On the Disturbance produced by an Element of a Plane Wave of Sound or Light. By A. B. BASSET.

[Read May 14th, 1891.]

1. The effect of the disturbance produced at a distant point by an element of a plane wave of light, was first investigated by Sir G. Stokes,* who showed that, upon the assumption that the equations of motion of the luminiferous ether are of the same form as those of an isotropic elastic solid, the disturbance produced at a distant point P by an element dS of the plane wave

$$\zeta = c \sin \frac{2\pi}{\lambda} (bt - x),$$

is given by the equation

$$\zeta' = \frac{cdS}{2\lambda r} (1 + \cos \theta) \sin \phi \cos \frac{2\pi}{\lambda} (bt - r) \dots\dots\dots (1),$$

where r is the distance of the element dS from P , and θ and ϕ are the angles which the direction of r makes with the direction of propagation and vibration respectively.

The method employed by Stokes is equally applicable to the electromagnetic theory of light, inasmuch as the equations of electric displacement (upon Maxwell's theory), are of the same form as those which are satisfied by those portions of the displacement of an elastic solid upon which distortion unaccompanied by dilatation depends.

The question of the resolution of plane waves has recently been discussed by Lord Rayleigh,† who has suggested that the method of resolution employed by Stokes is the simplest, but only one of an indefinite number which might be proposed, all of which are equally legitimate so long as the question is regarded as a purely mathematical one; and that the purely mathematical question has no definite answer. I propose, in the present paper, to consider the question raised by Lord Rayleigh in detail; but it will be convenient to commence with the case of waves of sound, because the analysis, depending as it does on a single function, is far more simple than in the case of light.

* "Dynamical Theory of Diffraction," *Trans. Camb. Phil. Soc.*, Vol. ix., p. 1; and *Math. and Phys. Papers*, Vol. II., p. 243.

† *Encycl. Brit.*, Art. "Wave Theory," pp. 452-454.

2. To fix our ideas, let us suppose that a source of sound acts for a very short time and then ceases. A spherical wave will be generated, which will be propagated through space, and the disturbance at any particular epoch will be confined to a thin sheet bounded by two spheres concentric with the source. Any point P outside this space will remain at rest until the outer boundary of the sheet reaches it; it will then be in a state of disturbance until the inner boundary reaches it, and, after the latter has passed over it, P will remain at rest for ever. If the source be periodic, the disturbance produced by it may be conceived to consist of a succession of very thin sheets, each one of which as it passes over P creates a disturbance, and then travels on, leaving no trace of its existence behind it.

Let us now take up our position at a point upon a *fixed* sphere whose centre is the source, and let t be the time which the wave occupies in travelling from this sphere to an external point P . Then the disturbance which actually exists at P at any particular instant, must depend upon that which existed at the surface of this sphere at a previous epoch t . And since every element of the sphere may be supposed to contribute something to the disturbance at P , it follows that the disturbance produced by an element dS must be equal to σdS , where σ is to be determined from the condition, that the value at P of the integral $\iint \sigma dS$ taken over the whole surface of the sphere, is equal to the disturbance which actually exists at P , whilst its value at any internal point is zero.

The function σ must be capable of being represented mathematically by a series of multiple sources of sound; and it therefore follows that, in order to determine the effect at P of the successive waves which are continually being transmitted across the sphere, we may, if we please, suppose the original source of sound to be removed, and a spherical sheet of multiple sources to be substituted for it, which at an external point produces the same effect as the original source, and whose effect at an internal point is zero.

The question we have to consider is, whether this distribution of multiple sources is unique, or whether there are an infinite number of distributions which will produce the required effect; and we shall show that the latter is the case.

3. The simplest case which we can have, is that of plane waves of sound which are travelling parallel to the axis of x , and whose velocity potential (the time factor $e^{i\omega t}$ being omitted) may be taken to be

$$\phi = e^{-i\omega x};$$

and we have to consider what distribution of sources over the plane $x = 0$ will produce this velocity potential when x is positive, and zero velocity potential when x is negative.

It is evident that the required distribution cannot consist of *simple* sources, since a simple source would produce a disturbance on the negative side of the plane; if, however, we combine a simple source with a double source whose axis is perpendicular to the plane, the required effect may be produced, and we shall now show that this will be the case if we take

$$\sigma = \frac{\kappa \epsilon^{-\kappa r}}{4\pi r} + \frac{\kappa x}{4\pi r^2} \left(1 + \frac{1}{\kappa r}\right) \epsilon^{-\kappa r} \dots\dots\dots (2),$$

where

$$\kappa = 2\pi/\lambda.$$

Restoring the time factor, the real part of this expression is

$$\sigma = -\frac{1}{2\lambda r} (1 + \cos \theta) \sin \frac{2\pi}{\lambda} (at - r) + \frac{\cos \theta}{4\pi r^2} \cos \frac{2\pi}{\lambda} (at - r) \dots\dots (3),$$

where θ is the angle which the line joining the element to P , makes with the normal to the wave front drawn in the direction of propagation.

Whence, if x is positive, $\cos \theta = x/r$, and

$$\begin{aligned} \iint \sigma dS &= -\frac{\pi}{\lambda} \int_x^\infty \left(1 + \frac{x}{r}\right) \sin \frac{2\pi}{\lambda} (at - r) dr + \frac{1}{2} \int_x^\infty \frac{1}{r^2} \cos \frac{2\pi}{\lambda} (at - r) dr \\ &= \cos \frac{2\pi}{\lambda} (at - x). \end{aligned}$$

If, however, x is negative, we must put $\cos \theta = -x/r$, and the integral will be zero.

4. We have now to consider whether this distribution is the only one which can produce the required effect.

The most general expression for a multiple divergent source which is symmetrical with respect to the axis of x is*

$$\Psi_n = \psi_n P_n,$$

where P_n is a zonal harmonic, and

$$\psi_n = \frac{\epsilon^{-\kappa r}}{r} f_n(\kappa r),$$

* Stokes, *Phil. Trans.*, 1868; and Lord Rayleigh, *Theory of Sound*, Chap. xvii.

where

$$f_n(x) = 1 + \frac{n(n+1)}{2x} + \frac{(n-1)n(n+1)(n+2)}{2 \cdot 4x^2} + \dots \\ \dots + \frac{1 \cdot 2 \cdot 3 \dots 2n}{2 \cdot 4 \cdot 6 \dots 2n x^n} \dots \dots \dots (4).$$

The function ψ can also be shown to satisfy the following equations,

$$\text{viz.,} \quad r \frac{d\psi_n}{dr} - n\psi_n = -\kappa r \psi_{n+1} \dots \dots \dots (5),$$

$$r \frac{d\psi_n}{dr} + (n+1)\psi_n = -\kappa r \psi_{n-1} \dots \dots \dots (6),$$

$$(2n+1)\psi_n = \kappa r (\psi_{n+1} - \psi_{n-1}) \dots \dots \dots (7),$$

which are frequently useful.

Now, if ϕ and ψ be any functions which satisfy the equation

$$(\nabla^2 + \kappa^2)\phi = 0,$$

and which vanish at infinity, it follows that, if we apply Green's theorem to any periphractic region,

$$\iint \phi \frac{d\psi}{d\nu} dS = \iint \psi \frac{d\phi}{d\nu} dS \dots \dots \dots (8),$$

where the surface integrals are to be taken over the internal boundaries. If, however, ϕ or ψ become infinite at any point of the region, we must describe a small sphere round such a point as centre, take the surface integrals over the surface of this sphere, and then make the radius of the latter zero.

Let

$$\phi = e^{-\kappa x},$$

$$\psi = A\psi_n P_n;$$

where x is the distance of any point P on the axis of x from the plane of resolution (that is, the plane $x = 0$), r is the distance of any point on this plane from P , and θ is the angle which the direction of r makes with the normal and the plane drawn in the direction of propagation.

Applying Green's theorem to the region lying on the positive side of the plane $x = 0$, it follows that, if P lies on the negative side of this plane, ψ does not become infinite at any point of the region considered; whence (8) becomes

$$\iint \phi \frac{d\psi}{d\nu} dS + \kappa \iint \psi dS = 0 \dots \dots \dots (9).$$

If, however, P lies on the positive side of the plane, ψ becomes infinite at P , whence (8) becomes

$$\iint \frac{d\psi}{d\nu} dS + \kappa \iint \psi dS = -c^2 \epsilon^{-\kappa x} \iint \epsilon^{\kappa c \cos \theta} \frac{d\psi}{dc} d\Omega \\ + c^2 \epsilon^{-\kappa x} \iint \psi \frac{d}{dc} (\epsilon^{\kappa c \cos \theta}) d\Omega \dots \dots \dots (10);$$

where dS is an element of the plane $x = 0$, and $d\Omega$ is the solid angle subtended at P by an element of the surface of a sphere of radius c described about P as centre.

Now* if

$$\mu = \cos \theta,$$

$$\epsilon^{\kappa c \cos \theta} = \sum_0^\infty F_i(c) P_i(\mu),$$

where

$$F_i(c) = \frac{(\kappa c)^i}{1 \cdot 3 \cdot 5 \dots (2i-1)} \left\{ 1 - \frac{\kappa^2 c^2}{2 \cdot 2i+3} + \frac{\kappa^4 c^4}{2 \cdot 4 \cdot 2i+3 \cdot 2i+5} - \dots \right\},$$

whence the right-hand side of (10) becomes

$$Ac^2 \epsilon^{-\kappa x} \iint \left\{ \psi_n P_n \Sigma \left(\frac{dF_i}{dc} P_i \right) - \frac{d\psi_n}{dc} P_n \Sigma (F_i P_i) \right\} d\Omega \\ = \frac{4\pi A c^2 \epsilon^{-\kappa x}}{2n+1} \left(\psi_n \frac{dF_n}{dc} - F_n \frac{d\psi_n}{dc} \right).$$

Substituting the values of ψ_n and F_n , and making $c = 0$, it will be found that the right-hand side is equal to $4\pi A \epsilon^{-\kappa x}$; and, since this is to be equal to $\epsilon^{-\kappa x}$, we must have $4\pi A = 1$, whence on the positive side of the plane

$$\iint \frac{d\psi}{d\nu} dS + \kappa \iint \psi dS = \epsilon^{-\kappa x} \dots \dots \dots (11),$$

where

$$\psi = \frac{\psi_n P_n}{4\pi} \dots \dots \dots (12).$$

Since $d/d\nu = -d/dx$, it follows that the effect of an element dS is represented by the expression

$$\left(\kappa \psi - \frac{d\psi}{dx} \right) dS \dots \dots \dots (13),$$

where ψ is given by (12). The first term of this expression represents

* Rayleigh, *Theory of Sound*, § 334.

a multiple source of the n^{th} order; to find what the second term is, we have

$$\frac{d}{dx} = \mu \frac{d}{dr} + \frac{1-\mu^2}{r} \frac{d}{d\mu};$$

whence, by (12),

$$\begin{aligned} 4\pi \frac{d\psi}{dx} &= \mu P_n \frac{d\psi_n}{dr} + \frac{1-\mu^2}{r} \psi_n \frac{dP_n}{d\mu} \\ &= -\kappa\mu P_n \psi_{n+1} + \frac{n\psi_n}{r} P_{n-1} \\ &= -\frac{\kappa}{2n+1} \{n\psi_{n-1} P_{n-1} + (n+1)\psi_{n+1} P_{n+1}\}. \end{aligned}$$

Accordingly, (13) becomes

$$\frac{\kappa dS}{4\pi (2n+1)} \{n\psi_{n-1} P_{n-1} + (2n+1)\psi_n P_n + (n+1)\psi_{n+1} P_{n+1}\} \dots\dots (14).$$

The effect of the element may, therefore, be represented by the combination of three multiple sources of orders $n-1$, n , $n+1$, where n is zero or any positive integer. If, therefore, the cause which produces the train of plane waves be removed, and the distribution of multiple sources given by (14) be substituted for it, the effect of this distribution at points lying on the positive side of the plane $x=0$ will be the same as that of the original train of waves, whilst the effect at points lying on the negative side will be zero.

If we put $n=0$, we shall fall back on our previous result. This result, therefore, gives the simplest way of resolving a plane wave, but it is only one out of an infinite number.

5. The preceding investigation renders it unnecessary to consider the corresponding optical problem in so much detail; but, inasmuch as the theory of singular points has proved of very great utility in a variety of branches of physics, it will be desirable to show that a similar theory exists in the case of elastic solids and other media whose motion is governed by similar equations.

The general equations of motion of an isotropic elastic solid are of the form

$$\frac{d^2 u}{dt^2} = (a^2 - b^2) \frac{d\delta}{dx} + b^2 \nabla^2 u \dots\dots\dots (15),$$

&c., &c.,

where a and b are the velocities of propagation of the dilatational and distortional waves respectively.

Assuming that all the quantities vary as $e^{i\delta}$, and putting $\kappa = s/b$, $h = s/a$, (15) becomes

$$(\nabla^2 + \kappa^2) u + \left(\frac{\kappa^2}{h^2} - 1 \right) \frac{d\delta}{dx} = 0 \dots \dots \dots (16).$$

The solution of (16) is

$$u = \frac{d\phi}{dx} + u' \dots \dots \dots (17),$$

where

$$(\nabla^2 + h^2) \phi = 0 \dots \dots \dots (18),$$

and u', v', w' each satisfy an equation of the form

$$(\nabla^2 + \kappa^2) u' = 0 \dots \dots \dots (19),$$

subject to the condition

$$\frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz} = 0 \dots \dots \dots (20).$$

The function ϕ determines the dilatational or sound wave which we have already discussed. We must now consider the distortional or optical wave, which depends upon the quantities u', v', w' .

It is known that (19) are satisfied by

$$u' = r^{-n} \psi_n X_n, \quad v' = r^{-n} \psi_n Y_n, \quad w' = r^{-n} \psi_n Z_n,$$

where ψ_n is the function $r^{-1} e^{-i\kappa r} f_n(\kappa r)$, and X_n, Y_n, Z_n are solid harmonics of positive degree n ; and we have to determine these harmonics, so that u', v', w' shall satisfy (20). By means of a process similar to that employed by Lamb,* it can be shown that the required value of u' is

$$u = \frac{\psi_n}{r^n} \left(\frac{d\phi_{n+1}}{dx} + y \frac{d\chi_n}{dz} - z \frac{d\chi_n}{dy} \right) + \frac{(n+1) \psi_{n+2} r^{n+3}}{n+2} \frac{d}{dx} \left(\frac{\phi_{n+1}}{r^{2n+3}} \right) \dots (21),$$

with symmetrical expressions for v', w' , where ϕ_n and χ_n are positive solid harmonics of degree n .

The right-hand side of (21) is easily seen to be a solution of (19), and by means of (5) and (6), and the known properties of solid harmonics, it can be verified by trial that these values satisfy (20).

6. The simplest solution is obtained by putting $n = 0$, in which case

$$\psi_0 = r^{-1} e^{-i\kappa r}, \quad \phi_1 = Ax + By + Cz, \quad \chi_0 = \text{const.};$$

* *Proc. Lond. Math. Soc.*, Vol. XIII., p. 51.

whence

$$u_0 = \frac{Ae^{-\kappa r}}{r} + \frac{e^{-\kappa r}}{2r^3} \left\{ 1 + \frac{3}{\kappa r} + \frac{3}{(\kappa r)^2} \right\} \left\{ Ar^2 - 3(Ax + By + Cz)x \right\} \dots\dots\dots (22).$$

This expression may be regarded as giving the value of u for a *simple source of light*, and it corresponds to a source in hydrodynamics, or to an electrified point. The expression is, however, more analogous to a doublet or a magnet, inasmuch as a simple source of light has *direction* as well as *magnitude*. The direction cosines of the axis of the source are proportional to A, B, C ; and, if we suppose that its axis is parallel to z , we shall have $A = B = 0$. Also, at a considerable distance from the source, λ is usually so small in optical problems, in comparison with r , that powers of $(\kappa r)^{-1}$ may be neglected; whence, writing $F = \frac{3}{2}C$, we shall obtain

$$\left. \begin{aligned} u_0 &= -\frac{Fxx}{r^3} e^{-\kappa r} \\ v_0 &= -\frac{Fyz}{r^3} e^{-\kappa r} \\ w_0 &= \frac{F(x^2 + y^2)}{r^3} e^{-\kappa r} \end{aligned} \right\} \dots\dots\dots (23).$$

In the figure let P be the point x, y, z ;

also let $\theta = POx, \phi = POz$;

draw PT perpendicular to OP in the plane POz . Then the preceding equations show that the direction of vibration is along PT , and its magnitude is equal to

$$\frac{F}{r} e^{-\kappa r} \sin \phi.$$

Restoring the time factor $e^{i\kappa t}$ and realising, this becomes

$$-\frac{F}{r} \sin \phi \cos \frac{2\pi}{\lambda} (bt - r) \dots\dots\dots (24).$$

This is the expression for the disturbance produced by a simple source of light at a point r , whose distance from the source is large compared with the wave length; and it is of the same form as the expression obtained by Lord Rayleigh,* for the effect of a periodic

* *Phil. Mag.*, Feb., 1871.

force which acts at the origin and in the direction of the axis of z . The motion is, as might be expected, symmetrical with respect to the axis of z , and vanishes on that axis where $\phi = 0$ or π , and it is a maximum on the plane xy where $\phi = \frac{1}{2}\pi$.

7. In order to obtain the most general expression for a singular point of the second order, we must put $n = 1$; whence

$$\phi_2 = (B-C)x^2 + (C-A)y^2 + (A-B)z^2 + 2A'yz + 2B'zx + 2C'xy,$$

$$\chi_1 = \alpha x + \beta y + \gamma z;$$

and

$$u_1 = \frac{\psi_1}{r} \left(\frac{d\phi_2}{dx} + \gamma y - \beta z \right) + \frac{2}{3} \psi_3 r^4 \frac{d}{dx} \left(\frac{\phi_2}{r^5} \right) \dots\dots\dots (25).$$

The expression for a singular point of the second order accordingly contains eight constants, and is therefore a function of considerable generality. Let us now suppose, as a particular case, that

$$\phi_2 = \frac{1}{16} F z x, \quad \chi_1 = \frac{1}{2} F y;$$

then, if we confine our attention to points at a considerable distance from the origin, we may put

$$\psi_1 = \psi_3 = r^{-1} \epsilon^{-i\alpha r},$$

whence

$$\left. \begin{aligned} u_1 &= -\frac{F x^2 z}{r^4} \epsilon^{-i\alpha r} \\ v_1 &= -\frac{F x y z}{r^4} \epsilon^{-i\alpha r} \\ w_1 &= \frac{F z (x^2 + y^2)}{r^4} \epsilon^{-i\alpha r} \end{aligned} \right\} \dots\dots\dots (26).$$

It therefore follows that the magnitude of the displacement represented by (26) is

$$\frac{F}{r} \epsilon^{-i\alpha r} \sin \phi \cos \theta \dots\dots\dots (27),$$

and that its direction is along PT . Restoring the time factor, adding (24) and (27), and writing $cdS/2\lambda$ for F , we obtain

$$\frac{cdS}{2\lambda r} (1 + \cos \theta) \sin \phi \cos \frac{2\pi}{\lambda} (bt - r),$$

which is Stokes' result.

We therefore see that Stokes' expression for the disturbance produced by an element of a plane wave of light is equivalent to the combination of a simple and a double source, whilst Lord Rayleigh's expression* represents a simple source alone. It appears to me that Stokes' result is the preferable one of the two; partly for the reason that a distribution of simple sources of light, like simple sources of sound, produces a motion behind the wave, whilst the combination of a simple and double source in the above expression, as Stokes has shown, gives zero motion behind the wave; and also because the effect of an element upon a distant point must not only depend upon the direction of vibration (*i.e.*, upon ϕ), but also on the inclination of the line joining the point and the element with the plane of the latter. At the same time, if we were to carry out the investigation on the same lines as has been done in the case of sound at the commencement of this paper, there can, I think, be little doubt that we should find that there is an infinite number of combinations of multiple sources which would produce the required effect, and consequently Stokes' law, although the simplest, is only one out of an infinite number. The question is not, however, of very much importance in the case of light, inasmuch as, in problems relating to diffraction, we may with sufficient accuracy take $\sin \phi = \cos \theta = 1$, in which case the disturbance due to the element will be

$$\frac{cdS}{\lambda r} \cos \frac{2\pi}{\lambda} (bt - r),$$

corresponding to the wave

$$w = c \sin \frac{2\pi}{\lambda} (bt - x).$$

[8.† By means of the general solution (21) combined with (17) which gives the dilatational wave, it would be possible to work out the problem of a spherical pendulum vibrating in ether, and also the scattering of a plane wave of light by a fixed or movable spherical obstacle. But, before attacking any problem of this nature, it would be first of all necessary to consider the boundary conditions.

With regard to these conditions three suppositions may be made.

1st. We may suppose that no slipping takes place, which requires that the velocity of the ether in contact with the sphere should be

* *Encycl. Brit.*, Art. "Wave Theory," p. 453, equation (20).

† Rewritten June 14th, 1891.

equal to that of the sphere itself; but, inasmuch as there are reasons for thinking that the amplitudes of the vibrations of the matter are very much smaller than those of the ether in contact with it, except in the extreme case in which one of the free periods of the matter is equal to the period of the ethereal waves, this hypothesis is improbable.

2nd. We may suppose that partial slipping takes place. This hypothesis is open to the objection that the law of slipping is unknown, and would therefore involve an additional assumption; and also that it would introduce frictional resistance. It might, at first sight, appear that frictional resistance might be explained by supposing that there is a conversion of energy into heat; but, in attempting to construct a molecular theory, we ought to regard radiant heat as consisting of ethereal waves of longer period than waves of light.

3rd. We may suppose that perfect slipping takes place. In this case the boundary conditions are continuity of normal motion, and zero tangential stress. This hypothesis has much to commend it on the ground of simplicity, since the action of the ether on the matter consists of a hydrostatic pressure, and in the case of a sphere is consequently reducible to a force; whereas, if no slipping or partial slipping took place, the action would (except in special cases) consist of a couple as well as a force.

The solution of the problems suggested would undoubtedly throw light on the phenomena of absorption and anomalous dispersion; but the question of absorption can be illustrated just as well, and more simply as far as analysis is concerned, by considering the corresponding acoustical problem.]

9. The theory of absorption, which was first suggested by Stokes,* and afterwards amplified and more fully developed by Sellmeier,† depends upon the theorem, that when a dynamical system is subjected to a force whose period is equal, or nearly so, to one of the free periods of the system, the amplitude of the corresponding forced vibration will be large.‡ In certain cases, the amplitude of the forced vibration

* *Phil. Mag.*, March 1860, p. 196.

† *Logg. Ann.*, Vol. cxlv., pp. 399, 520; cxlvii., pp. 386, 525.

‡ [It must be recollected that Herschel's theorem, that the period of the forced vibration is equal to that of the force, is only true when the molecular forces depend upon the first powers of displacements. If, for example, the equation of motion were of the form

$$\frac{d^2 y}{dt^2} + \mu^2 y^3 = \frac{2p^3}{3\mu\sqrt{3}} \cos 3pt,$$

becomes infinite when the periods are equal; and, in order to avoid this difficulty, Helmholtz* in his theory of anomalous dispersion introduced a viscous term (*i.e.*, a term proportional to the velocity), which he endeavoured to account for on the supposition that it represented a conversion of energy into heat. The introduction of this term is objected to by Sir W. Thomson,† on the ground that we have no right to suppose that the luminiferous ether is a medium which is capable of exerting any action in the nature of frictional resistance; but, on the other hand, Sir W. Thomson's molecular theory is open to the objection which Helmholtz sought to evade, inasmuch as the amplitudes of the vibrations of his molecules and the index of refraction become infinite when the period of the waves of light is equal to any one of the free periods of the matter. It will therefore be desirable to show that the viscous term introduced by Helmholtz may be accounted for on physical grounds, which are not open to Sir W. Thomson's objections.

If plane waves of sound whose period is τ impinge upon a spherical obstacle, whose radius is small in comparison with the wavelength of the waves of sound, and if the sphere be attached to a spring so as to be capable of vibrating parallel to the direction in which the waves are travelling, it can be shown that the effect of the waves will be to cause the sphere to perform small oscillations. The energy of the sphere is supplied in the first instance by the waves which impinge upon it, and the motion of the sphere generates divergent waves which travel away into space carrying energy with them. If, therefore, the incident waves ceased, the sphere would

the forced vibration would be

$$y = \frac{2\eta}{\mu\sqrt{3}} \cos pt;$$

whence the period of the forced vibration is three times that of the force. An equation of this kind illustrates the way in which ultra-violet waves might be converted into luminous waves. See Stokes, "On the Change of Refrangibility of Light," *Phil. Trans.*, 1852, p. 463.

The complementary function is

$$y = a \operatorname{cn}(a\mu t + \alpha), \quad k = 2^{-\frac{1}{2}},$$

where a and α are the constants of integration. The amplitude is a , and the period $4K/\mu a$. This result is an example of a vibration whose period depends upon its amplitude. If, therefore, we were to suppose that waves of dark heat falling upon a body were capable of producing vibrations of this form, in which a increases slowly with the time, the body would first become hot, then red-hot, and afterwards intensely luminous.—*Added June 14th, 1891.*]

* *Wissen. Abhand.*, Vol. II., p. 213.

† "Lectures on Molecular Dynamics," p. 149.

gradually get rid of all its energy by generating scattered waves, and would ultimately come to rest; hence the equation of motion of the sphere must contain a viscous term, notwithstanding the fact that the system is assumed to be devoid of viscosity.

Let τ be the period of the sound waves, τ' the free period of the sphere, M the mass of the latter, and M' the mass of the fluid displaced; also let the radius c of the sphere be so small in comparison with the wave-length of the sound waves, that powers of κc higher than the cube may be neglected. Then it can be shown that the equation of motion of the sphere is

$$\left\{ M + \frac{1}{4}M' (2 + \kappa^2 c^2) \right\} \ddot{\xi} + \frac{M'\pi}{2\tau} \kappa^3 c^3 \dot{\xi} + \frac{4\pi^2 M'}{a\tau^2} \left(\frac{3}{2} + \frac{1}{4}\kappa^2 c^2 \right) \epsilon^{2\pi i t/\tau} + \frac{4\pi^2 M}{\tau'^2} \xi = 0,$$

which puts in evidence the existence of the viscous term.

The amplitude A is determined by the equation

$$A = \frac{M'\tau'^2 \left(\frac{3}{2} + \frac{3}{4}\kappa^2 c^2 \right) / a}{M(\tau'^2 - \tau^2) + M' \left(\frac{1}{2} + \frac{1}{4}\kappa^2 c^2 \right) \tau'^2 - \frac{1}{4}\kappa^2 c^3 M'\tau'^2}.$$

Since this expression contains an imaginary term in the denominator, it follows that the amplitude can never become infinite for any real value of τ .

For all solids M is very much greater than M' ; hence, if the difference between τ and τ' be not very small, A will be very small; but, if $\tau = \tau'$, the large term in the denominator involving M disappears, and A is approximately equal to $3/a$.

To the above order of approximation, the amplitude will be greatest

$$\text{when} \quad \tau^2 = \tau'^2 + \frac{M'\tau'^2}{2M} (1 + \frac{1}{2}\kappa^2 c^2),$$

$$\text{in which case} \quad A = -\frac{3}{a} \left(1 + \frac{2}{\kappa^2 c^2} \right),$$

which is large on account of the smallness of $\kappa^2 c^2$.