u)} du \quad \left(k = \frac{1}{2}\right). \end{aligned}$$

Since, when $k=1$,

$$\int_0^{\pi} \sqrt{(\operatorname{dn} u)} du = \int_0^{\pi} \frac{d\theta}{\sqrt{(\cos \theta)}} = \sqrt{2} \cdot F^1\left(\frac{1}{\sqrt{2}}\right),$$

the third of these equations may be written

$$\int_0^{\pi} \int_0^{\pi} \phi(x^8-2x^4y^4+y^8) dx dy = \frac{1}{\sqrt{2}} C \cdot F^1\left(\frac{1}{\sqrt{2}}\right).$$

Also, putting $k=1$ in §7,

$$\int_0^{\pi} \int_0^{\pi} \phi(U) dx dy = \int_0^{\pi} \sqrt{(\operatorname{dn} u)} du \quad (k=1),$$

where

$$U = x^8 - x^4y^2 - x^2y^4 + y^8,$$

or

$$= x^8 + 2x^4y^2 + x^2y^4,$$

or

$$= x^8 - 6x^4y^2 + 9x^2y^4;$$

and, when $k=1$,
$$\int_0^{\pi} \sqrt{(\operatorname{dn} u)} du = \int_0^{\pi} \frac{d\theta}{(\cos \theta)^{\frac{1}{2}}}.$$

the value of which integral is*

$$\frac{3}{\sqrt[3]{3}} F^1(\sin 15^\circ) = \frac{3}{\sqrt[3]{3}} F^1\left\{\frac{1}{2}\sqrt{(2-\sqrt{3})}\right\}.$$

10. Taking $\phi(t) = e^{-t}$, or, as it is convenient to write it, $\exp(-t)$, since

$$\int_0^\infty \exp(-t^n) dt = \Gamma\left(1 + \frac{1}{n}\right),$$

we have

$$A = \Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi},$$

$$B = \Gamma\left(1 + \frac{1}{3}\right) = \frac{1}{3}\Gamma\left(\frac{1}{3}\right),$$

$$C = \Gamma\left(1 + \frac{1}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right);$$

and therefore

$$\int_0^\infty \int_0^\infty \exp\{-(x^4 + y^4)\} dx dy = \frac{1}{4}\sqrt{\pi} \cdot F^1\left(\frac{1}{\sqrt{2}}\right),$$

$$\int_0^\infty \int_0^\infty \exp\{-(x^6 + y^6)\} dx dy = \frac{1}{6}\Gamma\left(\frac{1}{3}\right) \cdot \int_0^K \sqrt[3]{(dn u)} du \quad \left(k = \frac{\sqrt{3}}{2}\right),$$

$$\int_0^\infty \int_0^\infty \exp\{-(x^8 - 2x^4y^4 + y^8)\} dx dy = \frac{1}{8}\Gamma\left(\frac{1}{4}\right) \cdot \sqrt{2} F^1\left(\frac{1}{\sqrt{2}}\right),$$

&c. &c.

$$\begin{aligned} \text{Since } \int_0^\infty \int_0^\infty \exp\{-(x^n + y^n)\} dx dy &= \int_0^\infty \exp(-x^n) dx \cdot \int_0^\infty \exp(-y^n) dy \\ &= \Gamma^2\left(1 + \frac{1}{n}\right), \end{aligned}$$

the first two of these formulæ give

$$\frac{1}{\sqrt[3]{8}} \Gamma^2\left(\frac{1}{4}\right) = \frac{1}{4}\sqrt{\pi} \cdot F^1\left(\frac{1}{\sqrt{2}}\right),$$

$$\frac{1}{\sqrt[3]{8}} \Gamma^2\left(\frac{1}{6}\right) = \frac{1}{6}\Gamma\left(\frac{1}{3}\right) \int_0^K \sqrt[3]{(dn u)} du \quad \left(k = \frac{\sqrt{3}}{2}\right);$$

whence

$$F^1\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4} \frac{\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{\pi}} = \frac{1}{4} \frac{\Gamma^2\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)},$$

$$\int_0^K \sqrt[3]{(dn u)} du = \frac{1}{6} \frac{\Gamma^2\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \quad \left(k = \frac{\sqrt{3}}{2}\right),$$

the first of which is a well-known result.

The integral in the second equation may be written also in the form

$$\int_0^{1\pi} \frac{d\theta}{\sqrt[3]{(1 - \frac{3}{4}\sin^2\theta)}}, \quad = \sqrt[3]{4} \cdot \int_0^{1\pi} \frac{d\theta}{\sqrt[3]{(1 + 3\sin^2\theta)}}.$$

11. It may be observed that, since

$$\int_0^\infty \int_0^\infty \exp\{-(a^n x^n + b^n y^n)\} dx dy = \frac{1}{ab} \Gamma^2\left(1 + \frac{1}{n}\right),$$

* Legendre, *Traité des Fonctions Elliptiques*, t. i. p. 67.

we have

$$\int_0^\infty \int_0^{2\pi} \exp \{ -r^n (a^n \cos^n \theta + b^n \sin^n \theta) \} r dr d\theta = \frac{1}{ab} \Gamma^3 \left(1 + \frac{1}{n} \right);$$

$$\begin{aligned} \text{and therefore } \int_0^{2\pi} \frac{d\theta}{\sqrt[n]{(a^n \cos^n \theta + b^n \sin^n \theta)}} &= \frac{2}{ab} \frac{\Gamma^3 \left(1 + \frac{1}{n} \right)}{\Gamma \left(1 + \frac{2}{n} \right)} \\ &= \frac{1}{nab} \frac{\Gamma^3 \left(\frac{1}{n} \right)}{\Gamma \left(\frac{2}{n} \right)}. \end{aligned}$$

On the Tangents drawn from a Point to a Nodal Cubic.

By R. A. ROBERTS, B.A.

[Read Jan. 13th, 1881.]

The equation of a nodal cubic, referred to the triangle formed by the inflexional tangents, may be written $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, or in tangential coordinates $\lambda^{-\frac{1}{3}} + \mu^{-\frac{1}{3}} + \nu^{-\frac{1}{3}} = 0$. If we combine with this last equation the equation of a point $x\lambda + y\mu + z\nu = 0$, we get a biquadratic which determines the tangents drawn from (x, y, z) to the cubic.

Let us consider the cubics $f = au^3 + bv^3 + cw^3$, $f' = a'u^3 + b'v^3 + c'w^3$, where $u + v + w = 0$. Then the discriminant of $f + kf'$ is seen to be $(a + ka')^{-\frac{1}{3}} + (b + kb')^{-\frac{1}{3}} + (c + kc')^{-\frac{1}{3}} = 0$. Let us suppose that this equation in k coincides with the biquadratic found above; we must have, then, $\lambda = a + ka'$, $\mu = b + kb'$, $\nu = c + kc'$; and, since $x\lambda + y\mu + z\nu = 0$, identically,

$$\frac{x}{(bc')} = \frac{y}{(ca')} = \frac{z}{(ab')} \dots\dots\dots (1).$$

Now, the invariants, S and T , of the equation in k are expressed in terms of the combinants, P and Q , of the cubics f and f' , thus (Salmon's *Higher Algebra*, Art. 204),

$$S = 3P(P^3 - 24Q),$$

$$T = -(P^6 - 36P^3Q + 216Q^2).$$

But

$$P = (bc') + (ca') + (ab'),$$

$$Q = (ab')(bc')(ca').$$

Hence, from (1),

$$P = x + y + z,$$

$$Q = xyz;$$

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