XXVI.—On a Group of Linear Differential Equations of the 2nd Order, including Professor Chrystal's Seiche-equations. By J. Halm, Ph.D., Lecturer on Astronomy in the University of Edinburgh.

(MS. received May 20, 1905. Read June 19, 1905. Issued separately July 31, 1905.)

It is readily seen that the two differential equations

$$(1-w^2) \frac{d^2y}{dw^2} + n(n-1)y = 0 \quad (1)$$

$$(1+x^2)^2 \frac{d^2y}{dx^2} + n(n+2)y = 0 \quad (2)$$

which play an important rôle in Professor CHRYSTAL'S mathematical theory of the Seiches, are special cases of the more general type

$$(1-w^2) \frac{d^2y}{dw^2} - (2\alpha+1)w \frac{dy}{dw} + n(n+2\alpha)y = 0. \quad (3)$$

With regard to the first, the Seiche-equation, this becomes at once apparent by writing $\alpha = -\frac{1}{2}$. Equation (2), on the other hand, which we may briefly call the STOKES-equation [see Professor CHRYSTAL'S paper on "Some further Results in the Mathematical Theory of Seiches," *Proc. Roy. Soc. Edin.*, vol. xxv.] will be recognised as a special case ($\alpha = +1$) of the equation

$$(1+x^2)^2 \frac{d^2y}{dx^2} - (2\alpha-2)x(1+x^2) \frac{dy}{dx} + n(n+2\alpha)y = 0 \quad (4)$$

which is transformed into (3) by the substitution $x = \frac{w}{(1-w^2)^{\frac{1}{2}}}$.

It appears, therefore, that the Seiche- as well as the STOKES-equation belong to the same family of differential equations whose general form is given by (3). We may write the latter also

$$\frac{d^2y}{dz^2} - 2\alpha \tan z \frac{dy}{dz} + n(n+2\alpha)y = 0 \quad (5)$$

if we substitute $w = \sin z$ or $x = \tan z$. Corresponding to this equation we have further :

$$\frac{d^2y}{dz^2} + 2\alpha \tanh z \frac{dy}{dz} + n(n+2\alpha)y = 0 \quad (6)$$

which for $w = \sinh z$ and $\alpha = -\frac{1}{2}$ leads to the hyperbolic Seiche-equation :

$$(1+w^2) \frac{d^2y}{dw^2} + n(n-1)y = 0 \quad (7)$$

and for $x = \tanh z$ and $\alpha = +1$ to the "hyperbolic" STOKES-equation :

$$(1-x^2)^2 \frac{d^2y}{dx^2} + n(n+2)y = 0. \quad (8)$$

From (5) and (6) the solutions of the STOKES-equation may at once be derived. If we write, in accordance with Professor CHRYSTAL's notations, $n(n+2) = 4\delta^2 - 1$, and consider that (5) is identical with

$$\frac{d^2[y \cos z]}{dz^2} + (n+1)^2 [y \cos z] = 0,$$

we find the general solution

$$y = \frac{1}{\cos z} \left[C \cos(2\delta z) + D \sin(2\delta z) \right],$$

or since $z = \tan^{-1} x$,

$$y = (1+x^2)^{\frac{1}{2}} [C \cos(2\delta \tan^{-1} x) + D \sin(2\delta \tan^{-1} x)]. \quad (9)$$

In the same way we find from (6)

$$\frac{d^2[y \cosh z]}{dz^2} + (n^2 - 2n - 1) [y \cosh z] = 0,$$

and, writing $n^2 + 2n - 1 = 4\delta^2$,

$$y = \frac{1}{\cosh z} \left[C' \cos(2\delta z) + D' \sin(2\delta z) \right]$$

which, since $z = \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$, becomes

$$y = (1-x^2)^{\frac{1}{2}} \left[C' \cos\left(\delta \log \frac{1+x}{1-x}\right) + D' \sin\left(\delta \log \frac{1+x}{1-x}\right) \right]. \quad (10)$$

(9) and (10) are identical with (26) and (12) of Professor CHRYSTAL's paper quoted above if $\frac{x}{a}$ is substituted for x .

It is also at once evident that if we express the STOKES-functions by means of the variable w instead of x , we find the general solutions:

$$y_1 = A \frac{d}{dw} \left[\cos(2\delta \sin^{-1} w) \right] + B \frac{d}{dw} \left[\sin(2\delta \sin^{-1} w) \right] = \frac{dY_1}{dw}, \text{ say,}$$

and

$$y_2 = A' \frac{d}{dw} \left[\cos(2\delta \sinh^{-1} w) \right] + B' \frac{d}{dw} \left[\sin(2\delta \sinh^{-1} w) \right] = \frac{dY_2}{dw}.$$

This result is made evident if we consider that Y_1 and Y_2 satisfy the well-known differential equations

$$(1-w^2) \frac{d^2 Y_1}{dw^2} - w \frac{dY_1}{dw} + 4\delta^2 Y_1 = 0$$

$$(1+w^2) \frac{d^2 Y_2}{dw^2} + w \frac{dY_2}{dw} + 4\delta^2 Y_2 = 0,$$

which, by differentiation with regard to w , assume the form:

$$(1-w^2) \frac{d^2 y_1}{dw^2} - 3w \frac{dy_1}{dw} + (4\delta^2 - 1)y_1 = 0$$

$$(1+w^2) \frac{d^2 y_2}{dw^2} + 3w \frac{dy_2}{dw} + (4\delta^2 + 1)y_2 = 0,$$

and become identical with the STOKES-equations if x is introduced instead of w .

Seeing that it is possible to refer the four differential equations of the Seiche-problem to one and the same general equation, viz., that given under (3), it may be of interest to discuss here the properties of this in many ways remarkable general type, and to derive its particular solutions. The results obtained seem, on the one hand, to be of practical value in regard to the computations involved in the evolution of the periods and nodes of the seiches, while, on the other hand, the mathematical investigation here given establishes an important relation between the Seiche-functions discovered by Professor CHRYSTAL and other functions frequently used in mathematical and physical problems, notably those of LEGENDRE, a relation which increases the importance of the Seiche-functions from the mathematical point of view. It will be recognised at once that equation (3) is a special case of the hypergeometric differential equation

$$v(1-v)\frac{d^2y}{dv^2} + [\gamma - (\alpha + \beta + 1)v]\frac{dy}{dv} - \alpha\beta y = 0, \quad (11)$$

whose solution is represented by the contour-integral

$$y = \text{const} \times \int_{\mathbf{c}} u^{\alpha-\gamma} (1-u)^{\gamma-\beta-1} (u-v)^{-\alpha} du \quad (12)$$

Indeed, if we substitute $v = \frac{1-w^2}{2}$ and $\alpha = n + 2\alpha$, $\beta = -n$, $\gamma = n + \frac{1}{2}$ we have

$$(1-w^2)\frac{d^2y}{dw^2} - (2n+1)\frac{dy}{dw} + n(n+2\alpha)y = 0 \quad (3)$$

which, by introducing $u = \frac{1-t}{2}$ under the sign of integration in (12), is satisfied by the integral

$$y = \text{const} \times \int_{\mathbf{c}} \frac{(1-t^2)^{n+\alpha-\frac{1}{2}}}{(t-w)^{n+2\alpha}} dt. \quad (13)$$

If we write

$$y = (1-w^2)^{\frac{1}{2}-\alpha} \cdot Y,$$

we find easily that Y is a solution of the equation :

$$(1-w^2)\frac{d^2Y}{dw^2} + (2\alpha-3)\frac{dY}{dw} + (n+1)(n+2\alpha-1)Y = 0,$$

and hence, substituting in (11) and (12) $\alpha = n + 1$, $\beta = 1 - n - 2\alpha$, $\gamma = \frac{3}{2} - \alpha$:

$$Y = \text{const} \times \int_{\mathbf{c}} \frac{(1-t^2)^{n+\alpha-\frac{1}{2}}}{(t-w)^{n+1}} dt,$$

so that we find as another solution of (3) :

$$y = \text{const} \times (1-w^2)^{\frac{1}{2}-\alpha} \int_{\mathbf{c}} \frac{(1-t^2)^{n+\alpha-\frac{1}{2}}}{(t-w)^{n+1}} dt. \quad (14)$$

For $\alpha = \frac{1}{2}$, the integrals (13) and (14) become identical, viz.,

$$y = \text{const} \times \int_{\mathbf{c}} \frac{(1-t^2)^n}{(t-w)^{n+1}} dt.$$

This is SCHLÄFLI'S contour-integral of the LEGENDRE-functions, which thus is seen to satisfy the differential equation :

$$(1-w^2)\frac{d^2y}{dw^2}-2\frac{dy}{dw}+n(n+1)y=0. \quad (15)$$

Our general equation (3) includes therefore also the LEGENDRE-functions. If we denote the solution of (3) by the symbol $C_n^a(w)$, the SEICHE-functions are represented by $C_n^{-\frac{1}{2}}(w)$, the LEGENDRE-functions by $C_n^{\frac{1}{2}}(w)$, and the STOKES-functions by $C_n^1(w)$.

Since (3) is a special case of the hypergeometric differential equation, its integrals can be at once expressed by hypergeometric series of the type $F(\alpha, \beta, \gamma; w)$. From JACOBI'S schematic table of particular solutions given in his "Untersuchungen über die Differentialgleichung der hypergeometrischen Reihe," *Crelle's Journal*, vol. lvi., we obtain thus the following 24 possible integrals of equation (3):—

GROUP I.

1. $F\left(n+2a, -n, a+\frac{1}{2}; \frac{1-w}{2}\right)$
2. $\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a} \times F\left(\frac{1}{2}-n-a, \frac{1}{2}+n+a, a+\frac{1}{2}; \frac{1-w}{2}\right)$
3. $\left(\frac{1+w}{2}\right)^{-n-2a} \times F\left(n+2a, \frac{1}{2}+n+a, a+\frac{1}{2}; \frac{w-1}{w+1}\right)$
4. $\left(\frac{1+w}{2}\right)^n \times F\left(-n, \frac{1}{2}-n-a, a+\frac{1}{2}; \frac{w-1}{w+1}\right)$

GROUP II.

1. $\left(\frac{1-w}{2}\right)^{-n-2a} \times F\left(n+2a, \frac{1}{2}+n+a, a+\frac{1}{2}; \frac{w+1}{w-1}\right)$
2. $\left(\frac{1-w}{2}\right)^n \times F\left(-n, \frac{1}{2}-n-a, a+\frac{1}{2}; \frac{w+1}{w-1}\right)$
3. $F\left(n+2a, -n, a+\frac{1}{2}; \frac{1+w}{2}\right)$
4. $\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a} \times F\left(\frac{1}{2}+n+a, \frac{1}{2}-n-a, a+\frac{1}{2}; \frac{1+w}{2}\right)$

GROUP III.

1. $\left(\frac{1-w}{2}\right)^{-n-2a} \times F\left(n+2a, \frac{1}{2}+n+a, 2n+2a+1; \frac{2}{1-w}\right)$
2. $\left(\frac{1-w}{2}\right)^{-n-a-\frac{1}{2}} \left(\frac{1+w}{2}\right)^{\frac{1}{2}-a} \times F\left(1+n, \frac{1}{2}+a+n, 2n+2a+1; \frac{2}{1-w}\right)$
3. $\left(\frac{1+w}{2}\right)^{-n-2a} \times F\left(n+2a, \frac{1}{2}+n+a, 2n+2a+1; \frac{2}{1+w}\right)$
4. $\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a} \left(\frac{1+w}{2}\right)^{-n-a-\frac{1}{2}} \times F\left(\frac{1}{2}+n+a, 1+n, 2n+2a+1; \frac{2}{1+w}\right)$

(16)

GROUP IV.

1. $\left(\frac{1-w}{2}\right)^n \times F\left(-n, \frac{1}{2}-n-a, 1-2n-2a; \frac{2}{1-w}\right)$
2. $\left(\frac{1-w}{2}\right)^{n+a-1} \left(\frac{1+w}{2}\right)^{1-a} \times F\left(1-n-2a, \frac{1}{2}-n-a, 1-2n-2a; \frac{2}{1-w}\right)$
3. $\left(\frac{1+w}{2}\right)^n \times F\left(-n, \frac{1}{2}-n-a, 1-2n-2a; \frac{2}{1+w}\right)$
4. $\left(\frac{1-w}{2}\right)^{1-a} \left(\frac{1+w}{2}\right)^{n+a-1} \times F\left(\frac{1}{2}-n-a, 1-n-2a, 1-2n-2a; \frac{2}{1+w}\right)$

GROUP V.

1. $\left(\frac{1-w}{2}\right)^{1-a} \times F\left(\frac{1}{2}+n+a, \frac{1}{2}-n-a, \frac{3}{2}-a; \frac{1-w}{2}\right)$
2. $\left(\frac{1-w}{2}\right)^{1-a} \left(\frac{1+w}{2}\right)^{1-a} \times F\left(1-n-2a, 1+n, \frac{3}{2}-a; \frac{1-w}{2}\right)$
3. $\left(\frac{1-w}{2}\right)^{1-a} \left(\frac{1+w}{2}\right)^{-1-n-a} \times F\left(\frac{1}{2}+n+a, 1+n, \frac{3}{2}-a; \frac{w-1}{w+1}\right)$
4. $\left(\frac{1-w}{2}\right)^{1-a} \left(\frac{1+w}{2}\right)^{n+a-1} \times F\left(\frac{1}{2}-n-a, 1-n-2a, \frac{3}{2}-a; \frac{w-1}{w+1}\right)$

GROUP VI.

1. $\left(\frac{1-w}{2}\right)^{n+a-1} \left(\frac{1+w}{2}\right)^{1-a} \times F\left(\frac{1}{2}-n-a, 1-n-2a, \frac{3}{2}-a; \frac{w+1}{w-1}\right)$
2. $\left(\frac{1-w}{2}\right)^{-n+a-1} \left(\frac{1+w}{2}\right)^{1-a} \times F\left(\frac{1}{2}+n+a, 1+n, \frac{3}{2}-a; \frac{w+1}{w-1}\right)$
3. $\left(\frac{1+w}{2}\right)^{1-a} \left(\frac{1-w}{2}\right)^{1-a} \times F\left(1-n-2a, 1+n, \frac{3}{2}-a; \frac{1+w}{2}\right)$
4. $\left(\frac{1+w}{2}\right)^{1-a} \times F\left(\frac{1}{2}-n-a, \frac{1}{2}+n+a, \frac{3}{2}-a; \frac{1+w}{2}\right)$

That these integrals cannot all exist at the same time is evident, the convergence of the F-series depending on the values of n and c . It is easy, however, to find in each single case those series which represent convergent solutions of the differential equation. With regard to this point I may refer to JACOBI'S treatise, from which the conditions of convergence may at once be obtained. We notice that the above solutions may be arranged in pairs, which differ only by $-w$ being substituted for the positive value. The necessity of the existence of such pairs is obvious, since the differential equation (3) remains unaltered if $-w$ is substituted for $+w$. Since we are also permitted to write $-n-2c$ for n without changing the equation (3), we have on the whole the following eight particular solutions, from which the others are obtained by the substitutions just mentioned.

$$\begin{aligned}
& F\left(n+2a, -n, a+\frac{1}{2}; \frac{1-w}{2}\right) \\
& \left(\frac{1+w}{2}\right)^{1-a} F\left(\frac{1}{2}+n+a, \frac{1}{2}-n-a, a+\frac{1}{2}; \frac{1-w}{2}\right) \\
& \left(\frac{1+w}{2}\right)^n F\left(-n, \frac{1}{2}-n-a, a+\frac{1}{2}; \frac{w-1}{w+1}\right) \\
& \left(\frac{1+w}{2}\right)^n F\left(-n, \frac{1}{2}-n-a, 1-2n-2a; \frac{2}{1+w}\right) \\
& \left(\frac{1-w}{2}\right)^{1-a} \left(\frac{1+w}{2}\right)^{n+a-\frac{1}{2}} F\left(\frac{1}{2}-n-a, 1-n-2a, 1-2n-2a; \frac{2}{1+w}\right) \\
& \left(\frac{1-w}{2}\right)^{1-a} \times F\left(\frac{1}{2}+n+a, \frac{1}{2}-n-a, \frac{3}{2}-a; \frac{1-w}{2}\right) \\
& \left(\frac{1-w}{2}\right)^{1-a} \left(\frac{1+w}{2}\right)^{1-a} \times F\left(1-n-2a, 1+n, \frac{3}{2}-a; \frac{1-w}{2}\right) \\
& \left(\frac{1-w}{2}\right)^{1-a} \left(\frac{1+w}{2}\right)^{n+a-\frac{1}{2}} \times F\left(1-n-2a, \frac{1}{2}-n-a, \frac{3}{2}-a; \frac{w-1}{w+1}\right)
\end{aligned} \tag{16a}$$

Note added on June 30.—The sixth integral in (16a) agrees with Professor CHRYSTAL's Lake function, which is obtained by substituting $a = -\frac{1}{2}$, $n(n-1) = c$ and $\frac{1-w}{2} = z$, so that $L(c, z) = z F(n, 1-n, 2; z)$. The other corresponding integral is represented by No. 1 of (16a), viz., $F(n-1, n, 0; z)$, but it belongs to the exceptional class $\gamma = 0$ of the hypergeometric series, and has a logarithmic form (see Professor CHRYSTAL "On the Hydrodynamical Theory of Seiches," § 41). The corresponding solution of the LEGENDRE-function ($\alpha = \frac{1}{2}$) is $F(n+1, -n, 1; \frac{1-w}{2})$, a well-known expression for the LEGENDRE-function of the first kind. (See WHITTAKER, *Modern Analysis*, § 118.) In this case Nos. 1 and 6 of (16a) are identical.

Reverting to equation (11), we notice that it may be transformed into (3) by still another substitution, viz., by $v = w^2$, $\alpha = \frac{n}{2} + a$, $\beta = -\frac{n}{2}$, and $\gamma = \frac{1}{2}$. Hence we obtain 24 other particular integrals of the following type:—

GROUP I.

1. $F\left(\frac{n}{2}+a, -\frac{n}{2}, \frac{1}{2}; w^2\right)$
2. $(1-w^2)^{1-a} \times F\left(\frac{1}{2}-\frac{n}{2}-a, \frac{1}{2}+\frac{n}{2}, \frac{1}{2}; w^2\right)$
3. $(1-w^2)^{\frac{n}{2}-a} \times F\left(\frac{n}{2}+a, \frac{1}{2}+\frac{n}{2}, \frac{1}{2}; \frac{w^2}{w^2-1}\right)$
4. $(1-w^2)^{\frac{n}{2}} \times F\left(-\frac{n}{2}, \frac{1}{2}-\frac{n}{2}-a, \frac{1}{2}; \frac{w^2}{w^2-1}\right)$

GROUP II.

1. $w^{-n-2a} \times F\left(\frac{n}{2} + a, \frac{1}{2} + \frac{n}{2} + a, \frac{1}{2} + a; \frac{w^2 - 1}{w^2}\right)$
2. $w^n \times F\left(-\frac{n}{2}, \frac{1}{2} - \frac{n}{2}, \frac{1}{2} + a; \frac{w^2 - 1}{w^2}\right)$
3. $F\left(\frac{n}{2} + a, -\frac{n}{2}, \frac{1}{2} + a; (1 - w^2)\right)$
4. $w \times F\left(\frac{1}{2} + \frac{n}{2} + a, \frac{1}{2} - \frac{n}{2}, \frac{1}{2} + a; (1 - w^2)\right)$

GROUP III.

1. $w^{-n-2a} \times F\left(\frac{n}{2} + a, \frac{1}{2} + \frac{n}{2} + a, n + a + 1; \frac{1}{w^2}\right)$
2. $w^{-n-1}(1 - w^2)^{1-a} \times F\left(1 + \frac{n}{2}, \frac{1}{2} + \frac{n}{2}, n + a + 1; \frac{1}{w^2}\right)$
3. $(1 - w^2)^{-\frac{n}{2}-a} \times F\left(\frac{n}{2} + a, \frac{1}{2} + \frac{n}{2}, n + a + 1; \frac{1}{1 - w^2}\right)$
4. $w(1 - w^2)^{-\frac{n}{2}-a-\frac{1}{2}} \times F\left(\frac{1}{2} + \frac{n}{2} + a, 1 + \frac{n}{2}, n + a + 1; \frac{1}{1 - w^2}\right)$

GROUP IV.

1. $w^n \times F\left(-\frac{n}{2}, \frac{1}{2} - \frac{n}{2}, 1 - n - a; \frac{1}{w^2}\right)$ (17)
2. $w^{n+2a-1}(1 - w^2)^{1-a} \times F\left(1 - \frac{n}{2} - a, \frac{1}{2} - \frac{n}{2} - a, 1 - n - a; \frac{1}{w^2}\right)$
3. $(1 - w^2)^{\frac{n}{2}} \times F\left(-\frac{n}{2}, \frac{1}{2} - \frac{n}{2} - a, 1 - n - a; \frac{1}{1 - w^2}\right)$
4. $w(1 - w^2)^{\frac{n}{2}-\frac{1}{2}} \times F\left(\frac{1}{2} - \frac{n}{2}, 1 - \frac{n}{2} - a, 1 - n - a; \frac{1}{1 - w^2}\right)$

GROUP V.

1. $w \cdot \times F\left(\frac{1}{2} + \frac{n}{2} + a, \frac{1}{2} - \frac{n}{2}, \frac{3}{2}; w^2\right)$
2. $w(1 - w^2)^{1-a} \times F\left(1 - \frac{n}{2} - a, 1 + \frac{n}{2}, \frac{3}{2}; w^2\right)$
3. $w(1 - w^2)^{-\frac{n}{2}-a-\frac{1}{2}} \times F\left(\frac{1}{2} + \frac{n}{2} + a, 1 + \frac{n}{2}, \frac{3}{2}; \frac{w^2}{w^2 - 1}\right)$
4. $w(1 - w^2)^{\frac{n}{2}-\frac{1}{2}} \times F\left(\frac{1}{2} - \frac{n}{2}, 1 - \frac{n}{2} - a, \frac{3}{2}; \frac{w^2}{w^2 - 1}\right)$

GROUP VI.

1. $w^{n+2a-1}(1 - w^2)^{1-a} \times F\left(\frac{1}{2} - \frac{n}{2} - a, 1 - \frac{n}{2} - a, \frac{3}{2} - a; \frac{w^2 - 1}{w^2}\right)$
2. $w^{-n-1}(1 - w^2)^{1-a} \times F\left(\frac{1}{2} + \frac{n}{2}, 1 + \frac{n}{2}, \frac{3}{2} - a; \frac{w^2 - 1}{w^2}\right)$
3. $(1 - w^2)^{1-a} \times F\left(\frac{1}{2} - \frac{n}{2} - a, \frac{1}{2} + \frac{n}{2}, \frac{3}{2} - a; (1 - w^2)\right)$
4. $w(1 - w^2)^{1-a} \times F\left(1 - \frac{n}{2} - a, 1 + \frac{n}{2}, \frac{3}{2} - a; (1 - w^2)\right)$

The first of Group I. corresponds to Professor CHRYSTAL's Seiche Cosine, and the first of Group V. to the Seiche Sine. Since for the Seiche-functions $\alpha = -\frac{1}{2}$, we have

$$\text{Seiche Cosine} = F\left(\frac{n-1}{2}, -\frac{n}{2}, \frac{1}{2}; w^2\right)$$

$$\text{Seiche Sine} = w F\left(\frac{n}{2}, \frac{1}{2}, -\frac{n}{2}, \frac{3}{2}; w^2\right)$$

or

$$\text{Seiche Cosine} = 1 - \frac{n(n-1)}{1.2}w^2 + \frac{n(n-1)[n(n-1)-1.2]}{1.2.3.4}w^4 - \frac{n(n-1)[n(n-1)-1.2][n(n-1)-3.4]}{1.1 \times 3.4 \times 5.6}w^6 + \dots$$

$$\text{Seiche Sine} = w - \frac{n(n-1)}{2.3}w^3 + \frac{n(n-1)[n(n-1)-2.3]}{2.3 \times 4.5}w^5 - \frac{n(n-1)[n(n-1)-2.3][n(n-1)-4.5]}{2.3 \times 4.5 \times 6.7}w^7 + \dots$$

or finally, if we write for $n(n-1)$, which is the factor of y in the Seiche-equation, the symbol c , in accordance with Professor CHRYSTAL's notation :

$$\text{Seiche Cosine} = 1 - \frac{c}{1.2}w^2 + \frac{c(c-1.2)}{1.2 \times 3.4}w^4 - \frac{c(c-1.2)(c-3.4)}{1.2 \times 3.4 \times 5.6}w^6 + \dots$$

$$\text{Seiche Sine} = w - \frac{c}{2.3}w^3 + \frac{c(c-2.3)}{2.3 \times 4.5}w^5 - \frac{c(c-2.3)(c-4.5)}{2.3 \times 4.5 \times 6.7}w^7 + \dots$$

Let us denote generally

$$\text{Cos}_a(w) = F\left(\frac{n}{2} + a, -\frac{n}{2}, \frac{1}{2}; w^2\right)$$

$$\text{Sin}_a(w) = w F\left(\frac{1}{2} + \frac{n}{2} + a, \frac{1}{2}, -\frac{n}{2}, \frac{3}{2}; w^2\right) \quad (18)$$

then we have :

$$\text{Cos}_{-1}(w) = \text{Seiche Cosine} = 1 - \frac{(c+1.0)}{1.2}w^2 + \frac{(c+1.0)(c-1.2)}{1.2 \times 3.4}w^4 - \frac{(c+1.0)(c-1.2)(c-3.4)}{1.2 \times 3.4 \times 5.6}w^6 + \dots$$

$c = n(n-1)$

$$\text{Cos}_0(w) = \text{Cos}(n \sin^{-1} w) = 1 - \frac{(c+0.0)}{1.2}w^2 + \frac{(c+0.0)(c-2.2)}{1.2 \times 3.4}w^4 - \frac{(c+0.0)(c-2.2)(c-4.4)}{1.2 \times 3.4 \times 5.6}w^6 + \dots$$

$c = n^2$

$$\text{Cos}_1(w) = \text{Legendre Cosine} = 1 - \frac{(c-1.0)}{1.2}w^2 + \frac{(c-1.0)(c-3.2)}{1.2 \times 3.4}w^4 - \frac{(c-1.0)(c-3.2)(c-5.4)}{1.2 \times 3.4 \times 5.6}w^6 + \dots$$

$c = n(n+1)$

$$\text{Cos}_1(w) = \text{Stokes Cosine} = 1 - \frac{(c-2.0)}{1.2}w^2 + \frac{(c-2.0)(c-4.2)}{1.2 \times 3.4}w^4 - \frac{(c-2.0)(c-4.2)(c-6.4)}{1.2 \times 3.4 \times 5.6}w^6 + \dots$$

$c = n(n+2)$ (19)

$$\text{Sin}_{-1}(w) = \text{Seiche Sine} = w - \frac{(c-0.1)}{2.3}w^3 + \frac{(c-0.1)(c-2.3)}{2.3 \times 4.5}w^5 - \frac{(c-0.1)(c-2.3)(c-4.5)}{2.3 \times 4.5 \times 6.7}w^7 + \dots$$

$c = n(n-1)$

$$\text{Sin}_0(w) = \frac{1}{n} \sin(n \sin^{-1} w) = w - \frac{(c-1.1)}{2.3}w^3 + \frac{(c-1.1)(c-3.3)}{2.3 \times 4.5}w^5 - \frac{(c-1.1)(c-3.3)(c-5.5)}{2.3 \times 4.5 \times 6.7}w^7 + \dots$$

$c = n^2$

$$\text{Sin}_1(w) = \text{Legendre Sine} = w - \frac{(c-2.1)}{2.3}w^3 + \frac{(c-2.1)(c-4.3)}{2.3 \times 4.5}w^5 - \frac{(c-2.1)(c-4.3)(c-6.5)}{2.3 \times 4.5 \times 6.7}w^7 + \dots$$

$c = n(n+1)$

$$\text{Sin}_1(w) = \text{Stokes Sine} = w - \frac{(c-3.1)}{2.3}w^3 + \frac{(c-3.1)(c-5.3)}{2.3 \times 4.5}w^5 - \frac{(c-3.1)(c-5.3)(c-7.5)}{2.3 \times 4.5 \times 6.7}w^7 + \dots$$

$c = n(n+2)$

From (18) we conclude that

$$\begin{aligned}\text{Cos}_a(1) &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-a)}{\Gamma(\frac{1}{2}-\frac{n}{2}-a)\Gamma(\frac{1}{2}+\frac{n}{2})} \\ \text{Sin}_a(1) &= \frac{1}{2} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-a)}{\Gamma(1-\frac{n}{2}-a)\Gamma(1+\frac{n}{2})}\end{aligned}\quad (20)$$

Hence

$$\begin{aligned}\text{Cos}_0(1) = \cos n \frac{\pi}{2} &= \frac{(\Gamma(\frac{1}{2}))^2}{\Gamma(\frac{1}{2}-\frac{n}{2})\Gamma(\frac{1}{2}+\frac{n}{2})}; \quad \Gamma(\frac{1}{2}-\frac{n}{2})\Gamma(\frac{1}{2}+\frac{n}{2}) = \frac{\pi}{\cos \frac{n}{2}\pi} \\ \text{Sin}_0(1) = \frac{1}{n} \sin n \frac{\pi}{2} &= \frac{(\Gamma(\frac{1}{2}))^2}{n\Gamma(1-\frac{n}{2})\Gamma(\frac{n}{2})}; \quad \Gamma(1-\frac{n}{2})\Gamma(\frac{n}{2}) = \frac{\pi}{\sin \frac{n}{2}\pi}\end{aligned}$$

which are well-known relations between the Γ -functions.

From (20) we see at once that

$$\begin{aligned}\text{Cos}_a(1) &= 0, \text{ if } n(n+2a) = 1, 3, 5, \dots \\ \text{Sin}_a(1) &= 0, \text{ if } n(n+2a) = 2, 4, 6, \dots\end{aligned}$$

Applying this result to the Seiche-functions, we find

$$\begin{aligned}\text{Cos}_{-1}(c, 1) &= 0 \text{ for } c = 1, 2, 3, 4, 5, 6, \dots \\ \text{Sin}_{-1}(c, 1) &= 0 \text{ for } c = 2, 3, 4, 5, 6, 7, \dots\end{aligned}$$

If in equation (3) n represents a positive integer, the functions $C_n^a(w)$ have a peculiar significance. They are then the coefficients of the powers h^n in the series

$$(1-2hw+h^2)^{-a} = \sum_0^\infty h^n C_n^a(w).$$

Now, according to an important theorem

$$\frac{d^n f(w)}{dw^n} = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-w)^{n+1}} dt,$$

if $f(t)$ represents a function which is regular within the contour C (see WHITTAKER, *Modern Analysis*, p. 53). Since we had before (by (13) and (14)) that

$$C_n^a(w) = \text{const} \times \int_C \frac{(1-t^2)^{n+a-1} dt}{(t-w)^{n+2a}} = \text{const} (1-w^2)^{1-a} \int_C \frac{(1-t^2)^{n+a-1}}{(t-w)^{n+1}} dt,$$

we notice that $C_n^a(w)$ is proportional to

$$\frac{d^{n+2a-1}}{dw^{n+2a-1}} \left\{ (1-w^2)^{n+a-1} \right\}$$

and also to

$$(1-w^2)^{1-a} \frac{d^n}{dw^n} \left\{ (1-w^2)^{n+a-1} \right\}.$$

We verify without difficulty the following relations:—

$$\begin{aligned}C_n^a(w) &= (-2)^n \frac{a(a+1)(a+2) \dots (a+n-1)}{1.2.3 \dots (2a+2n-1)} (-1)^{n-1} \frac{d^{n+2a-1}}{dw^{n+2a-1}} \left\{ (1-w^2)^{n+a-1} \right\} \\ &= (-2)^n \frac{a(a+1)(a+2) \dots (a+n-1)}{n!(2n+2a-1)(2n+2a-2) \dots (n+2a)} (1-w^2)^{1-a} \frac{d^n}{dw^n} \left\{ (1-w^2)^{n+a-1} \right\}\end{aligned}\quad (21)$$

The Seiche-polynomials $C_n^{-1}(w)$ are therefore represented by the coefficients of the ascending powers in the series

$$(1 - 2hw + h^2)^{-1} = \sum_0^{\infty} h^n C_n^{-1}(w)$$

where

$$\begin{aligned} C_n^{-1}(w) &= (-1)^{n-1} \frac{1.3.5.7 \dots (2n-3)}{1.2.3 \dots (2n-2)} \frac{d^{n-2}}{dw^{n-2}} \left\{ (1-w^2)^{n-1} \right\} \\ &= \frac{(-1)^{n-1}}{(n-1)!} \cdot \frac{1.3.5.7 \dots (2n-3)}{1.2.3 \dots (2n-2)} (1-w^2) \frac{d^n}{dw^n} \left\{ (1-w^2)^{n-1} \right\} \end{aligned} \quad (22)$$

The following relations between contiguous C-functions may be mentioned (see WHITTAKER, *l.c.*, p. 236).

$$\begin{aligned} w C_{n-1}^{1+a}(w) - C_{n-2}^{1+a}(w) &= \frac{n}{2a} C_n^a(w) \\ \frac{dC_n^a(w)}{dw} &= 2a C_{n-1}^{1+a}(w) \\ C_n^{1+a}(w) - w C_{n-1}^{1+a}(w) &= -\frac{n+2a}{2a} C_n^a(w) \\ n C_n^a(w) &= (n-1+2a) w C_{n-1}^a(w) - 2a(1-w^2) C_{n-2}^{a-1}(w). \end{aligned} \quad (23)$$

In the case $a = 0$ the functions $C_n^0(w)$ are of course zero, since

$$(1 - 2hw + h^2)^0 = 1.$$

But it can be shown that the limiting values of $\frac{C_n^0(w)}{a}$ ($\lim a = 0$) represent the coefficients in the expansion

$$\frac{1}{2} \log(1 - 2hw + h^2),$$

and we derive the following relation:

$$\begin{aligned} \log \sqrt{1 - 2h \sin z + h^2} &= \sum_{n=1}^{\infty} (-1)^n \left[\frac{h^{2n-1}}{2n-1} \sin(2n-1)z - \frac{h^{2n}}{2n} \cos 2nz \right] \\ w &= \sin z. \end{aligned} \quad (24)$$

The relations (24) are of particular interest in the Seiche-theory because they lead to elegant expressions for the horizontal and vertical displacements ξ and ζ . If we write the Seiche-equation referring to a parabolic concave lake

$$(1-w^2) \frac{d^2 P}{dw^2} + n(n-1)P = 0$$

we have, using Professor CHRYSTAL's notations,

$$\begin{aligned} (-1)^{n-1} 2.4 \dots (2n-2) \cdot \frac{\zeta}{\sin nt} &= -\frac{2A}{h(1-w^2)} \frac{d^{n-2}}{dw^{n-2}} \left\{ (1-w^2)^{n-1} \right\} = \frac{2A}{hn(n-1)} \frac{d^n}{dw^n} \left\{ (1-w^2)^{n-1} \right\} \\ (-1)^{n-1} 2.4 \dots (2n-2) \cdot \frac{\xi}{\sin nt} &= \frac{2A}{a} \frac{d^{n-1}}{dw^{n-1}} \left\{ (1-w^2)^{n-1} \right\} \end{aligned}$$

UNINODAL SEICHE.

$$n = 2, n(n-1) = 1.2$$

$$2\xi = -\frac{2\Lambda}{2h} \cdot \frac{d^2}{dw^2} \left\{ (1-w^2) \right\} \sin n_1 t, \xi = \frac{\Lambda}{h} \sin n_1 t$$

$$2\zeta = -\frac{2\Lambda}{a} \cdot \frac{d}{dw} \left\{ (1-w^2) \right\} \sin n_1 t, \zeta = \frac{2\Lambda w}{a} \sin n_1 t.$$

BINODAL SEICHE.

$$n = 3, n(n-1) = 2.3$$

$$2.4\xi = \frac{2\Lambda}{6h} \cdot \frac{d^3}{dw^3} \left\{ (1-w^2)^2 \right\} \sin n_2 t, \xi = \frac{\Lambda}{h} w \sin n_2 t$$

$$2.4\zeta = \frac{2\Lambda}{a} \cdot \frac{d^2}{dw^2} \left\{ (1-w^2)^2 \right\} \sin n_2 t, \zeta = \frac{\Lambda}{a} (3w^2 - 1) \sin n_2 t.$$

TRINODAL SEICHE.

$$n = 4, n(n-1) = 4.3$$

$$2.4.6 \xi = -\frac{\Lambda}{6} \cdot \frac{d^4}{dw^4} \left\{ (1-w^2)^3 \right\} \sin n_3 t; \quad \xi = -\frac{\Lambda}{4h} (1-5w^2) \sin n_3 t \quad (25)$$

$$2.4.6 \zeta = -\frac{2\Lambda}{a} \cdot \frac{d^3}{dw^3} \left\{ (1-w^2)^3 \right\} \sin n_3 t; \quad \zeta = -\frac{\Lambda}{4a} (12-20w^2)w \sin n_3 t.$$

QUADRINODAL SEICHE.

$$n = 5, n(n-1) = 5.4$$

$$2.4.6.8 \xi = \frac{\Lambda}{10h} \cdot \frac{d^5}{dw^5} \left\{ (1-w^2)^4 \right\} \sin n_4 t; \quad \xi = \frac{\Lambda}{4h} w(7w^2-3) \sin n_4 t$$

$$2.4.6.8 \zeta = \frac{2\Lambda}{a} \cdot \frac{d^4}{dw^4} \left\{ (1-w^2)^4 \right\} \sin n_4 t; \quad \zeta = \frac{\Lambda}{4a} (35w^4-30w^2+3) \sin n_4 t.$$

The positions of the nodes are found from the equation $\zeta = 0$, and hence $\frac{d^n}{dw^n} \left\{ (1-w^2)^n \right\} = 0$ for n -nodal seiche, and since this differential-quotient is proportional to the LEGENDRE-polynomial $P_n(w)$, the nodes are also determined by the equation

$$P_n(w) = 0,$$

where $P_n(w)$ may be defined as the coefficient of h^n in the expansion

$$(1-2hw+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(w)h^n.$$

Turning now to the hyperbolic Seiche-functions, we obtain convenient expressions in the form of hypergeometric series by substituting in equation (3) wi for w and $ni - a$ for n , i being the imaginary unit root $\sqrt{-1}$. We have then

$$(1+w^2) \frac{d^2 y}{dw^2} + (2a+1)w \frac{dy}{dw} + (n^2 + a^2)y = 0, \quad (26)$$

the particular solutions of which may at once be taken from (17). Most of these series

are, however, unsuitable on account of the presence of complex factors. The hyperbolic Sine- and Cosine-functions are again represented by the first series of Groups I. and V. These series contain only real terms, as was shown already by Professor CHRYSTAL. We find for the hyperbolic Cosine-function :

$$\begin{aligned} & F\left(\frac{\alpha}{2} + \frac{ni}{2}, \frac{\alpha}{2} - \frac{ni}{2}, \frac{1}{2}; -w^2\right) \\ &= 1 - \frac{n^2 + \alpha^2}{1.2} w^2 + \frac{(n^2 + \alpha^2)[(n^2 + \alpha^2) + 4(\alpha + 1)]}{1.2.3.4} w^4 - \frac{(n^2 + \alpha^2)[n^2 + \alpha^2 + 4(\alpha + 1)][n^2 + \alpha^2 + 8(\alpha + 2)]}{1.2.3.4.5.6} w^6 - \dots \end{aligned}$$

The Seiche Cosine is obtained by substituting $\alpha = -\frac{1}{2}$ and $n^2 + \frac{1}{4} = c$, hence

$$\text{hyp. Seiche Cosine } \mathfrak{C}(c, w) = 1 - \frac{c}{1.2} w^2 + \frac{c(c+1.2)}{1.2.3.4} w^4 - \frac{c(c+1.2)(c+3.4)}{1.2.3.4.5.6} w^6 + \dots \quad (27)$$

while for the hyperbolic Seiche Sine we have

$$\begin{aligned} \mathfrak{S}(c, w) &= w F\left(\frac{1}{4} + \frac{ni}{2}, \frac{1}{4} - \frac{ni}{2}, \frac{3}{2}; -w^2\right) \\ &= w - \frac{c}{2.3} w^3 + \frac{c(c+2.3)}{2.3.4.5} w^5 - \frac{c(c+2.3)(c+4.5)}{2.3.4.5.6.7} w^7 + \dots \end{aligned}$$

Similar expressions are obtained for the corresponding hyperbolic LEGENDRE- and STOKES-functions. The series, by which the functions are represented, converge very slowly. For this reason the theory of convex lakes with parabolic floor has so far remained incomplete, chiefly owing to the difficulty in determining the roots of the equations $\mathfrak{C}(c, 1) = 0$ and $\mathfrak{S}(c, 1) = 0$. To avoid this difficulty Professor CHRYSTAL, in his second communication, proposed a different assumption as to the contour of the floor of the lakes, by which he was enabled to express the problem by the solutions of the STOKES-equation

$$(1 \pm x^2) \frac{d^2 y}{dx^2} + cy = 0,$$

which, as we have seen, are represented by elementary transcendents. In this case, curiously, the convex lake offers the least difficulty, doubtless owing to the fact that its equation belongs to the class which also contains the more tractable parabolic concave lakes. A disadvantage of the STOKES-equations, however, is that the quartic lake profile seems to be a less close approximation to the actual conditions than the parabolic. Even apart from this, it is certainly of importance to discuss the problem on at least two different assumptions in order to estimate the influence of the form of the lake on the periods and nodes. Now the preceding investigation enables us to find any of the infinite number of roots of the equations $\mathfrak{C}(c, 1) = 0$ and $\mathfrak{S}(c, 1) = 0$ with at least a sufficient degree of approximation, and thus places us in a position to find also without any difficulty the periods and nodes of the seiches in lakes with *parabolic convex* floor.

From equation (26) we see that the Seiche-equation, for which $\alpha = -\frac{1}{2}$, lies between the two equations :

$$\left. \begin{aligned} (1+w^2) \frac{d^2 y}{dw^2} - w \frac{dy}{dw} + (n^2+1)y &= 0 & c_{-1} &= n^2+1 \\ (1+w^2) \frac{d^2 y}{dw^2} + w \frac{dy}{dw} + n^2 y &= 0 & c_0 &= n^2 \end{aligned} \right\} \quad (28)$$

The particular solutions of these corresponding to the hyperbolic Seiche Cosine and Seiche Sine are respectively :

$$\left. \begin{aligned} \mathfrak{S}_{-1}(w) &= -\frac{w}{n} \sin [n \log (w + \sqrt{1+w^2})] + \sqrt{1+w^2} \cos [n \log (w + \sqrt{1+w^2})] \\ \mathfrak{S}_1(w) &= \cos [n \log (w + \sqrt{1+w^2})] \\ \text{and} \\ \left(n + \frac{1}{n}\right) \mathfrak{S}_{-1}(w) &= \frac{w}{n} \cos [n \log (w + \sqrt{1+w^2})] + \sqrt{1+w^2} \sin [n \log (w + \sqrt{1+w^2})] \\ n \mathfrak{S}_0(w) &= \sin [n \log (w + \sqrt{1+w^2})] \end{aligned} \right\} \quad (29)$$

We may also at once write down the corresponding particular integrals of the two further differential equations ($\alpha = 1$ and 2) :

$$\left. \begin{aligned} (1+w^2) \frac{d^2 y}{dw^2} + 3w \frac{dy}{dw} + (n^2+1)y &= 0 & c_1 &= n^2+1 \\ (1+w^2) \frac{d^2 y}{dw^2} + 5w \frac{dy}{dw} + (n^2+4)y &= 0, & c_2 &= n^2+4 \end{aligned} \right\} \quad (30)$$

which are respectively :

$$\left. \begin{aligned} \mathfrak{S}_1(w) &= (1+w^2)^{-\frac{1}{2}} \cos [n \log (w + \sqrt{1+w^2})] \\ \mathfrak{S}_2(w) &= (1+w^2)^{-\frac{3}{2}} \left\{ \sqrt{1+w^2} \cos [n \log (w + \sqrt{1+w^2})] - \frac{w}{n} \sin [n \log (w + \sqrt{1+w^2})] \right\} \\ \text{and} \\ n \mathfrak{S}_1(w) &= (1+w^2)^{-\frac{1}{2}} \sin [n \log (w + \sqrt{1+w^2})] \\ \left(n + \frac{1}{n}\right) \mathfrak{S}_2(w) &= (1+w^2)^{-\frac{3}{2}} \left\{ \sqrt{1+w^2} \sin [n \log (w + \sqrt{1+w^2})] + \frac{w}{n} \cos [n \log (w + \sqrt{1+w^2})] \right\}. \end{aligned} \right\} \quad (31)$$

On the other hand, the solutions (29) and (31) are expressed by the following hypergeometric series :

(S-FUNCTIONS.

$$\begin{aligned} 1 - \frac{c_{-1}}{1.2} w^2 + \frac{c_{-1} \cdot c_{-1}}{1.2.3.4} w^4 - \frac{c_{-1} \cdot c_{-1} \cdot (c_{-1} + 2.4)}{1.2.3.4.5.6} w^6 + \dots \\ 1 - \frac{c_0}{1.2} w^2 + \frac{c_0(c_0+2.2)}{1.2.3.4} w^4 - \frac{c_0(c_0+2.2)(c_0+4.4)}{1.2.3.4.5.6} w^6 + \dots \\ 1 - \frac{c_1}{1.2} w^2 + \frac{c_1(c_1+4.2)}{1.2.3.4} w^4 - \frac{c_1(c_1+4.2)(c_1+6.4)}{1.2.3.4.5.6} w^6 + \dots \\ 1 - \frac{c_2}{1.2} w^2 + \frac{c_2(c_2+6.2)}{1.2.3.4} w^4 - \frac{c_2(c_2+6.2)(c_2+8.4)}{1.2.3.4.5.6} w^6 + \dots \end{aligned}$$

$\mathfrak{S}$ -FUNCTIONS.

$$\begin{aligned}
w - \frac{(c_{-1} - 1.1)}{2.3}w^3 + \frac{(c_{-1} - 1.1)(c_{-1} + 1.3)}{2.3.4.5}w^5 - \frac{(c_{-1} - 1.1)(c_{-1} + 1.3)(c_{-1} + 3.5)}{2.3.4.5.6.7}w^7 + \dots & \quad (32) \\
w - \frac{(c_0 + 1.1)}{2.3}w^3 + \frac{(c_0 + 1.1)(c_0 + 3.3)}{2.3.4.5}w^5 - \frac{(c_0 + 1.1)(c_0 + 3.3)(c_0 + 5.5)}{2.3.4.5.6.7}w^7 + \dots \\
w - \frac{(c_1 + 3.1)}{2.3}w^3 + \frac{(c_1 + 3.1)(c_1 + 5.3)}{2.3.4.5}w^5 - \frac{(c_1 + 3.1)(c_1 + 5.3)(c_1 + 7.5)}{2.3.4.5.6.7}w^7 + \dots \\
w - \frac{(c_2 + 5.1)}{2.3}w^3 + \frac{(c_2 + 5.1)(c_2 + 7.3)}{2.3.4.5}w^5 - \frac{(c_2 + 5.1)(c_2 + 7.3)(c_2 + 9.5)}{2.3.4.5.6.7}w^7 + \dots
\end{aligned}$$

Let us now find the roots of the equations $\mathfrak{S}_a(c_a, 1) = 0$ and $\mathfrak{S}_a(c_a, 1) = 0$ for the eight functions given in (29) and (31). For the $\mathfrak{S}$ -functions we have the conditions:

$$\begin{aligned}
\tan[n \log(1 + \sqrt{2})] &= n\sqrt{2}; c_{-1} = n^2 + 1. \\
\cos[n \log(1 + \sqrt{2})] &= 0; c_0 = n^2. \\
\cos[n \log(1 + \sqrt{2})] &= 0; c_1 = n^2 + 1. \\
\tan[n \log(1 + \sqrt{2})] &= n\sqrt{2}; c_2 = n^2 + 4.
\end{aligned} \quad (33)$$

and for the $\mathfrak{S}$ -functions:

$$\begin{aligned}
\cotan[n \log(1 + \sqrt{2})] &= -n\sqrt{2}; c_{-1} = n^2 + 1. \\
\sin[n \log(1 + \sqrt{2})] &= 0; c_0 = n^2. \\
\sin[n \log(1 + \sqrt{2})] &= 0; c_1 = n^2 + 1. \\
\cotan[n \log(1 + \sqrt{2})] &= -n\sqrt{2}; c_2 = n^2 + 4.
\end{aligned}$$

From these conditions the roots c_a are easily obtained. We find

 $\mathfrak{S}$ -FUNCTIONS. $\mathfrak{S}$ -FUNCTIONS.

$$\begin{array}{ll}
c_{-1} = 2.35012; 27.9681; 78.7976; \dots & c_{-1} = 12.06756; 50.2082; \dots \\
c_0 = 3.17627; 28.5865; 79.4068; \dots & c_0 = 12.70508; 50.8204; \dots \\
c_{+1} = 4.17627; 29.5865; 80.4068; \dots & c_{+1} = 13.70508; 51.8204; \dots \\
c_{+2} = 5.35012; 30.9681; 81.7976; \dots & c_{+2} = 15.06756; 53.2082; \dots
\end{array} \quad (34)$$

But obviously the roots in each vertical row may be considered as special values of a certain unknown function of a , so that generally

$$c_a = f(a),$$

from which equation, if $f(a)$ were known, we would obtain the values c_{-1} , c_0 , etc. by substituting $a = -1$, 0 , etc. Now, as long as $f(a)$ may be considered as finite and continuous, the well-known formulæ of numerical interpolation allow us to find intermediate values of c from the given data, without knowing the analytical character of $f(a)$, if a sufficient number of equidistant values of c are at our disposal. We are thus in a position to determine, at least approximately, the roots c_{-1} of the hyperbolic Seiche-functions $\mathfrak{G}$ and $\mathfrak{S}$ by interpolating between the numbers of the above table (34). Performing the necessary calculations, we obtain the following roots:

$$\begin{aligned}
\mathfrak{G}_{-1}(c, 1) = 0; c_{-1} &= 2.74 \dots, 28.23 \dots, 79.05 \dots \\
\mathfrak{S}_{-1}(c, 1) = 0; c_{-1} &= 12.34 \dots, 50.46 \dots
\end{aligned} \quad (35)$$

These interpolated values of the roots of the hyperbolic Seiche-functions are in all probability sufficiently accurate for practical purposes. The smallest roots of each

function have also been computed directly from the series by Professor CHRYSTAL assisted by Dr BURGESS, Professor GIBSON, and Mr HORSEBURGH. They were found to be 2.76 and 12.34, and hence are remarkably close to the values given in (35). The root 28.23 was also verified by Dr BURGESS, Professor GIBSON, and myself. I have convinced myself that the exact value must certainly lie between 28.229 and 28.230, so that the second decimal place of the interpolated value appears to be correct. The method of interpolation here adopted places us therefore in a position to determine the infinite number of roots of the two transcendental equations $\mathfrak{G}_{-1}(c, 1) = 0$, and $\mathfrak{Z}_{-1}(c, 1) = 0$, and thus enables us to determine the periods of the seiches in lakes with convex parabolic floors from Professor CHRYSTAL'S equation:— $T_s = 2\pi a \sqrt{c/g}$ (see *Proc. R.S.E.*, vol. xxv. p. 332).

A similar method of interpolation may be employed in the calculation of the position of the nodes. These are found by first forming the equations

$$\begin{aligned} \frac{d\mathfrak{G}_{-1}(w)}{dw} = 0, \quad \frac{d\mathfrak{G}_0(w)}{dw} = 0, \quad \frac{d\mathfrak{G}_1(w)}{dw} = 0, \quad \frac{d\mathfrak{G}_2(w)}{dw} = 0, \\ \frac{d\mathfrak{Z}_{-1}(w)}{dw} = 0, \quad \frac{d\mathfrak{Z}_0(w)}{dw} = 0, \quad \frac{d\mathfrak{Z}_1(w)}{dw} = 0, \quad \frac{d\mathfrak{Z}_2(w)}{dw} = 0, \end{aligned} \quad (36)$$

and by determining the values of w satisfying these equations under the condition that c_a has the values mentioned in (35). By interpolating between the w thus obtained, we find with sufficient approximation the successive values which satisfy the two Seiche-equations

$$\frac{d\mathfrak{G}_{-1}(w)}{dw} = 0 \quad \text{and} \quad \frac{d\mathfrak{Z}_{-1}(w)}{dw} = 0,$$

and which therefore fulfil the required condition that the vertical displacement $\zeta = 0$. In this way I have found the following positions of the nodes in convex parabolic lakes:

Uninodal Seiche $w = 0$
 Binodal Seiche $w = \pm 0.473 \dots$
 Trinodal Seiche $w = 0; \pm 0.632 \dots$
 Quadrinodal Seiche $w = \pm 0.224 \dots; \pm 0.717 \dots$

It is interesting now to investigate the positions of the nodes under the various assumptions which have been made with regard to the curvature of the floor of the lake. From Professor CHRYSTAL'S investigations and the preceding discussion we find for

Lake with	Uninodal Seiche.	Binodal Seiche.	Trinodal Seiche.	Quadrinodal Seiche.
concave parabolic floor,	$w = 0$	$\pm .577$	$0; \pm .775$	$\pm .340; \pm .862$
plain horizontal ,, .	0	$\pm .500$	$0; \pm .667$	$\pm .350; \pm .750$
convex parabolic ,, .	0	$\pm .473$	$0; \pm .632$	$\pm .224; \pm .717$
convex quartic ,, .	0	$\pm .447$	$0; \pm .600$	$\pm .202; \pm .684$

(37)

The figures show clearly that in lakes with curved floors the nodes are always displaced towards the *shallow* water.

As regards the constants c_v which determine the periods of the various seiches, we have the following values :

Lake with	Uninodal Seiche.	Binodal Seiche.	Trinodal Seiche.	Quadrinodal Seiche.
concave parabolic floor, . . .	2.00	6.00	12.0	20.0
plain horizontal „ . . .	2.47	9.87	22.2	39.5
convex parabolic „ . . .	2.76	12.34	28.2	50.5
convex quartic „ . . .	3.00	15.00	35.0	63.0

(38)

and by virtue of the equation $T_v/T_1 = \sqrt{c_1/c_v}$:

	T_2/T_1	T_3/T_1	T_4/T_1
concave parabolic :	.577	.408	.317
plain horizontal :	.500	.333	.250
convex parabolic :	.472	.312	.234
convex quartic :	.447	.293	.218

(39)

We recognise here, in a more general form, the law found by Professor CHRYSTAL, that in concave lakes the ratio T_v/T_1 is greater and in convex lakes smaller than the corresponding ratio in a lake with plain horizontal floor

The positions of the nodes may be represented in a convenient graphical form, which not only shows clearly their dependence on the curvature of the lake, but at the same time enables us to find the nodes for the curves lying between those here discussed, which are not amenable to direct analytical treatment. In fig. 1 are shown the halves of the vertical longitudinal sections of symmetric lakes. O B represents a , the half-length, and O A the central depth, h , of the lake, whereas A B, A C, A D, and A E signify the intersections of the vertical plane with the concave-parabolic, the plane-horizontal, the convex-parabolic, and the convex-quartic floors. Now, on each of these curves the nodes have been marked by the points B_2, B_3, B_4 , etc., in such a way that for instance the distance of B_2 from A O agrees with the value of w in (43) which refers to the binodal seiche in a lake with concave parabolic floor, *i.e.* $w = \frac{x}{a} = 0.577$. In the same way E_2 is

drawn at a distance 0.447 from A O, thus representing the position of the binode in a convex-quartic lake. Having secured the corresponding four points on each of the curves A B, A C, A D, and A E, we draw the curved lines $B_2 E_2, B_3 E_3$, and $B_4 E_4$, and these lines are obviously the *loci* of the nodes. We recognise at once in all cases the displacements of the nodal points towards the shallow water, a phenomenon specially marked in concave lakes. Let us now take, for instance, a convex lake whose depth is represented by $h\sqrt{1+w^2}$ and is indicated in our diagram by the dotted curve A F. The solutions of the corresponding differential equation

$$\sqrt{1+w^2} \frac{d^2 y}{dm^2} + cy = 0$$

are not known, and hence we are not in a position to compute the nodes and periods of these particular seiches by analytical methods. But approximately the nodes may be now directly found from the diagram, being represented by the points of intersection between the *loci* B E and the curve A F. It would seem, therefore, that by the preceding

investigation we have gained considerable freedom in the selection of curves representing the curvature-conditions of the lake floors, so that we may, by the suggested interpolatory method, assign approximate values to the nodes, and by means of (38) also to the periods, even under conditions which are unmanageable from the rigorous mathematical point of view.*

FIG. 1.

The advantage of the preceding analysis of Professor CHRYSTAL's Seiche-equations lies in the fact that they belong to a class which includes cases where the solutions are expressible by means of simple transcendents. If we write the general differential equations in their most symmetric form :

$$\begin{aligned} (1-w^2)\frac{d^2y}{dw^2} - (\gamma+1)w\frac{dy}{dw} + \left(n^2 - \frac{\gamma^2}{4}\right)y &= 0 \\ (1+w^2)\frac{d^2y}{dw^2} + (\gamma+1)w\frac{dy}{dw} + \left(n^2 + \frac{\gamma^2}{4}\right)y &= 0 \end{aligned} \quad (40)$$

* In § 45 of his Hydrodynamical Theory, Professor CHRYSTAL investigates the case of a rectilinear lake, the floor of which would be represented in the above diagram by the straight line A B. The directions of the nodal loci E B in this figure suggest that the nodes in rectilinear lakes should be more displaced towards the shallow water than in any of the cases here considered, and this conclusion is indeed supported by the numerical results given in § 49 of Professor CHRYSTAL's paper.—(Note added on June 30.)

we may say quite generally, that the solutions can always be expressed by simple transcendents if γ is an *even* integer. The Seiche- and LEGENDRE-functions, on the other hand, belong to differential equations of the same class for which γ is an *odd* integer. Hence each equation of the latter class is included between two equations of the former, and we have seen that through this remarkable property we were enabled, by studying the behaviour of the neighbouring simple transcendents, to form conclusions with regard to the far more difficult Seiche-functions.

In the theory of lakes, the floors of which are composed of two or more parabola with different parameters, the evaluation of the two functions $C(c, 1)$ and $S(c, 1)$ for any given value of c becomes important. The calculation of the series

$$C(c, 1) = 1 - \frac{c}{1.2} + \frac{c(c-2)}{1.2.3.4} - \frac{c(c-2)(c-12)}{1.2.3.4.5.6} + \dots$$

$$S(c, 1) = 1 - \frac{c}{2.3} + \frac{c(c-6)}{2.3.4.5} - \frac{c(c-6)(c-20)}{2.3.4.5.6.7} + \dots$$

is always an exceedingly troublesome process, especially for great values of c . In many cases even the calculation of 100 terms is not sufficient. But the foregoing investigation immediately suggests a rigorous and extremely simple method of computing these quantities. Reverting to equations (18) and (20) we find for the two Seiche-functions ($u = -\frac{1}{2}$) the relations:

$$C(c, 1) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1 - \frac{n}{2}) \Gamma(\frac{1}{2} + \frac{n}{2})}$$

$$S(c, 1) = \frac{1}{2} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2} - \frac{n}{2}) \Gamma(1 + \frac{n}{2})}$$

$; n = \frac{1}{2} + \sqrt{c + \frac{1}{4}}.$

But considering that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

we find

$$C(c, 1) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \frac{\sin n \cdot \frac{\pi}{2}}{\sqrt{\pi}}$$

$$S(c, 1) = -\frac{2}{n(n-1)} \cdot \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \cdot \frac{\cos n \cdot \frac{\pi}{2}}{\sqrt{\pi}}. \quad (41)$$

These expressions are very convenient for computation. Let us take, for instance, $c = 105.0176$, to which corresponds $n = 10.76$. We have

$$C(105.0176; 1) = -\frac{\Gamma(5.38) \sin 68^\circ.4}{\Gamma(5.88) \sqrt{\pi}}$$

$$= -\frac{1.38 \times 2.38 \times 3.38 \times 4.38}{1.88 \times 2.88 \times 3.88 \times 4.88} \frac{\Gamma(1.38)}{\Gamma(1.88)} \cdot \frac{\sin 68^\circ.4}{\sqrt{\pi}}$$

$$S(105.0176; 1) = \frac{2}{105.0176} \cdot \frac{1.88 \times 2.88 \times 3.88 \times 4.88}{1.38 \times 2.38 \times 3.38 \times 4.38} \frac{\Gamma(1.88)}{\Gamma(1.38)} \frac{\cos 68^\circ.4}{\sqrt{\pi}};$$

and, since

$$\log \Gamma(1.38) = \bar{1}.94868 \quad \log \Gamma(1.88) = \bar{1}.94004,$$

we find

$$C(105.0176; 1) = -0.231467; \quad S(105.0176; 1) = +0.008964$$

I need scarcely say that in this case a satisfactory calculation by means of the series is quite impossible. For smaller values of c the computations are of course still more convenient. If, for instance, $c = 14.96$, or $n = 4.4$, we have

$$C(14.96; 1) = \frac{\Gamma(2.2)}{\Gamma(2.7)} \cdot \frac{\sin 36^\circ}{\sqrt{\pi}} = \frac{1.2}{1.7} \frac{\Gamma(1.2)}{\Gamma(1.7)} \frac{\sin 36^\circ}{\sqrt{\pi}} = +0.23654.$$

We notice also the relation

$$S(c, 1) = -\frac{\sin n\pi}{\pi} \cdot \frac{1}{n(n-1)C(c, 1)}. \quad (42)$$

Note added on June 30.—The relations (41) and (42), which follow here immediately from a well-known property of the hypergeometric series $F(a, \beta, \gamma; 1)$, were originally derived by Professor CHRYSTAL from a different point of view. (See §§ 25 and 41 of the *Hydrodynamical Theory*.) His important relation (53) in § 41 may be also obtained from the equation

$$L(c, 1) = F(n, 1-n, 2; 1) = 1/\Gamma(1+n)\Gamma(2-n) = -\frac{\sin n\pi}{n(n-1)\pi},$$

and, in consequence of (42) :

$$L(c, 1) = C(c, 1) \cdot S(c, 1).$$

At the request of Professor CHRYSTAL I subjoin tables from which the numerical values of $C(c, 1)$ and $S(c, 1)$ may be taken for any value of c . If we write

$$\begin{aligned} a &= \frac{1}{4} + \frac{1}{2}\sqrt{c + \frac{1}{4}} \\ \Theta(a) &= \frac{\Gamma(a)}{\Gamma(a + \frac{1}{2})} \frac{\sin a\pi}{\sqrt{\pi}} \\ \Sigma(a) &= -\frac{\Gamma(a + \frac{1}{2})}{\Gamma(a)} \frac{\cos a\pi}{\sqrt{\pi}}, \end{aligned}$$

we find easily from the above formulæ the following relations :

$$\begin{aligned} 0 < a < 1 : C(c, 1) &= -\frac{a + \frac{1}{2}}{a} \cdot \Theta(a+1); \quad \frac{c}{2} S(c, 1) = -\frac{a}{a + \frac{1}{2}} \Sigma(a+1) \\ 1 < a < 2 : C(c, 1) &= \Theta(a) \quad \frac{c}{2} S(c, 1) = \Sigma(a) \\ 2 < a < 3 : C(c, 1) &= -\frac{a - \frac{1}{2}}{a - \frac{1}{2}} \Theta(a-1); \quad \frac{c}{2} S(c, 1) = -\frac{a - \frac{1}{2}}{a - 1} \Sigma(a-1); \\ 3 < a < 4 : C(c, 1) &= \frac{(a-2)(a-1)}{(a-\frac{3}{2})(a-\frac{1}{2})} \Theta(a-2); \quad \frac{c}{2} S(c, 1) = \frac{(a-\frac{3}{2})(a-\frac{1}{2})}{(a-2)(a-1)} \Sigma(a-2), \\ &\dots \dots \dots \end{aligned}$$

where the values of θ and Σ are to be taken from the following Table¹.

TABLE I.

a	$\Theta(a)$	a	$\Theta(a)$	a	$\Theta(a)$	a	$\Theta(a)$	a	$\Theta(a)$
1.00 - 0.00000		1.20 - 0.33511		1.40 - 0.49501		1.60 - 0.45815		1.80 - 0.26474	
1.01 - 0.01988		1.21 - 0.34769		1.41 - 0.49775		1.61 - 0.45163		1.81 - 0.25237	
1.02 - 0.03949		1.22 - 0.35984		1.42 - 0.49999		1.62 - 0.44471		1.82 - 0.23983	
1.03 - 0.05884		1.23 - 0.37152		1.43 - 0.50171		1.63 - 0.43741		1.83 - 0.22712	
1.04 - 0.07790		1.24 - 0.38271		1.44 - 0.50294		1.64 - 0.42975		1.84 - 0.21430	
1.05 - 0.09666		1.25 - 0.39346		1.45 - 0.50366		1.65 - 0.42171		1.85 - 0.20133	
1.06 - 0.11512		1.26 - 0.40370		1.46 - 0.50391		1.66 - 0.41332		1.86 - 0.18824	
1.07 - 0.13326		1.27 - 0.41347		1.47 - 0.50365		1.67 - 0.40457		1.87 - 0.17505	
1.08 - 0.15106		1.28 - 0.42275		1.48 - 0.50292		1.68 - 0.39552		1.88 - 0.16177	
1.09 - 0.16852		1.29 - 0.43153		1.49 - 0.50171		1.69 - 0.38612		1.89 - 0.14842	
1.10 - 0.18563		1.30 - 0.43982		1.50 - 0.50000		1.70 - 0.37643		1.90 - 0.13499	
1.11 - 0.20237		1.31 - 0.44762		1.51 - 0.49785		1.71 - 0.36641		1.91 - 0.12151	
1.12 - 0.21874		1.32 - 0.45490		1.52 - 0.49520		1.72 - 0.35612		1.92 - 0.10800	
1.13 - 0.23473		1.33 - 0.46170		1.53 - 0.49211		1.73 - 0.34555		1.93 - 0.09445	
1.14 - 0.25032		1.34 - 0.46798		1.54 - 0.48858		1.74 - 0.33472		1.94 - 0.08090	
1.15 - 0.26551		1.35 - 0.47375		1.55 - 0.48458		1.75 - 0.32362		1.95 - 0.06734	
1.16 - 0.28029		1.36 - 0.47902		1.56 - 0.48014		1.76 - 0.31228		1.96 - 0.05380	
1.17 - 0.29464		1.37 - 0.48378		1.57 - 0.47527		1.77 - 0.30071		1.97 - 0.04028	
1.18 - 0.30855		1.38 - 0.48804		1.58 - 0.46998		1.78 - 0.28892		1.98 - 0.02680	
1.19 - 0.32206		1.39 - 0.49177		1.59 - 0.46426		1.79 - 0.27692		1.99 - 0.01337	
a	$\Sigma(a)$	a	$\Sigma(a)$	a	$\Sigma(a)$	a	$\Sigma(a)$	a	$\Sigma(a)$
1.00 + 0.50000		1.20 + 0.45171		1.40 + 0.18899		1.60 - 0.20419		1.80 - 0.57178	
1.01 + 0.50281		1.21 + 0.44336		1.41 + 0.17133		1.61 - 0.22463		1.81 - 0.58638	
1.02 + 0.50510		1.22 + 0.43447		1.42 + 0.15336		1.62 - 0.24499		1.82 - 0.60047	
1.03 + 0.50687		1.23 + 0.42503		1.43 + 0.13507		1.63 - 0.26525		1.83 - 0.61406	
1.04 + 0.50811		1.24 + 0.41505		1.44 + 0.11649		1.64 - 0.28536		1.84 - 0.62708	
1.05 + 0.50880		1.25 + 0.40453		1.45 + 0.09765		1.65 - 0.30533		1.85 - 0.63956	
1.06 + 0.50894		1.26 + 0.39347		1.46 + 0.07855		1.66 - 0.32513		1.86 - 0.65148	
1.07 + 0.50853		1.27 + 0.38190		1.47 + 0.05921		1.67 - 0.34473		1.87 - 0.66279	
1.08 + 0.50758		1.28 + 0.36982		1.48 + 0.03966		1.68 - 0.36410		1.88 - 0.67348	
1.09 + 0.50604		1.29 + 0.35723		1.49 + 0.01992		1.69 - 0.38325		1.89 - 0.68355	
1.10 + 0.50396		1.30 + 0.34416		1.50 - 0.00000		1.70 - 0.40213		1.90 - 0.69303	
1.11 + 0.50130		1.31 + 0.33060		1.51 - 0.02007		1.71 - 0.42072		1.91 - 0.70183	
1.12 + 0.49807		1.32 + 0.31657		1.52 - 0.04028		1.72 - 0.43901		1.92 - 0.70998	
1.13 + 0.49428		1.33 + 0.30207		1.53 - 0.06060		1.73 - 0.45697		1.93 - 0.71744	
1.14 + 0.48991		1.34 + 0.28716		1.54 - 0.08101		1.74 - 0.47456		1.94 - 0.72423	
1.15 + 0.48496		1.35 + 0.27179		1.55 - 0.10149		1.75 - 0.49181		1.95 - 0.73032	
1.16 + 0.47946		1.36 + 0.25601		1.56 - 0.12203		1.76 - 0.50866		1.96 - 0.73570	
1.17 + 0.47335		1.37 + 0.23982		1.57 - 0.14259		1.77 - 0.52511		1.97 - 0.74037	
1.18 + 0.46671		1.38 + 0.22324		1.58 - 0.16315		1.78 - 0.54111		1.98 - 0.74432	
1.19 + 0.45948		1.39 + 0.20630		1.59 - 0.18369		1.79 - 0.55667		1.99 - 0.74755	

More convenient, however, for practical use may be Table II., from which the numerical values of $C(c, 1)$ and $S(c, 1)$ are directly obtained for values of c included between 0 and 30.0. The intervals chosen are sufficiently close to permit an easy

interpolation of these functions for intermediate values of c . If required, the table may be extended by means of Table I. and the preceding formulæ.

TABLE II.

c	$C(c, 1)$	$S(c, 1)$	c	$C(c, 1)$	$S(c, 1)$
0.0	+1.00000	+1.00000	5.6	-0.50392	+0.02851
0.2	+0.86568	+0.94030	5.8	-0.50292	+0.01375
0.4	+0.71008	+0.88216	6.0	-0.50000	0.00000
0.6	+0.62247	+0.82674	7.0	-0.46052	-0.05615
0.8	+0.51257	+0.77390	8.0	-0.38976	-0.09397
1.0	+0.41034	+0.72302	9.0	-0.29986	-0.11695
1.2	+0.31593	+0.67440	10.0	-0.20033	-0.12809
1.4	+0.22694	+0.62778	11.0	-0.09837	-0.13006
1.6	+0.14511	+0.58321	12.0	0.00000	-0.12500
1.8	+0.06956	+0.54071	13.0	+0.09082	-0.11479
2.0	0.00000	+0.50000	14.0	+0.17106	-0.10101
2.2	-0.06385	+0.46111	15.0	+0.23802	-0.08524
2.4	-0.12221	+0.42406	16.0	+0.29323	-0.06753
2.6	-0.17561	+0.38864	17.0	+0.33377	-0.04971
2.8	-0.22395	+0.35492	18.0	+0.36035	-0.03222
3.0	-0.26758	+0.32282	19.0	+0.37382	-0.01550
3.2	-0.30673	+0.29227	20.0	+0.37500	0.00000
3.4	-0.34177	+0.26317	21.0	+0.36508	+0.01404
3.6	-0.37273	+0.23555	22.0	+0.34531	+0.02640
3.8	-0.39987	+0.20933	23.0	+0.31710	+0.03698
4.0	-0.42343	+0.18442	24.0	+0.28192	+0.04572
4.2	-0.44360	+0.16083	25.0	+0.24103	+0.05266
4.4	-0.46051	+0.13853	26.0	+0.19611	+0.05779
4.6	-0.47438	+0.11741	27.0	+0.14830	+0.06123
4.8	-0.48542	+0.09744	28.0	+0.09889	+0.06308
5.0	-0.49366	+0.07865	29.0	+0.04919	+0.06345
5.2	-0.49941	+0.06088	30.0	0.00000	+0.06250
5.4	-0.50277	+0.04422			

As regards the hyperbolic functions $\mathcal{C}(c, 1)$ and $\mathcal{S}(c, 1)$, it seems, however, impossible to calculate their values by means of the corresponding relations, because the Γ -functions, which appear in the expressions, have imaginary arguments. But fortunately, in this case, the series are far more manageable than those of the C - and S -functions, so that the labour involved in their direct calculation is not nearly so stupendous as it may appear at first sight, even if an accuracy within the fifth decimal place is desired. To show this let us first consider the series

$$1 - \frac{c+1}{2.3} + \frac{(c+1)(c+9)}{2.3.4.5} - \frac{(c+1)(c+9)(c+25)}{2.3.4.5.6.7} + \dots$$

which, as we know from the preceding investigations (see (29)), is rigorously represented by the transcendent $\frac{1}{\sqrt{c}} \sin(\sqrt{c} \log(1 + \sqrt{2}))$. We want to inquire how many terms of the series are necessary to obtain the accurate value of this transcendent within, say, five places after the decimal point. If we call $\Sigma(x)$ the sum of the x first terms, and if

we plot the successive Σ as ordinates, taking the corresponding x as abscissæ, we find that, owing to the continuous change of sign, the Σ are represented as the corner-points of a zigzag line; and it can be shown that for sufficiently great values of x this line oscillates along a straight line parallel to the axis of abscissæ, and at a distance from it equal to the exact value of the series, *i.e.* to $\frac{1}{\sqrt{c}} \sin (\sqrt{c} \log (1+\sqrt{2}))$. Let us demonstrate this for a certain value of c , e.g. $c=24$. We find

x	$\Sigma(x)$	x	$\Sigma(x)$
1	+1.00000	12	-2.43133
2	-3.16667	13	+1.88211
3	+3.70833	14	-2.10568
4	-4.31251	15	+1.59235
5	+3.81975	16	-1.84723
6	-3.94285	17	+1.36108
7	+3.27238	18	-1.63963
8	-3.35875	19	+1.17409
9	+2.71167	20	-1.47067
10	-2.84399	21	+1.02088
11	+2.24871	22	-1.33133

If now we take the arithmetical means of each two consecutive $\Sigma(x)$, which we may call $\Sigma_1(x)$, these may again be represented by a zigzag line, whose excursions from the horizontal line are obviously much smaller, and if we repeat this process of forming arithmetical means, the fourth operation of this kind will lead to the following set of values for $\Sigma_4(x)$:

x	$\Sigma_4(x)$	x	$\Sigma_4(x)$
13	-0.18840	17	-0.18841
14	-0.18842	18	-0.18844
15	-0.18840	19	-0.18842
16	-0.18844	20	-0.18844

Now we convince ourselves, by computing the higher terms of the series for $x > 22$, that however far we may extend the calculations, the values of $\Sigma_4(x)$ will always be found between the last two figures of the preceding table, and will more and more converge towards their arithmetical mean, *viz.*, -0.18843 . But the exact value of the series should be

$$\frac{1}{\sqrt{24}} \sin \left(\sqrt{24} \log (1+\sqrt{2}) \right),$$

or, if the angle be expressed in degrees,

$$\frac{1}{\sqrt{24}} \sin \left(\sqrt{24} \times 50^\circ.499 \right) = -0.188438.$$

We see, then, that in this case the first twenty terms of the series are quite sufficient to find the value of our series with an error of less than one unit of the fifth

decimal place. The same number of terms is therefore also sufficient for the calculation of the more convergent Seiche-function

$$\Xi(c, 1) = 1 - \frac{c}{2.3} + \frac{c(c+6)}{2.3.4.5} - \frac{c(c+6)(c+20)}{2.3.4.5.6.7} + \dots$$

for the same value of c . This method enables us to compute with sufficient accuracy the Ξ - and ζ -functions, even for great values of the argument c , from a comparatively limited number of terms of the series. I subjoin the results of my computations in the following Table III. :

TABLE III.

c	$\zeta(c, 1)$	$\Xi(c, 1)$	$\Delta\zeta$	$\Delta\Xi$
0	+1.0000	+1.0000	-0.0014	+0.0025
1	+0.5894	+0.8731	-0.0010	+0.0024
2	+0.2328	+0.7558	-0.0006	+0.0025
3	-0.0748	+0.6479	-0.0003	+0.0023
4	-0.3368	+0.5490	-0.0002	+0.0025
5	-0.5568	+0.4580	0.0000	+0.0024
6	-0.7388	+0.3748	+0.0001	+0.0025
8	-1.0007	+0.2299	+0.0003	+0.0024
10	-1.1468	+0.1108	+0.0003	+0.0024
12	-1.1985	+0.0147	+0.0002	+0.0025
16	-1.0908	-0.1208	0.0000	+0.0022
20	-0.8019	-0.1956	-0.0010	+0.0019
24	-0.4235	-0.2255	-0.0021	+0.0017
28	-0.0226	-0.2234	-0.0033	+0.0015

But for the construction of final tables, from which the ζ - and Ξ -functions might be obtained for any value of c by interpolation, the preceding Table III. is not sufficient. In order to facilitate the further computations, I have searched for convenient formulæ of interpolation, so that the still troublesome direct calculations of the series may be avoided, and I have found that the following consideration leads to a convenient result.

It may be easily shown that the differential equation

$$(1+w^2) \frac{d^2y}{dw^2} + c \left[1 - \frac{\frac{3}{4}}{c(1+w^2)} \right] y = 0$$

is satisfied by the particular solutions

$$(1+w^2)^{\frac{1}{4}} \cos(\sqrt{c-\frac{1}{4}} \log(w+\sqrt{1+w^2})) \quad \text{and} \quad \frac{(1+w^2)^{\frac{1}{4}}}{\sqrt{c-\frac{1}{4}}} \sin(\sqrt{c-\frac{1}{4}} \log(w+\sqrt{1+w^2})).$$

Now, the close similarity between this differential equation and the hyperbolic Seiche-equation, into which it converges if c is increased indefinitely, makes it probable that the solutions of the latter, *i.e.* the ζ - and Ξ -functions, are also closely related to the

two simple transcendents just now mentioned. It is, indeed, easily seen that the values of $\mathfrak{G}(c, 1)$ and $\mathfrak{S}(c, 1)$ are approximately represented by

$$2^{\frac{1}{2}} \cos[\sqrt{c+\gamma} \log(1+\sqrt{2})] \quad \text{and} \quad \frac{2^{\frac{1}{2}}}{\sqrt{c+\gamma}} \sin[\sqrt{c+\gamma} \log(1+\sqrt{2})]$$

where γ is a certain constant.

By slightly altering the constants, these two expressions can be made to represent the $\mathfrak{G}$ - and $\mathfrak{S}$ -functions even more closely. We have then approximately

$$\begin{aligned} \mathfrak{G}(c, 1) &= 1.19942 \times \cos[\sqrt{c+0.438} \times 50^\circ.458] \\ \sqrt{c+0.438} \mathfrak{S}(c, 1) &= 1.19942 \times \sin[\sqrt{c+0.438} \times 50^\circ.458]. \end{aligned}$$

If we calculate the expressions on the right-hand side for the values of c given in Table III., by comparison of the calculated figures with the exact values of $\mathfrak{G}(c, 1)$ and $\mathfrak{S}(c, 1)$ shown in the same table, we find the corrections $\Delta\mathfrak{G}$ and $\Delta\mathfrak{S}$ which have to be added to the computed transcendents. These corrections are conveniently small to permit an easy interpolation, and we are thus in a position to ascertain the correct numerical values of $\mathfrak{G}(c, 1)$ and $\mathfrak{S}(c, 1)$ for values of c lying between those mentioned in Table III. by simply computing the corresponding values of the right-hand sides of the above approximate equations and by adding the corresponding corrections $\Delta\mathfrak{G}$ and $\Delta\mathfrak{S}$ interpolated from the figures of the same table. In this way the following Table IV. has been constructed, which shows, for conveniently small intervals, the $\mathfrak{G}(c, 1)$ and $\mathfrak{S}(c, 1)$ for the arguments c between zero and 30.0. To exhibit at the same time, more concretely, the character of the functions $\mathfrak{C}(c, 1)$, $\mathfrak{S}(c, 1)$, $\mathfrak{G}(c, 1)$ and $\mathfrak{S}(c, 1)$, their graphs are given in fig. 2.

TABLE IV.

c	$\mathfrak{G}(c, 1)$	$\mathfrak{S}(c, 1)$	c	$\mathfrak{G}(c, 1)$	$\mathfrak{S}(c, 1)$
0.0	+1.0000	+1.0000	13.0	-1.1946	-0.0259
0.5	+0.7877	+0.9353	14.0	-1.1741	-0.0619
1.0	+0.5894	+0.8731	15.0	-1.1389	-0.0932
1.5	+0.4047	+0.8133	16.0	-1.0908	-0.1208
2.0	+0.2328	+0.7557	17.0	-1.0314	-0.1446
2.5	+0.0730	+0.7008	18.0	-0.9626	-0.1648
3.0	-0.0748	+0.6480	19.0	-0.8854	-0.1817
3.5	-0.2111	+0.5974	20.0	-0.8019	-0.1956
4.0	-0.3368	+0.5489	21.0	-0.7126	-0.2067
4.5	-0.4519	+0.5024	22.0	-0.6192	-0.2153
5.0	-0.5568	+0.4580	23.0	-0.5224	-0.2214
6.0	-0.7388	+0.3747	24.0	-0.4235	-0.2255
7.0	-0.8856	+0.2989	25.0	-0.3229	-0.2275
8.0	-1.0007	+0.2299	26.0	-0.2225	-0.2277
9.0	-1.0868	+0.1673	27.0	-0.1219	-0.2264
10.0	-1.1468	+0.1108	28.0	-0.0226	-0.2234
11.0	-1.1831	+0.0600	29.0	+0.0755	-0.2192
12.0	-1.1984	+0.0146	30.0	+0.1707	-0.2137

FIG. 2.

An extension of Table IV. beyond $c=30$ is probably unnecessary for practical purposes. That the approximate formulæ for $G(c, 1)$ and $S(c, 1)$ are still valid for $c>30$ may, however, be concluded from the fact that the higher roots of the equations

$$\begin{aligned}\cos[\sqrt{c+0.438} \times 50^\circ.458] &= 0 \quad \text{and} \\ \sin[\sqrt{c+0.438} \times 50^\circ.458] &= 0\end{aligned}$$

agree very closely with those of the corresponding Seiche-equations $G(c, 1)=0$ and $S(c, 1)=0$. We find for the former 79.099 and 50.465, whereas the corresponding roots of the latter were previously found to be 79.053 and 50.466. As regards the lower roots which are included in Table III. we have in the case of the hyperbolic Seiche Cosine

$$1.19942 \cos[\sqrt{c+0.438} \times 50^\circ.458] - 0.0004 = 0; \quad c = 2.742$$

$$1.19942 \cos[\sqrt{c+0.438} \times 50^\circ.458] - 0.0034 = 0; \quad c = 28.230$$

and for the hyperbolic Seiche Sine

$$\frac{1.19942}{\sqrt{c+0.438}} \sin[\sqrt{c+0.438} \times 50^\circ.458] + 0.0024 = 0; \quad c = 12.341.$$

These roots agree almost exactly with those previously found by an entirely different approximative method (see (35)).

In conclusion, I wish to express my great indebtedness to Professor CHRYSTAL for the interest he has taken in this investigation, and especially for having kindly permitted me to publish this mathematical discourse on the differential equations of his Seiche-problem along with his own physical and mathematical researches. The general type

of differential equations, of which the Seiche-equations are special cases, is well known to mathematicians through the important rôle it plays in the theory of the associated LEGENDRE-functions. But, as far as I am aware, its solutions have been investigated only in the special cases where n represents integers (see WHITTAKER, *Modern Analysis*, p. 235). In the theory of the $C_n^a(w)$ polynomials, which are particular integrals of the general equations here considered, the two synectic integrals, with which we were particularly concerned in this investigation, are of less importance than some of the other hypergeometric series. But their property as synectic solutions renders them particularly useful in the special physical problem before us. It is not improbable that in other problems—for instance, in such which are based on the differential equation of the LEGENDRE-functions ($a = \frac{1}{2}$)—the corresponding synectic integrals, the LEGENDRE Sine and Cosine, might also be of special importance. Since the latter, as well as their associated functions, may be derived from the Seiche-functions by differentiation, we may consider these as the typical representatives of the whole class; and this fact, doubtless, gives to Professor CHRYSTAL'S investigation a considerable importance from the mathematical point of view,—an importance still more enhanced, on the one hand, by the introduction of the corresponding hyperbolic functions, to which he has now for the first time directed the attention of mathematicians, and, on the other, by the remarkable relations between this general class of functions and another class represented by simple transcendents.