ON CRITERIA FOR THE FINITENESS OF THE ORDER OF A GROUP OF LINEAR SUBSTITUTIONS

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1. The coefficients in the substitutions of a group of homogeneous linear substitutions are in general complex quantities of the form $\alpha+\beta\sqrt{(-1)}$, where α and β are real numbers. If the group is one of finite order, there is a finite number of coefficients, and there must therefore be a finite positive number M, such that for each coefficient

$$|\alpha| < M, \qquad |\beta| < M. \tag{i}$$

Similarly there must be another positive number m, such that for each coefficient either

$$\left.\begin{array}{lll} \alpha = 0 & \text{or} & |\alpha| > m; \\ \beta = 0 & \text{or} & |\beta| > m \end{array}\right\}. \tag{ii}$$

and

The existence of these two numbers M and m is a necessary condition if the group is a group of finite order.

I propose to shew that, if two positive numbers M and m can be assigned so that each coefficient in every substitution of a group satisfies the system of relations (i) and (ii), then the group is a group of finite order.

The equations
$$x_i' = \sum_{j=1}^{j=n} a_{ij}x_j \quad (i = 1, 2, \ldots, n),$$

defining a linear substitution of a group G, imply the equations

$$\bar{x}_i' = \sum_{j=1}^{j=n} \bar{a}_{ij}\bar{x}_j \quad (i = 1, 2, \ldots, n),$$

where x and $\bar{x}$ denote conjugate imaginaries. These $2n$ equations are equivalent to

$$\left.\begin{aligned} x_i'+\bar{x}_i' &= \sum_{j=1}^{j=n} \tfrac{1}{2}(a_{ij}+\bar{a}_{ij})(x_j+\bar{x}_j) \\ &\quad - \sum_{j=1}^{j=n} \frac{1}{2\sqrt{(-1)}}(a_{ij}-\bar{a}_{ij})\frac{1}{\sqrt{(-1)}}(x_j-\bar{x}_j) \\ \frac{1}{\sqrt{(-1)}}(x_i'-\bar{x}_i') &= \sum_{j=1}^{j=n} \frac{1}{2\sqrt{(-1)}}(a_{ij}-\bar{a}_{ij})(x_j+\bar{x}_j) \\ &\quad + \sum_{j=1}^{j=n} \tfrac{1}{2}(a_{ij}+a_{ij})\frac{1}{\sqrt{(-1)}}(x_j-\bar{x}_j) \end{aligned}\right\} \quad (i = 1, 2, \ldots n),$$

which define a linear substitution with real coefficients on the $2n$ variables

$$x_i+\bar{x}_i, \qquad \frac{1}{\sqrt{(-1)}}(x_i-\bar{x}_i), \quad (i = 1, 2, \ldots, n).$$

It may be easily verified by direct calculation that the system of linear substitutions thus formed from all the substitutions of G have the group property and constitute a group G' simply isomorphic with G.

Moreover, if the coefficients of G satisfy the system of relations (i) and (ii), then the real coefficients of G' are such that each one of them is either zero or numerically (*i.e.*, apart from sign) less than M and greater than m.

As the succeeding argument does not depend in any way on the number of variables being even, their number will still be represented by n, and any substitution of G' by

$$x_i = \sum_{j=1}^{j=n} a_{ij}x_j \quad (i = 1, 2, \ldots, n),$$

where now a_{ij} is a real number, such that either

$$a_{ij} = 0$$

or

$$M > |a_{ij}| > m.$$

The sum of the squares of the coefficients of any substitution is certainly less than n^2M^2. Hence, if $a_{11}, a_{12}, \ldots, a_{nn}$ be regarded as the rectangular co-ordinates of a point P in space of n^2 dimensions, the point is restricted to be within a sphere of radius nM with the origin as centre. Consider now the aggregate of points P which correspond to the totality of the substitutions of G'. If the aggregate is a finite one, the number of substitutions of G', *i.e.*, its order, is finite. If the aggregate is not finite, then, since the points are restricted to be within a certain sphere of finite radius, the aggregate must have at least one limit point. Let

$$A_{11}, A_{12}, \ldots, A_{nn}$$

be the coordinates of a limit-point. Then, if $\frac{1}{2}\epsilon$ is any given small positive quantity, an infinite number of points of the aggregate lie within a sphere of radius $\frac{1}{2}\epsilon$ described about the limit-point as centre. If

$$a_{11}, a_{12}, \ldots, a_{nn}$$

be an assigned one of these points, and

$$b_{11}, b_{12}, \ldots, b_{nn}$$

be any other, then

$$\sum_{ij}(a_{ij}-b_{ij})^2 < \epsilon^2,$$

and therefore for all suffixes $|a_{ij}-b_{ij}| < \epsilon$.

Let A and B be the two substitutions of G' which correspond to these points, and denote by C with coefficients c_{ij} the substitution $A^{-1}B$. Then, since

$$b_{ij} = a_{ij} + \eta,$$

where η lies between $-\epsilon$ and $+\epsilon$,

$$| c_{ii} - 1 | < nM\epsilon,$$
$$| c_{ij} | < nM\epsilon \quad (i \neq j);$$

and G' has an infinite number of substitutions whose coefficients satisfy these inequalities, however small ϵ may be. If, for every one of them c_{ij} $(i \neq j)$ were zero, every one of them would replace each of the symbols by a real multiple of itself. Since there is an infinite number of these substitutions, the multipliers cannot all be either $+1$ or -1. But, if any multiplier is different from $+1$ and -1, a sufficiently high power of the corresponding substitution would have a coefficient which is either greater than M, or without being zero less than m. If, on the other hand, some c_{ij} is not zero, then, by making ϵ small enough, it will be numerically less than m. In either case the supposition that the aggregate of points P is not finite leads to a contradiction. It follows therefore that, with the limitation of the coefficients given by the relations (i) and (ii), the order of G', and necessarily also the order of G, is finite.

It is perhaps worth remarking that either relation (i) by itself, or relation (ii) by itself, is not sufficient to ensure that the order shall be finite. In fact the group

$$x' = \alpha x + \beta y,$$
$$y' = \gamma x + \delta y,$$

where α, β, γ, δ are real integers such that

$$\alpha\delta - \beta\gamma = 1$$

is not of finite order, and has no coefficient other than zero numerically less than unity. Again, the group

$$x' = \cos n\theta\, x - \sin n\theta\, y,$$
$$y' = \sin n\theta\, x + \cos n\theta\, y,$$

where n is an integer and π/θ is not rational, is not of finite order and has no coefficient numerically greater than unity.

2. If the order of a group of linear substitutions is finite, the order of every substitution of the group is a finite number equal to or less than the order of the group. I propose to show that the converse of this theorem may be proved in the form :—

If, in a group of linear substitutions on a finite number n of symbols, the order of every substitution is equal to or less than a finite number m, then the group is a group of finite order.

As above, any substitution A of the group is given by the system of equations

$$x_i' = \sum_{j=1}^{j=n} a_{ij} x_j \quad (i = 1, 2, \ldots, n).$$

The substitution A being one of finite order, equal to or less than m, its characteristic

$$a_{11} + a_{22} + \ldots + a_{nn}$$

is the sum of n roots of unity, the index of each root that occurs being equal to or less than m. Such a set of roots of unity can only be chosen in a finite number of ways, and therefore there is only a finite number of distinct characteristics. Now the characteristic of AB, where the coefficients of B are β_{ij}, is

$$\sum_{ij} a_{ij} \beta_{ji},$$

and therefore, whatever substitution of the group B may be, its coefficients satisfy the equation

$$\sum_{ij} a_{ij} \beta_{ji} = \chi,$$

where χ is one of the finite number of distinct characteristics.

Suppose now that it is possible to choose n^2 distinct substitutions of the group, $A_1, A_2, \ldots, A_{n^2}$, such that

$$\begin{vmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \ldots & a_{nn}^{(1)} \\ a_{11}^{(2)} & a_{22}^{(2)} & \ldots & a_{nn}^{(2)} \\ \ldots & \ldots & & \ldots \\ a_{11}^{(n^2)} & a_{12}^{(n^2)} & \ldots & a_{nn}^{(n^2)} \end{vmatrix} \neq 0;$$

then the coefficients of any other substitution B satisfy a system of n^2 linear equations of non-vanishing determinant, the right-hand sides of which can take only a finite number of distinct values. In this case then the number of substitutions, and therefore the order of the group, is finite.

If no such set of n^2 substitutions as

$$A_1, A_2, \ldots, A_{n^2}$$

exists, there must be one or more homogeneous linear equations

$$\Sigma B_{ij} \beta_{ij} = 0$$

satisfied by the coefficients of every substitution of G. This, however, is known to imply that the group is reducible. Hence under the conditions assumed the group is either reducible, or, if irreducible, of finite order.

From this it immediately follows that it must be of finite order in any case. Suppose, in fact, that of the n variables r are transformed irreducibly among themselves. So far as it affects these r symbols the group is of finite order, and therefore it has a self-conjugate sub-group of finite index for which each of the r symbols is unchanged. The substitutions on the remaining $n-r$ symbols, if in the equations defining them the first r symbols are all made zero, give rise to a group with the properties of the original one in a smaller number of symbols, which may or may not be reducible. If it is not, it again is a group of finite order; and G has a self-conjugate sub-group of finite index, whose substitutions are of the form

$$x_s' = x_s \qquad (s = 1, 2, \ldots, r),$$

$$x_{r+s}' = x_{r+s} + \sum_1^r a_{st} x_t \qquad (s = 1, 2, \ldots, n-r).$$

Unless the coefficients a_{st} are all zero, so that the sub-group reduces to the identical operation, the substitutions of this sub-group are clearly not of finite order.

If the group on the $n-r$ last symbols is reducible, it may be treated as the original group. Hence, finally, in any case, whether reducible or not, the group is necessarily of finite order.

3. A group of linear substitutions of finite order has a finite number of distinct sets of conjugate substitutions. The converse of this theorem is also true in the form :—

If a group of linear substitutions on a finite number of symbols has a finite number of distinct sets of conjugate substitutions, the order of the group is finite.

The characteristics of two conjugate substitutions (*i.e.*, the sums of the coefficients in the leading diagonals) are the same. Hence, if the number of conjugate sets is finite, the number of characteristics is finite. When the group is irreducible, the argument of the preceding paragraph then shows that the number of substitutions is also finite. When the group is reducible, the same argument shows that the group must have a self-conjugate sub-group H of finite index, whose substitutions are of the form

$$x_s' = x_s \qquad (s = 1, 2, \ldots, r_1),$$

$$x_{r_1+s}' = x_{r_1+s} + \sum_{t=1}^{t=r_1} a_{st} x_t \qquad (s = 1, 2, \ldots r_2),$$

$$x_{r_1+r_2+s}' = x_{r_1+r_2+s} + \sum_{u=1}^{u=r_2} \beta_{su} x_{r_1+u} + \sum_{v=1}^{v=r_1} \gamma_{sv} x_v \qquad (s = 1, 2, \ldots, r_3).$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

If this sub-group actually exists, *i.e.*, if it does not consist of the identical substitution only, it is necessarily *not* of finite order.

Now, so far as the first r_1+r_2 symbols are concerned, it is an Abelian group, whose substitutions can only be conjugate in respect of the substitutions of G not contained in H. Of these there are only a finite number of classes, and therefore unless all the α's are zero the group would contain an infinite number of sets of conjugate substitutions. But, if the α's are all zero, the group, so far as it affects the first $r_1+r_2+r_3$ symbols, is Abelian; and the same reasoning shows that all the β's and γ's must vanish. Hence, if the group contains only a finite number of distinct conjugate sets, H must consist of the identical substitution only, and G must be a group of finite order.