Determination of the Volumes of certain Species of Tetrahedra without employment of the Method of Limits.

By M. J. M. Hill, M.A., D.Sc., F.R.S., Professor of Mathematics at University College, London. Received October 22nd, 1895. Read November 14th, 1895.

The object of this communication is to prove the existence of certain species of tetrahedra whose volumes are determinable without employment of the method of limits.

Abstract.

Art. 1. It is shown that symmetrical tetrahedra have equal volumes.

Art. 2. It is shown that tetrahedra which are images of one another with regard to a common face have equal volumes.

Art. 3. It is shown that a tetrahedron in which the straight line bisecting a pair of opposite edges is perpendicular to those edges can be bisected into two superposable tetrahedra, by drawing a plane through either of these edges and the middle point of the other.

Art. 4. If now (see the figure of Art. 4) $ABCD$ be a tetrahedron, and DE, CF be drawn equal and parallel to BA, and if EA, AF, FE, EC be joined, then it follows by Art. 2, if BE be perpendicular to the plane ACD, that the tetrahedra $ABCD$, $ACDE$ are equal.

In like manner, if DF be perpendicular to the plane ACE, the tetrahedra $ADCE$, $AECF$ are equal.

Hence the tetrahedron $ABCD$ is a third of the prism $BDCFEA$.

The two conditions (each of which amounts to two restrictions) (1) that BE is perpendicular to the plane ACD and (2) that DF is perpendicular to the plane ACE result in the expression of the lengths of the six edges of the tetrahedron in terms of two positive quantities a, r as follows:—

$$AC = a\sqrt{9-3r^2},$$

$$AD = BC = 2a,$$

$$AB = BD = DC = a\sqrt{1+r^2}.$$

Tetrahedra of this kind will be called tetrahedra of the first type.

Art. 5. It is shown that the tetrahedron $ABCD$ of the first type can be bisected into two superposable tetrahedra by a plane drawn through BD and the middle point J of AC; or by a plane through AC and the middle point K of BD.

Arts. 6, 7. The last article leads to the second and third types of tetrahedra.

The tetrahedron $BDJA$ of the second type has its edges

$$AD = 2a,$$
$$AJ = \tfrac{1}{2}a\sqrt{9-3r^2},$$
$$BA = BD = a\sqrt{1+r^2}.$$
$$JB = JD \doteq \tfrac{1}{2}a\sqrt{1+5r^2}.$$

The tetrahedron $ACKB$ of the third type has its edges

$$AC = a\sqrt{9-3r^2},$$
$$BA = 2BK = a\sqrt{1+r^2},$$
$$BC = 2a,$$
$$KA = KC = \tfrac{1}{2}a\sqrt{9+r^2}.$$

Art. 8. It is shown that the three types of tetrahedra are distinct.

Art. 9. In the special case $r^2 = 2$, it is possible by Art. 2 to bisect the tetrahedron $ABCD$ of the first type into two superposable tetrahedra by a plane through AD and the middle point O of BC.

The tetrahedron $ABOD$ is therefore one whose volume is known.

The edges are
$$AO = OD = a\sqrt{2},$$
$$AB = BD = a\sqrt{3},$$
$$AD = 2\,(OB) = 2a.$$

It is not included in any of the three preceding types, but by drawing a plane through BO and the middle point of AD it can be divided into two equal tetrahedra of the first type for which $r^2 = 1$.

Again, all the faces of the tetrahedron of the first type for which $r^2 = 2$ are equal, and all the tetrahedra which have a common vertex at the centre M of the sphere circumscribing that tetrahedron and which stand on the faces of the tetrahedron are equal. Hence the tetrahedron $MABD$ is one whose volume is known.

In this case $MA = MB = MD = \frac{1}{2}a\sqrt{5}$,

$$AB = BD = a\sqrt{3},$$

$$AD = 2a.$$

This does not belong to any of the previous types. But a plane through MB and the middle point of AD divides it into two equal tetrahedra of the third type for which $r^2 = 1$.

Art. 1. *To prove that symmetrical tetrahedra have equal volumes.*

Let $ABCD$ be a tetrahedron, and let the edges BA, CA, DA meeting at the vertex A be produced through A respectively to E, F, G, so that

$$BA = AE, \quad CA = AF, \quad DA = AG;$$

and let EF, FG, GE be joined; then the tetrahedra $ABCD$, $AEFG$ will be shown to be equal.

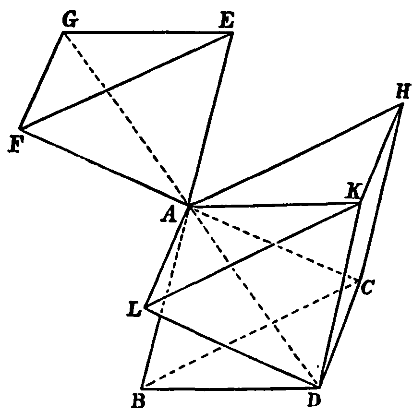

Fig. 1.

Draw CH, DK equal and parallel to BA; and DL equal and parallel to CA.

Join AH, HK, KA, AL, LK.

Then $BCDHKA$ is a prism on base BCD, and its altitude is the perpendicular from A on BCD.

Also $DLKHCA$ is a prism on base ACH, and its altitude is the perpendicular from D on ACH. It is therefore equal to a prism on base ABC, with altitude equal to the perpendicular from D on ABC.

Now the base BCD : the base BCA

= the perpendicular from D on BC : the perpendicular from A on BC

= the perpendicular from D on ABC : the perpendicular from A on BCD.

Hence the prism on base BCD whose altitude is the perpendicular from A on BCD is equal to the prism on base ABC whose altitude is the perpendicular from D on ABC.

Hence the prisms $BCDHKA$, $DLKHCA$ are equal.

The prisms have common the pyramid whose vertex is A, and whose base is the parallelogram $DCKH$.

Taking this away from each prism, it follows that the tetrahedra $ABCD$, $ADKL$ are equal.

Now DA, DL, DK are equal and parallel to AG, AF, AE, respectively, and are drawn in the same directions.

Hence the tetrahedron $ADKL$ can be superposed on the tetrahedron $AEFG$.

Therefore the tetrahedra $ABCD$, $AEFG$ are equal in volume. They are not superposable, but are said to be symmetrically equal.

Art. 2. *To prove that tetrahedra which are images of one another with regard to a common face have equal volumes.*

Let $ABCD$ be a tetrahedron, and let the edge DA be perpendicular to the edges BA, CA, and let DA be produced to E, so that

$$DA = AE,$$

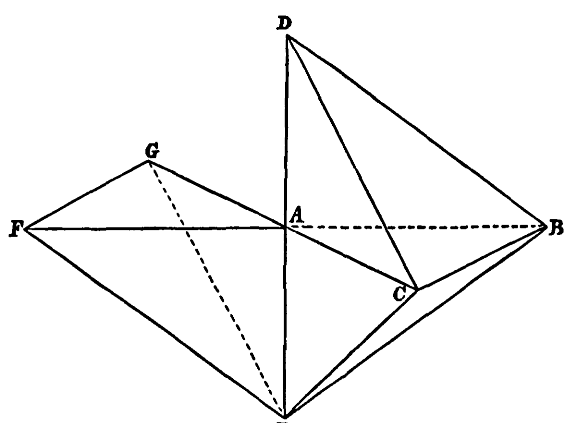

Fig. 2.

and let EB, EC be joined; then the tetrahedra $ABCD$, $ABCE$ will be shown to be equal.

Produce BA, CA to F and G, respectively, so that

$$BA = AF, \quad CA = AG.$$

Then, by the last article, the tetrahedra $ABCD$, $AEFG$ are equal.

But the tetrahedron $AEFG$ can be superposed on the tetrahedron $ABCE$.

Hence the tetrahedra $ABCD$, $ABCE$ are equal.

If, now, ABC be any plane triangle, and P, Q two points on opposite sides of its plane such that PQ is bisected by the plane of ABC at R, then, by the above, the tetrahedra $PRBC$, $PRCA$, $PRAB$ are respectively equal to $QRBC$, $QRCA$, $QRAB$.

Hence the tetrahedra $PABC$, $QABC$ are equal, and these two tetrahedra are images of one another with regard to the common face ABC.

Art. 3. *To prove that a tetrahedron in which the straight line bisecting a pair of opposite edges is perpendicular to those edges can be bisected into two superposable tetrahedra by drawing a plane through either of these edges and the middle point of the other.*

Let AB bisect at right angles CD, EF, a pair of opposite edges o_ the tetrahedron $CDEF$; then the tetrahedra $CDBF$, $CDBE$ are superposable; as also are the tetrahedra $EFAC$, $EFAD$.

Let the figure revolve through two right angles about the straight line AB, which is supposed fixed.

Then the final positions of the points A, B, C, D, E, F are A, B, D, C, F, E, respectively.

Hence the tetrahedron $CDBF$ can be superposed on the tetrahedron $DCBE$.

Hence each of them is equal to half the tetrahedron $CDEF$.

Also the tetrahedron $EFAC$ can be superposed on the tetrahedron $FEAD$.

Hence each of them is equal to half the tetrahedron $CDEF$.

Art. 4. *To construct a tetrahedron whose volume can be determined without using the proposition that tetrahedra on equal bases and having equal altitudes are equal, except in the special form demonstrated in Art.* 2.

(The demonstration of the general case of the proposition just mentioned has not been effected hitherto without employing the method of limits.)

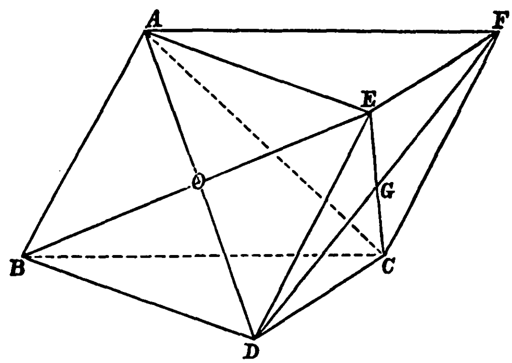

Fig. 3.

Let $ABCD$ be a tetrahedron.

Draw CF, DE equal and parallel to BA.

Join AE, AF, FE, EC.

Let BE and AD meet at O, DF and CE at G.

Then BE, AD bisect each other at O; DF and CE bisect each other at G.

The tetrahedra $ACDB$, $ACDE$ are on the base ACD and have equal altitudes.

The tetrahedra $AECD$, $AECF$ are on the base AEC and have equal altitudes.

Now, if BE be perpendicular to the plane ACD, then the tetrahedra $ADCE$, $ADCB$ are images of one another with regard to their common face ADC. Hence they are equal.

Again, if DF be perpendicular to the plane AEC, it can be shown in the same way that the tetrahedra $ADCE$, $AECF$ are equal.

Hence, if BE be perpendicular to the plane ACD, and if also DF be perpendicular to the plane ACE, then the tetrahedra $ABCD$, $AECD$, $AECF$ are all equal.

Hence the tetrahedron $ABCD$ is equal to one-third of the prism $BCDEFA$.

It remains to find the possible forms of the tetrahedron $ABCD$ which satisfy the conditions that BE, DF are perpendicular to the planes ACD, ACE, respectively.

The length of CE will first be found,

$$CE^2+CB^2 = 2CO^2+2OB^2;$$

therefore
$$CE^2+CB^2+AD^2 = 2\,(CO^2+OD^2)+2\,(OB^2+OD^2)$$
$$= AC^2+CD^2+AB^2+BD^2;$$

therefore $CE^2 = (AB^2+CD^2)+(AC^2+BD^2)-(BC^2+AD^2)$ ......(I.).

Next, if BE be perpendicular to the plane ACD, BE is perpendicular to AD and OC.

If BE, *i.e.*, BO, be perpendicular to AD, then

$$AB = BD \text{ ....(II.).}$$

If BE be perpendicular to OC, then

$$BC = CE \text{ ....(III.).}$$

Hence, by (I.),

$$AD^2+2BC^2 = (AB^2+CD^2)+(AC^2+BD^2) \text{ ....(IV.).}$$

In like manner DF will be perpendicular to the plane AEC, if

$$DC = DE \text{ ....(V.),}$$

and $$CE^2+2AD^2 = AE^2+AC^2+DE^2+DC^2 \text{ ....(VI.).}$$

But $$DE = AB,$$

$$CE = CB,$$

$$AE = DB.$$

Hence (V.) and (VI.) become

$$DC = AB \text{ ....(VII.),}$$

$$BC^2+2AD^2 = (DB^2+AC^2)+(AB^2+DC^2) \text{ ....(VIII.).}$$

From (II.) and (VII.),

$$AB = BD = DC \text{ ....(IX.).}$$

From (IV.) and (VIII.),

$$BC = AD \text{ ....(X.).}$$

From (VIII.), (IX.) and (X.),

$$3BC^2 = 3AB^2+AC^2 \text{ ....(XI.).}$$

Now put $$BC = AD = 2a,$$

and since $$BD+DC > BC,$$

therefore $$2DC > 2a;$$

therefore $$DC > a.$$

Hence it is possible to put

$$DC = a\sqrt{1+r^2};$$

therefore $$AB = BD = DC = a\sqrt{1+r^2}.$$

Then, by (XI.), $AC = a\sqrt{9-3r^2}$.

Hence the edges of the tetrahedron $ABCD$ are given by

$$AB = BD = DC = a\sqrt{1+r^2},$$

$$BC = AD = 2a,$$

$$AC = a\sqrt{9-3r^2}.$$

It is obvious that $0 < r^2 < 3$, and that $BO = ar$.

Any tetrahedron of this kind may be called a tetrahedron of the first type.

It may be shown that the dihedral angles whose edges are BC, AD are right angles.*

The dihedral angle whose edge is CA is 60°.

The cosines of the dihedral angles whose edges are AB, CD are each $\frac{1}{2}\sqrt{3-r^2}$.

The cosine of the dihedral angle whose edge is BD is $\frac{1}{2}(r^2-1)$.

The volume of the tetrahedron will now be found.

It is one-third of the base multiplied by the height, and any side may be taken as base.

Take ABD as base. This face is perpendicular to the face ACD by construction.

Hence volume of $ABCD$

$$= \tfrac{1}{3}\,(\text{area of } ABD) \times (\text{perpendicular from } C \text{ on } AD).$$

Now $$\cos CAD = \frac{9-3r^2+4-(1+r^2)}{4\sqrt{9-3r^2}} = \frac{\sqrt{3-r^2}}{\sqrt{3}}$$

therefore $$\sin CAD = \frac{r}{\sqrt{3}};$$

therefore perpendicular from C on $AD = ar\sqrt{3-r^2}$.

Since ABD is an isosceles triangle,

the perpendicular from B on $AD = ar$;

therefore $$\text{area of } ABD = a^2 r.$$

---

* In any tetrahedron $ABCD$ the cosine of the dihedral angle between the planes BCA and BCD is

$$\frac{(AC^2-AB^2)(DB^2-DC^2)+(BC)^2\left[AC^2+AB^2+BD^2+DC^2-BC^2-2AD^2\right]}{16\,(\text{area of } ABC)(\text{area of } DBC)}.$$

Hence the volume of $ABCD$

$$= \tfrac{1}{3}a^3r^2\sqrt{3-r^2}.$$

It may be noticed that the three tetrahedra $ABCD$, $ACDE$, $ACEF$ have their edges all equal in this particular case. It is possible to superpose $ABCD$ on $ACEF$ by rotation round AC through 120°. But $ACDE$ is symmetrically equal to either of the other two tetrahedra.

It will now be shown that the tetrahedron $ABCD$ of the first type can be dissected into eight equal tetrahedra of which six are superposable; the other two are also superposable, but they are symmetrically equal with regard to any one of the six.

As in the third proposition of Euclid's Twelfth Book (see Fig. 4),

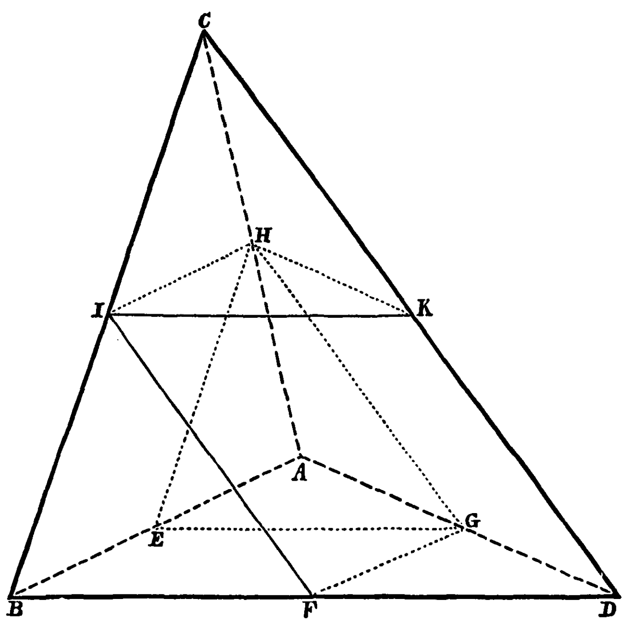

FIG. 4.

take E, F, G, H, I, K the middle points of AB, BD, DA, AC, BC, DC, respectively. Then the tetrahedron $ABCD$ breaks up into the two tetrahedra $CIHK$, $AHEG$, and the two prisms $FDGHIK$, $BFIHGE$.

The tetrahedra $CIHK$, $AHEG$ are superposable to one another and similar to $ABCD$.

On examining Fig. 3 it will be seen that the prism *BDCFEA* of that figure is drawn on the isosceles base *BDC* with its edges parallel to *BA*, the side of the tetrahedron equal to either of the equal sides of the isosceles base.

Now, taking in Fig. 4

$$AB = BD = DC = a\sqrt{1+r^2},$$

$$AD = BC = 2a,$$

$$AC = a\sqrt{9-3r^2},$$

it is at once seen that the prism *FDGHIK* stands on the isosceles triangle *FGD* as base, and has its edges parallel to *DK*, which is equal to the equal sides of the isosceles base *FGD*. Hence this prism breaks up into the tetrahedron *FDGK*, one superposable to it, and one symmetrically equal to it.

In like manner, the tetrahedron *BFIHGE* stands on the isosceles triangle *BFI* as base, and has its edges parallel to *BE*, which is equal to the equal sides of the isosceles base *BFI*. Hence this prism is decomposable into the tetrahedron *BIEF*, one superposable to it, and one symmetrically equal to it.

The tetrahedra *FDGK*, *BIEF*, *CIHK*, *AHEG* are superposable to one another, and similar to the whole tetrahedron.

Hence the whole tetrahedron *ABCD* is decomposable into six tetrahedra equal and superposable to *AHEG*, and two tetrahedra symmetrically equal to *AHEG*. All these tetrahedra are equal. Their linear dimensions are half of those of *ABCD*.

Art. 5. *To derive other types of tetrahedra from the first type.*

The tetrahedron *ABCD* is symmetrical with regard to the line joining the middle points of *AC*, *BD*.

Hence this joining line is perpendicular to *AC*, *BD*.

It will be useful to prove this in another way.

Let *J*, *K* be the middle points of *AC*, *BD*, respectively.

Then $$2JD^2+2AJ^2 = AD^2+DC^2,$$

$$2JB^2+2AJ^2 = AB^2+BC^2.$$

Now $$AD = BC = 2a,$$

$$AJ = \tfrac{1}{2}a\sqrt{9-3r^2},$$

$$AB = DC = a\sqrt{1+r^2};$$

therefore $$JB = JD = \frac{a}{2}\sqrt{1+5r^2}.$$

Also

$$KB = KD,$$
$$JK = JK;$$

therefore JK is perpendicular to BD.

Again

$$2KA^2 + 2KB^2 = AB^2 + AD^2,$$
$$2KC^2 + 2KB^2 = BC^2 + CD^2,$$
$$AD = BC = 2a,$$
$$KB = \tfrac{1}{2}a\sqrt{1+r^2},$$
$$AB = CD = a\sqrt{1+r^2};$$

therefore

$$KA = KC = \frac{a}{2}\sqrt{9+r^2}.$$

Also

$$JA = JC,$$
$$KJ = KJ;$$

therefore KJ is perpendicular to AC.

Hence KJ is perpendicular to the opposite edges AC and BD and bisects them both.

Hence, applying the proposition of Art. 3, it follows that the tetrahedra $ACKB$, $ACKD$, $BDJA$, $BDJC$ are each half the tetrahedron $ABCD$.

Hence the tetrahedra $BDJA$, $ACKB$ (which are respectively superposable on the other two) will give the two new types.

Art. 6. *The second type of tetrahedron.*

Consider first the tetrahedron $BDJA$, which will be referred to as the tetrahedron of the second type.

The sides are

$$AD = 2a,$$
$$AJ = \tfrac{1}{2}a\sqrt{9-3r^2},$$
$$BA = BD = a\sqrt{1+r^2},$$
$$JB = JD = \tfrac{1}{2}a\sqrt{1+5r^2}.$$

The dihedral angle whose edge is AD is a right angle.

The dihedral angle whose edge is AJ is 60°.

The cosine of the dihedral angle whose edge is AB is $\tfrac{1}{2}\sqrt{3-r^2}$.

The cosine of the dihedral angle whose edge is DB is $\tfrac{1}{2}\sqrt{1+r^2}$.

The cosine of the dihedral angle whose edge is JD is $\dfrac{\sqrt{3-r^2}}{2\sqrt{1+r^2}}$.

The cosine of the dihedral angle whose edge is JB is $-\dfrac{\sqrt{3-r^2}}{2\sqrt{1+r^2}}$.

[The tetrahedron of the second type may be derived from that of the first type in other ways, for example:—

If O be the middle point of AD, and CO be produced to H, so that $CO = OH$, then $BCDH$ is a tetrahedron of the second type.

Its sides are
$$HC = 2a\sqrt{4-r^2},$$
$$HD = a\sqrt{9-3r^2},$$
$$BC = BH = 2a,$$
$$BD = DC = a\sqrt{1+r^2}.$$

Putting
$$a = \tfrac{1}{2}A\sqrt{1+R^2},$$
$$r^2 = \frac{4R^2}{1+R^2},$$
$$HC = 2A,$$
$$HD = \tfrac{1}{2}A\sqrt{9-3R^2}$$
$$BC = BH = A\sqrt{1+R^2},$$
$$BD = DC = \tfrac{1}{2}A\sqrt{1+5R^2},$$

showing that the tetrahedron belongs to the second type.]

Art. 7. *The third type of tetrahedron.*

Consider next the tetrahedron $ACKB$, which will be referred to as the tetrahedron of the third type.

Its sides are
$$AC = a\sqrt{9-3r^2},$$
$$BK = \tfrac{1}{2}a\sqrt{1+r^2},$$
$$AB = a\sqrt{1+r^2},$$
$$BC = 2a,$$
$$KA = KC = \tfrac{1}{2}a\sqrt{9+r^2}.$$

The dihedral angle whose edge is BC is a right angle.

The dihedral angle whose edge is AC is 30°.

The cosine of the dihedral angle whose edge is AB is $\tfrac{1}{2}\sqrt{3-r^2}$.

The cosine of the dihedral angle whose edge is BK is $\tfrac{1}{2}(r^2-1)$.

The cosine of the dihedral angle whose edge is KC is $\dfrac{\sqrt{3-r^2}}{2\sqrt{3}}$.

The cosine of the dihedral angle whose edge is KA is $-\dfrac{\sqrt{3-r^2}}{2\sqrt{3}}$.

Art. 8. *Comparison of the three types of tetrahedra.*

The tetrahedron of the first type has three equal sides, a character not possessed *in general* by either of the tetrahedra of the second and third types.

Hence the tetrahedron of the first type is distinct from the tetrahedra of the second and third types.

The tetrahedron of the second type has two isosceles faces, whilst that of the third type has only one isosceles face.

Hence the three types are all distinct.

Art. 9. *Special tetrahedra not included in any of the previously mentioned types.*

(A) A very simple case of the tetrahedron of the first type is when $r^2 = 2$.

Then $$AB = BD = DC = CA = a\sqrt{3},$$

$$AD = BC = 2a.$$

Let O be the middle point of BC.

Then BC is perpendicular to the plane DOA, and O is the middle point of BC.

Hence the tetrahedra $ABDO$, $ACDO$ are equal by Art. 2.

Hence the tetrahedron $ABDO$ is one whose volume can be found without using the method of limits.

Its sides are $$AO = OD = a\sqrt{2},$$

$$AB = BD = a\sqrt{3},$$

$$AD = 2a,$$

$$OB = a.$$

This tetrahedron does not belong to any of the preceding types.

But, if it be bisected by drawing a plane through BO and the middle point E of AD, then the tetrahedron $ABEO$ has the sides

$$BO = OE = EA = a,$$

$$AO = BE = a\sqrt{2},$$

$$AB = a\sqrt{3},$$

and this belongs to the first type, r^2 being equal to 1.

(B) All the faces of the tetrahedron of the first type when $r^2 = 2$ are equal. Hence from the symmetry of the figure it follows that the four tetrahedra which have a common vertex at the centre M of the circumscribing sphere are all equal, *i.e.*, superposable or symmetrically equal.

The radius of the circle circumscribing ABD is

$$\frac{AB \,.\, BD \,.\, DA}{4 \text{ area of } ABD} = \tfrac{3}{4}a\sqrt{2}.$$

Hence the distance of this centre from the middle point P of AD is $\frac{1}{4}a\sqrt{2}$.

Hence the distance of the centre of the circumscribing sphere from P is $\frac{1}{2}a$.

Hence the radius of the circumscribing sphere is $\frac{1}{2}a\sqrt{5}$.

The tetrahedra whose bases are ABD, ACD, and which have a common vertex at the centre of the circumscribing sphere, though not superposable, are symmetrically equal.

Hence the tetrahedron $MABD$ is one whose volume is known, where

$$MA = MB = MD = \tfrac{1}{2}a\sqrt{5},$$

$$AB = BD = a\sqrt{3},$$

$$AD = 2a.$$

But, if a plane be drawn through MB and P the middle point of AD, then $MABD$ is divided into two equal tetrahedra. Consider $MPDB$.

$$MP = \tfrac{1}{2}a,$$

$$PD = a,$$

$$MB = MD = \tfrac{1}{2}a\sqrt{5},$$

$$PB = a\sqrt{2},$$

$$BD = a\sqrt{3}.$$

Putting

$$A\sqrt{9-3R^2} = a\sqrt{3},$$

$$A\sqrt{1+R^2} = a,$$

$$2A = a\sqrt{2},$$

$$A\sqrt{9+R^2} = a\sqrt{5},$$

all which are satisfied by putting

$$A = a/\sqrt{2}$$

and $$R^2 = 1,$$

showing that it is a tetrahedron of the third type.

Supplementary Remark.—If $ABCD$ be a tetrahedron of the first type, and if P be the middle point of BC and O the middle point of AD, then the tetrahedron $APOC$ belongs to the first type.

In this case $$PD = OB = ar,$$

$$OC = PA = a\sqrt{4-r^2},$$

whence $$PO = a.$$

The sides of $APOC$ are therefore

$$AO = OP = PC = a,$$

$$AP = OC = a\sqrt{4-r^2},$$

$$AC = a\sqrt{9-3r^2},$$

or, putting $$a = A\sqrt{1+R^2},$$

$$r^2 = \frac{4R^2}{1+R^2},$$

$$a\sqrt{4-r^2} \text{ becomes } 2A,$$

and $$a\sqrt{9-3r^2} \text{ becomes } A\sqrt{9-3R^2},$$

showing that it belongs to the first type.

Note by Lieut.-Col. Allan Cunningham, R.E.

Lt.-Col. Allan Cunningham, R.E., announced the discovery of a new criterion to determine when 2 is a 16-ic residue of a prime (p), viz.:

p must be of the two quadratic forms

$$p = (8\alpha \pm 1)^2 + (16\beta')^2 = (8\gamma \pm 1)^2 + 2\,(4\delta)^2,$$