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LXXVIII. *On the Depression due to a Load at the Centre of an Elastic Chain tightly stretched between Two Points in the same Horizontal Plane.* By CHARLES H. LEES, D.Sc.*

THE ease and convenience of determining Young's Modulus for a material by stretching a wire between two supports in the same horizontal plane, and measuring the depression at its centre produced by a load suspended there †, has led me to enquire more carefully into the theory of the method, with a view to defining the limits within which the method is applicable. In the theoretical work known to me, the weight and flexural rigidity of the wire are neglected, so that it may be taken as two straight lines extending from the point at which the load is applied to the two supports.

When the weight of the wire is taken into account but its flexural rigidity still neglected, the case becomes that of a chain, the links of which increase in length under longitudinal stress by stretching of their longitudinal parts without bending of their transverse parts. The curve of equilibrium of the chain on each side of the load is then a catenary, the lowest point of which is beyond the point of application of the load. When the load is withdrawn the whole chain becomes a catenary with its central lowest point at a distance below its ends which depends on the weight of the chain and the stretching force applied.

Let AB (fig. 1) be the two points of support at a distance $2l$ apart, C_1 the centre of the chain, M the mass suspended at it, D the lowest point of the catenary of which BC_1 forms an arc. Let the axis EO of the catenary be at a distance c below D, and let $EO = e$.

The equation to the catenary in terms of rectangular coordinates with O as origin is

$$y = c \cosh \frac{e + x}{c}.$$

* Communicated by the Author.

† Mr. G. F. Stradling of Philadelphia has been good enough to furnish me with some notes on the history of this method. It was first used by s'Gravesande, *Physices Elementa Mathematica*, 1742, to determine the breaking stress in wires; then by E. Tacke, *Dissertation*, Griefswald, 1889, who considered it unreliable; and afterwards by Stradling, *Ann. der Physik*, xli. p. 330 (1890), who found it both reliable and convenient. Mr. Grime and I came to the same conclusion as the result of the experiments we described in the February number of this Magazine.

Hence if y_1 is the ordinate of C_1 and $y_1 + b$ that of B , we have

$$\begin{aligned} b &= c \left(\cosh \frac{e+l}{c} - \cosh \frac{e}{c} \right) \\ &= 2c \sinh \frac{l}{2c} \sinh \frac{2e+l}{2c}. \quad . \quad . \quad . \quad (1) \end{aligned}$$

Fig. 1.

If s is the length of the curve from C_1 to B

$$s = c \left(\sinh \frac{e+l}{c} - \sinh \frac{e}{c} \right) = 2c \sinh \frac{l}{2c} \cosh \frac{2e+l}{2c}. \quad (2)$$

If θ is the angle of inclination of the catenary at any point to the x axis

$$\tan \theta = \frac{dy}{dx} = \sinh \frac{e+x}{c}.$$

Also if T is the stretching force at any point

$$T = \sigma g y = \sigma g c \cosh \frac{e+x}{c}, \quad . \quad . \quad . \quad (3)$$

where σ is the mass of the chain per unit length.

When the chain is unloaded the centre of the catenary is

at C_0 , a point at a distance b_0 below C, and

$$b_0 = c_0 \left(\cosh \frac{l}{c_0} - 1 \right) \quad . \quad . \quad . \quad (1')$$

$$s_0 = c_0 \sinh \frac{l}{c_0} \quad . \quad . \quad . \quad (2')$$

$$\text{and} \quad T_0 = \sigma g c_0 \cosh \frac{x}{c_0} \quad . \quad . \quad . \quad (3')$$

Since the object of the experimental method is to secure a comparatively large depression CC_1 for a small increase in the length of AC_1B , the initial stretching force T_0 should be such as to make the chain nearly straight. The maximum value of T must be within the elastic limit of the material of the chain. In most cases this will mean that the extension cannot exceed $1/1000$ of the original length. The value of CC_1 cannot therefore exceed $1/\sqrt{1000} = 1/32$ of CB , *i. e.* the upper limit to $\tan \theta$ will be $1/32$.

Hence the upper limit of $\sinh \frac{e}{c}$ and therefore of $\frac{e}{c} = 1/32$.

Since the difference in height of C_1 and B is small we may take the stretching force T to be constant throughout the chain, and equal to its value T' at C' , a point half way between B and C, *i. e.* for which $x = l/2$.

$$\text{Hence} \quad T = T' = \sigma g c \cosh \frac{2e + l}{2c}.$$

Since there is equilibrium at C' we must have

$$2T' \sin \theta' = (M + \sigma l)g,$$

where θ' is the inclination of the curve at C' to the axis of X.

But

$$\tan \theta' = \sinh \frac{2e + l}{2c}.$$

Hence

$$\sin \theta' = \tanh \frac{2e + l}{2c}.$$

Therefore

$$2T' \tanh \frac{2e + l}{2c} = (M + \sigma l)g, \quad . \quad . \quad . \quad (4)$$

and

$$c \sinh \frac{2e + l}{2c} = \frac{M + \sigma l}{2\sigma} \quad . \quad . \quad . \quad (5)$$

When the chain is unloaded these equations become

$$2T'_0 \tanh \frac{l}{2c} = \sigma l g, \quad . \quad . \quad . \quad (4')$$

and

$$c_0 \sinh \frac{l}{2c} = \frac{l}{2} \quad . \quad . \quad . \quad (5')$$

If ϵ is Young's Modulus for the material of the chain, a the total area of the transverse section of the longitudinal part of a link, T_0' the initial stretching force, T' that when a mass M is suspended from the centre of the chain,

$$\frac{s}{1 + \frac{T'}{a\epsilon}} = \frac{s_0}{1 + \frac{T_0'}{a\epsilon}}$$

where s_0 and s are the half-lengths of the chain in the two cases.

Since $\frac{T'}{a\epsilon}$ never exceeds $1/1000$ we may write this equation

$$s = s_0 \left(1 + \frac{T' - T_0'}{a\epsilon} \right).$$

Or

$$T' - T_0' = a\epsilon \left(\frac{s}{s_0} - 1 \right). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence substituting from equations (4), (4'), (2), and (2'), we have

$$\frac{(M + \sigma l)g}{2} \coth \frac{2e + l}{2c} - \frac{\sigma l g}{2} \coth \frac{l}{2c_0} = a\epsilon \left\{ \frac{c \sinh \frac{l}{2c} \cosh \frac{2e + l}{2c}}{c_0 \sinh \frac{l}{2c_0} \cosh \frac{l}{2c_0}} - 1 \right\}.$$

Or, expanding the hyperbolic functions,

$$\begin{aligned} \frac{(M + \sigma l)g}{2} \frac{2c}{2e + l} \left(1 + \frac{1}{3} \left(\frac{2e + l}{2c} \right)^2 \dots \right) - \frac{\sigma l g}{2} \cdot \frac{2c_0}{l} \left(1 + \frac{1}{3} \left(\frac{l}{2c_0} \right)^2 \dots \right) \\ = a\epsilon \left\{ \frac{1 + \frac{1}{2} \left(\frac{2e + l}{2c} \right)^2 \dots}{1 + \frac{1}{2} \left(\frac{l}{2c_0} \right)^2 \dots} - 1 \right\}. \end{aligned}$$

Hence, since $\left(\frac{2e + l}{2c} \right)^2$ never exceeds $1/1000$,

$$\frac{(M + \sigma l)g}{2} \frac{2c}{2e + l} - \sigma g c_0 = \frac{a\epsilon}{2} \left\{ \left(\frac{2e + l}{2c} \right)^2 - \left(\frac{l}{2c_0} \right)^2 \right\}. \quad . \quad (7)$$

Expanding similarly equations (5) and (1), we have

$$c \frac{2e + l}{2c} \left(1 + \frac{1}{3} \left(\frac{2e + l}{2c} \right)^2 \right) = \frac{M + \sigma l}{2\sigma}, \quad \text{i. e.} \quad \frac{2e + l}{2} \left(1 + \frac{1}{3} \left(\frac{2e + l}{2c} \right)^2 \right) = \frac{M + \sigma l}{2\sigma},$$

$$\text{and} \quad 2c \cdot \frac{l}{2c} \left(1 + \frac{1}{3} \left(\frac{l}{2c} \right)^2 \dots \right) \cdot \frac{2e + l}{2c} \left(1 + \frac{1}{3} \left(\frac{2e + l}{2c} \right)^2 \dots \right) = b,$$

$$\text{i. e.} \quad \frac{2e + l}{2c} \left(1 + \frac{1}{3} \frac{(2e + l)^2 + l^2}{4c^2} + \dots \right) = \frac{b}{l}.$$

Hence so long as $\frac{b}{l}$ does not exceed $1/30$ we may write

$$e = \frac{M}{2\sigma},$$

and
$$\frac{2e+l}{2e} = \frac{b}{l},$$

i. e.
$$\frac{e}{c} = \frac{b}{l} \frac{2e}{2e+l} = \frac{b}{l} \cdot \frac{\frac{M}{2\sigma}}{\frac{M}{2\sigma} + l} = \frac{b}{l} \cdot \frac{M}{M + \sigma l},$$

From equation (1') we have similarly

$$c_0 = \frac{l^2}{2b_0}.$$

Hence equation (7) becomes

$$\frac{(M + \sigma l) g l}{2} \frac{1}{b} - \sigma g \frac{l^2}{2b_0} = \frac{a\epsilon}{2} \left\{ \frac{b^2}{l^2} - \frac{b_0^2}{l^2} \right\}.$$

Let $\sigma l = m$ half the mass of the chain, then

$$\frac{(M + m)}{b} - \frac{m}{b_0} = \frac{a\epsilon}{gl^3} (b^2 - b_0^2), \quad \dots \quad (8)$$

a cubic equation to determine the depression b if M , m , a , ϵ , l , and b_0 are known. From it we have:—

$$\epsilon = \frac{gl^3}{a} \frac{\frac{M+m}{b} - \frac{m}{b_0}}{b^2 - b_0^2} \dots \dots \dots (9)$$

Or if M_1 and M_2 are any two masses suspended, b_1 , b_2 , the depressions produced

$$\epsilon = \frac{gl^3}{a} \frac{\frac{M_1+m}{b_1} - \frac{M_2+m}{b_2}}{b_1^2 - b_2^2}, \quad \dots \quad (10)$$

an equation to determine ϵ from observation of b .

From equation (10) it is evident that the simple theory which considers the chain to be without mass will give values for ϵ correct to about 1 part in 300 for depressions which do not exceed $\frac{1}{30}$ of the length of the chain, if the masses actually suspended from the centre of the chain be supposed increased by half the mass of the chain itself, and are not too nearly alike to make the difference between the values of $\frac{M+m}{b}$ less than one-third that of either.

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The distances b in the above equation are measured vertically from the horizontal straight line joining the points of support. The position of this zero point on the scale on which the depressions are measured is best determined by tilting the frame holding the chain or wire till the chain or wire is vertical, clamping it in position under a tension sufficient to keep it straight, then taking the measurements while the chain or wire is still vertical. If the clamping and measurement cannot be done with the wire in a vertical position, the zero may be determined with the chain or wire, the frame and the scale on which the measurements are taken all horizontal, so that the plane of bending of the wire is at right angles to the line of measurement.

In the case of the thin wires used by Mr. Grime and myself* the mass of the wire was neglected. If taken into account it makes a difference in the final value of ϵ which in most cases is less than 1 per cent. The experiment with the thickest wire used, no. 27 S.W.G. copper, and the lowest initial tension $T_0 = 380$ g.† is recalculated below, the zero from which the depressions are measured being calculated from the value of the initial tension T_0 .

Copper wire, no. 27 S.W.G., half length 26.50 cm.

Mean diameter .0412 cm. Area of cross-section .00133 sq. cm.

Density of copper 8.95. Mass per cm. $\sigma = .0119$.

Mass of half the wire $m = 315$ grams.

$$c_0 = \frac{T_0}{\sigma g} = 31930 \text{ cms.}$$

$$b_0 = c_0 \left(\cosh \frac{l}{c_0} - 1 \right) = \frac{l^2}{2c_0} = .0110 \text{ cm.}$$

M.	M + m.	b	M + m.	M ₁ + m	M ₂ + m	b ² .	b ₁ ² - b ₂ ² .	$\frac{M_1 + m}{b_1} - \frac{M_2 + m}{b_2}$	ϵ .
grs.	grs.	cms.	$\frac{M + m}{b}$	$\frac{M_1 + m}{b_1}$	$\frac{M_2 + m}{b_2}$			$\frac{b_1}{b_1^2 - b_2^2} - \frac{b_2}{b_1^2 - b_2^2}$	
0	315	.0110	28.64			.0001			
15	15.32	.3740	40.96	21.64		.1399	.2534	85.4	1.17×10^{12}
25	25.32	.5035	50.28	18.05		.2535	.2183	82.7	1.13 „
35	35.32	.5985	59.01			.3582			

Mean $\epsilon \dots 1.15 \times 10^{12}$.

The result obtained by neglecting the mass of the wire was 1.17×10^{12} . In this case the more accurate gives a result 2 per cent. less than that obtained by the simpler theory.

* Phil. Mag. ix. p. 258 (1905).

† Ibid. p. 264.