JOURNAL
OF THE
FRANKLIN INSTITUTE.
OF THE STATE OF PENNSYLVANIA,
FOR THE PROMOTION OF THE MECHANIC ARTS.

VOL. CXIV.

OCTOBER, 1882.

No. 4.

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MOHR'S GRAPHICAL THEORY OF EARTH PRESSURE.

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The subject of the pressure of earth has been worn nearly threadbare. It has been discussed from all points of view, again and again, until there seems no room for anything new to be presented. And yet, although our technical journals have had their full share of the discussion, the writer does not remember ever to have seen in English a presentation of the graphical method given by Professor Mohr in 1871. The analytical methods in general use, either the ones founded on the wedge of greatest pressure, or on the principles of internal stress, as first presented by Rankine, although simple enough, and although leading to formulas which can easily be interpreted graphically, often leave some confusion in the mind; and they are far inferior, as regards giving a clear and connected view of the subject, to Professor Mohr's method. The practical engineer who should happen to be unacquainted with the method will be glad to learn that the whole subject of the pressures in a mass of earth bounded above by a plane can be represented clearly and concisely without the use of any but the simplest mathematics, and with the aid of only the most elementary theorem of statics; and that the method leads to very simple methods of

Fig. 3, in which $EF = T$; $FG = V$; $GU = H \tan a$; $UV = N \sec a$; $VW = S \sec a$; $WE = T \tan a$.

Supposing these forces to be laid down correctly, we can investigate some of their properties which will enable us to draw the figures.

1. E and G are in the same vertical line; GE is the intensity of the pressure on $AC = \gamma y$.

2. Produce GF and UV to K . Then $GK = GU \cot a = H \tan a \cot a = H$.

3. Produce EF and WV to I . Then $EI = EW \cot a = T \tan a \cot a = T = EF$.

It follows that the points E , F , G , K and I are independent of a . If those points are given in the case of the plane AB , they will be the same for any other.

4. The angles IFK and IVK are each 90° . Hence a circle can be drawn through I , F and K , which will contain all the points V , for every plane; IK is the diameter, and M the centre of the circle.

5. Draw VO parallel to FG . Then $VO = VU \cos a = N \sec a \cos a = N$.

6. Draw VP parallel to EF , to meet EW produced. Then $VP = VW \cos a = S \sec a \cos a = S$.

7. It follows from the above that if once the circle IFK be drawn, then in order to find the intensity of the pressure on any plane AB , we have only to draw IV parallel to AB and VO and VP are the normal and tangential intensities of stress on this plane.

Draw $HEWPM L$ and HI . The latter is vertical. HV is the intensity of the stress on AB ; the angle PHV is δ , or the angle that stress makes with the normal to AB .

8. As the plane AB is changed, or rotated about A , the point V moves around the circumference of the circle. The intensity of stress on AB has for its maximum value HL , on a plane parallel to IL and for its minimum value HN , on a plane parallel to IN . On these planes the stress is normal; hence these stresses are the "principal stresses." The planes IL and IN are at right angles. Hence the planes on which the principal stresses act are at right angles to each other.

9. The angle δ has its maximum value for a plane parallel to IR , HR being tangent to the circle; as well as for a second plane parallel to IQ , HQ being also tangent to the circle.

10. The line WU is parallel to the actual pressure on the plane AB .

It is clear from the above that if once the circle is constructed, the

stress on any plane is easily found. It only remains to draw the circle.

This, however, we are unable to do in general, but we can locate it within limits. We know that the angle δ can never exceed the angle of friction φ , or angle of repose of the earth. Hence, if we draw from H two lines (Fig. 4) making angles φ on either side of HE , we know that the circle must be within those lines. Any circle through I , F , and K , which fulfills this condition is a possible circle of equilibrium, if we may so call a circle corresponding to a possible state of equilibrium of the mass of earth.

If the earth is just on the point of slipping, $\delta=\varphi$, and the circle is tangent to the two lines HR and HQ (Fig. 4). Two circles, $N_1 Q_1 L_1 R_1$ and $N_2 Q_2 L_2 R_2$ fulfill this condition, hence they represent the two limiting states of equilibrium when the earth is just ready to slip. It is with these cases only that the engineer has to deal. The larger

circle represents the case where the maximum principal pressure HL (Fig. 3) is increased until the limiting condition is reached. This we may call the *passive* earth pressure; it occurs when a thrust or pressure, as of an arch or building, is exerted on the earth until it is just ready to give way. The smaller circle represents the case where the minimum principal pressure HN (Fig. 3) is decreased until the limiting condition is reached. This we may call the *active* earth pressure; it occurs when the thrust of a mass of earth is resisted, as by a retaining wall, and is, therefore, the most usual in the practice of the engineer. For the passive earth pressure the maximum pressure is HL_2 on a plane parallel to IL_2 , and the minimum pressure HN_2 on a plane parallel to IN_2 . For the active earth pressure the maximum pressure is HL_1 , on a plane parallel to IL_1 , and the minimum pressure HN_1 on a plane parallel to IN_1 .

The maximum shear occurs on planes parallel to IR_2 and IQ_2 in the first case; IR_1 and IQ_1 in the second.

In the special case of a horizontal surface the force on AC is vertical, hence $EF = EI = 0$, and the two circles are tangent. (Fig. 5). If the surface is inclined at the angle of repose φ , HQ is vertical, and coincides with HI , hence only a single condition of equilibrium is possible (Fig. 6).

In all the previous figures we have taken H in the line SS , which represents the surface. IH is the intensity of the pressure on a plane parallel to the surface, hence it equals γy . If the point considered is at a normal distance, IH , from the surface, then $y = IH$, and $\gamma y = \gamma.IH$. Hence IH represents a column of earth whose weight would produce the pressure really existing on a plane AC at a normal distance, IH , from the surface, and all the other lines in the figure which represent forces, such as GU , UV , etc., represent columns of earth which would produce the pressures they stand for, all for a point at a normal distance, IH , from the surface.

To find the total pressure on any plane.

The following constructions, which follow at once from the preceding paragraphs, are given by Mohr for finding the total pressure on any plane, such as the back of a retaining wall.

First Method. (Figure 7.)

The first step is to draw the circle ILK . This can easily be done geometrically, but the following method is simpler: Draw SS , representing the surface, and HL perpendicular to it at any point H , making HL any convenient distance, say 5 inches; make

$$HN = HL \cdot \tan^2 \left(45^\circ - \frac{\alpha}{2} \right)$$

according to a formula yet to be explained. NL is the diameter of the circle. Draw HI vertically, the diameter IK , and a line IVW parallel to the wall: then draw KVU and UW , and the construction is complete. HV represents the intensity of the pressure at a point whose normal distance from the surface is $HI = HO$, and the pressure acts parallel to UW . No simpler construction could be desired. We have, then, if AB represents the wall, only to lay off HV at $H'V'$, perpendicular to AB , and knowing that the pressure is proportional to the depth below the surface, to draw the line $V'A$. The area of the triangle ACB , multiplied by γ , gives the pressure on the wall. If the

back of the wall is not plane, but composed of a number of planes, it is easy to find the pressure on each. Thus, in the above figure, the pressure on the part EF of the plane is represented by the area $EFPQ$. And finally, if the back is curved, we must divide it into a number of parts, and assume each to be plane. It is not necessary to go into farther details.

Second Method. (Figure 8.)

Draw the circle as before, and find first the pressure on a vertical plane. HV is the intensity of the pressure on such a plane at a point A , distant HI normally from the surface. If AD is the vertical plane the total pressure it supports is $\frac{r \cdot AD}{2} HV$, acting parallel to HD .

If we combine with this the weight of the triangle ABD , AB being

the real wall, we shall have the pressure on the latter. The weight of $ABD = \frac{\gamma \cdot AD}{2} BE$. We, therefore, lay off $HG = HV$, and $HK = BE$; and then the pressure on AB acts parallel to GK , and is equal to $\frac{\gamma \cdot AD}{2} GK$.

If the surface of the earth is loaded, reduce the load to the specific gravity of the earth, and let h be the height representing it (Fig. 9). The problem is, therefore, to find the pressure on a plane ab , as though the surface of the earth were $S'S'$. It will be represented by a trapezoid, $a a' b' b$.

That the pressure on a plane varies directly as the distance below the surface is easily seen when we remember that the pressure on a plane parallel to the surface follows that law, and that consequently the other pressures must do the same, inasmuch as Fig. 3 holds for any point, only the scale of forces changing. The resultant pressure

on a plane wall extending to the surface is, therefore, applied two-thirds of the distance from the top to the bottom.

Third Method. (Figure 10.)

This method depends upon the theorem of conjugate pressures; the pressure on a given plane ab having a certain direction, p , the pressure on a plane bc parallel to p is parallel to ab . (Fig. 10). The stresses on the four planes in Fig. 10 make, therefore, the same angle β with the normal. Now, if we draw (Fig. 11) the lines HT , HT' , both making angles of β with HL , then the pressure on each of the four planes IV_1 , IV_2 , IV_3 , IV_4 , will make the angle β with the normal. The stress on IV_1 must be parallel to either IV_2 , IV_3 , or IV_4 , and as the angle $V_1IV_3 = V_1V_4V_3 = 90^\circ - \beta$, it is clear that it must be parallel to IV_3 , so that IV_1 and IV_3 , as well as IV_2 and IV_4 , are a pair of conjugate planes. To find the direction of the stress on IV_1 , then, it is only necessary to lay off the angle $LHT' = LHV_1$, and draw IV_3 .

But it so happens that if we draw V_1V_3 , crossing HL in S , we have the triangles HV_2M and HV_1S similar, and $HS : HV_1 :: HV_2 : HM$ ∴

$HS = \frac{HV_1 \cdot HV_2}{HM} = \frac{HQ^2}{HM}$, so that S is the point when QR meets HL ,

and is, hence, independent of V_1 . In order, then, to find the direction of the pressure on a plane IV_1 , we need only to draw V_1SV_3 , and IV_3 is the required direction.

The preceding method will by its simplicity and accuracy commend itself to all. Founded on the condition of equilibrium of a particle of earth, it ought to give results agreeing with the analytical theory first

given by Rankine. The following equations, which the reader will easily demonstrate, will show the complete agreement of the two.

$$\sin.\varphi = \frac{N_{\max.} - N_{\min.}}{N_{\max.} + N_{\min.}} \quad (1)$$

$$\frac{1 - \sin.\varphi}{1 + \sin.\varphi} = \tan.^2\left(45^\circ - \frac{\varphi}{2}\right) = \frac{N_{\min.}}{N_{\max.}} \quad (2)$$

Call δ the angle between the surface and the direction of $N_{\max}$; then from the triangles HIM_1 and HIM_2 . (Fig. 12.)

$$\frac{M_1 I}{M_1 H} = \sin.\varphi = \frac{\sin.\epsilon}{\sin.(2\delta_1 + \epsilon)} \quad (3)$$

$$\frac{M_2 I}{M_2 H} = \sin.\varphi = \frac{\sin.\epsilon}{\sin.(2\delta_2 + \epsilon)} \quad (4)$$

$$\sin.(2\delta + \epsilon) = \frac{\sin.\epsilon}{\sin.\varphi} \quad (5)$$

$$\sin.2\delta = \frac{\sin.\epsilon}{\sin.\varphi} \left[\cos.\epsilon \pm \sqrt{\cos.^2\epsilon - \cos.^2\varphi} \right] \quad (6)$$

Let y = normal distance from surface; then

$$\begin{aligned} HM_1 &= \frac{1}{2}(N_{1\max.} + N_{1\min.}) = HI \cdot \frac{\sin.(2\delta_1 + \epsilon)}{\sin.2\delta_1} \\ &= y \frac{\sin.(2\delta_1 + \epsilon)}{\sin.2\delta_1} \end{aligned} \quad (7)$$

$$IM_1 = y \frac{\sin.\epsilon}{\sin.2\delta_1} \quad (8)$$

$$HM_2 = y \frac{\sin.(2\delta_2 + \epsilon)}{\sin.2\delta_2} \quad (9)$$

$$IM_2 = y \frac{\sin.\epsilon}{\sin.2\delta_2} \quad (10)$$

$$\text{From (5), } (2\delta_1 + \epsilon) + (2\delta_2 + \epsilon) = 180^\circ; \delta_1 + \delta_2 = 90^\circ - \epsilon \quad (11)$$

In (6), one of the signs of the double signs must apply to δ_1 and the other to δ_2 ; one cannot apply to both, for then we might have $\sin.2\delta_1 = \sin.2\delta_2$ or $\delta_1 + \delta_2 = 90^\circ$, which is contradictory of (11). We have, then,

$$\begin{aligned} N_{1\max.} &= \gamma.HL_1 = \gamma.(HM_1 + IM_1) = \gamma y \cdot \left[\frac{\sin.(2\delta_1 + \epsilon)}{\sin.2\delta_1} + \frac{\sin.\epsilon}{\sin.2\delta_1} \right] \\ &= \gamma y \frac{1 + \sin.\varphi}{\cos.\epsilon + \sqrt{\cos.^2\epsilon - \cos.^2\varphi}} \end{aligned}$$

$$= \gamma y \frac{\cos.\varepsilon - \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}}{1 - \sin.\varphi} \quad (12)$$

Similarly,

$$N_{1\min.} = \gamma y \frac{1 - \sin.\varphi}{\cos.\varepsilon + \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}} = \gamma y \frac{\cos.\varepsilon - \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}}{1 + \sin.\varphi} \quad (13)$$

$$N_{2\max.} = \gamma y \frac{1 + \sin.\varphi}{\cos.\varepsilon - \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}} = \gamma y \frac{\cos.\varepsilon + \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}}{1 - \sin.\varphi} \quad (14)$$

$$N_{2\min.} = \gamma y \frac{1 - \sin.\varphi}{\cos.\varepsilon - \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}} = \gamma y \frac{\cos.\varepsilon + \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}}{1 + \sin.\varphi} \quad (15)$$

The intensity of the pressure on a vertical plane is represented for the passive state by $HI_2 = p_2$, and for the active state by $HI_1 = p_1$, and we have

$$\begin{aligned} p_1.HI &= p_1.\gamma.y = \overline{Q_1 H^2} = N_{1\max.} N_{1\min.} \\ &= \gamma^2 y^2 \cdot \frac{\cos.\varepsilon - \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}}{\cos.\varepsilon + \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}}, \text{ hence} \\ p_1 &= \gamma y \frac{\cos.\varepsilon - \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}}{\cos.\varepsilon + \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}} \end{aligned} \quad (16)$$

Similarly,

$$p_2 = \gamma y \frac{\cos.\varepsilon + \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}}{\cos.\varepsilon - \sqrt{\cos.^2\varepsilon - \cos.^2\varphi}} \quad (17)$$

These equations show the complete agreement of the two methods.

Mohr has shown how the influence of the cohesion of the earth may be taken into account, but as this has little practical importance it is not necessary to do more than allude to it.

Finally, a few words regarding the limits within which the theory or method is applicable. We have assumed a plane upper surface. For this case our results are absolutely exact, but if the top surface is not plane, it is generally conceded that an exact solution of the problem of finding the pressure on any plane is impossible. But even

with a plane upper surface, we have seen that on any plane except one parallel to the surface, there are various pressures which are compatible with equilibrium. We have shown how to find the pressures for the two limiting conditions, when the earth is just ready to slide; or, in other words, we have found for any plane the least and greatest pressures consistent with equilibrium. In applying the method, then, it is only necessary to be sure that we know just what case we have to do with. In the case of a retaining wall we wish to find how it must be proportioned in order to prevent the mass of earth from sliding, *i. e.*, we wish to find the least pressure which would hold the earth in equilibrium, and make our wall so that the pressure will not displace it. Now, although by the method given we can find the least pressure on any plane in an unlimited mass of earth, yet it is not on every plane that we can replace the pressures by the resistance of a wall. Suppose, for instance, we made a wall parallel to the surface, removing the earth above; this is entirely without meaning, for then there would be no earth above that plane, and the earth would stand without the wall. Other positions can readily be imagined, in which a wall could be placed without experiencing the pressure which would act on that plane in an unlimited mass of earth. The following limitation, given by Mohr, will now be readily seen to be reasonable:

“A mass of earth supported by a wall is in the active condition as regards pressure, or at the lower limit of equilibrium, and in order to determine the pressure on the wall, the theory of earth pressure in an unlimited mass of earth may be applied, provided that the straight line, which, according to that theory, gives the direction of the maximum principal stress, at the foot of the wall, lies within the mass of earth.” In other cases, as, for instance, where the earth slopes downward from the wall, Mohr advocates the use of the old theory, so ably discussed in these pages not long ago by Prof. Du Bois, the pressure, however, being assumed perpendicular to the wall. Such cases, however, will not often occur in the practice of the engineer. In cases where the surface of the earth is not plane, the method can be used, and probably is as correct as any other, if the real surface be replaced by a plane. The engineer will readily see in each particular case how such a plane can best be assumed.