the general problem dealt with in the previously quoted paper, viz., the formation of a function w with given infinities, such that the imaginary part of $we^{i\theta_r}$ is constant at the circumference of the circle C_r ($r = 1, 2, \dots n+1$), it will be found that, even when the differences $\theta_r - \theta_s$ are commensurable with π , the terms in the infinite series for w which contain the same exponential do not arise from one of them by the operations of any sub-group of the original group; and therefore, that w cannot be represented as the sum of a finite number of ψ -functions formed from the substitutions of sub-groups in a manner analogous to that proved to be possible when the group arose from a single fundamental substitution.

Note on the Motion of a Fluid Ellipsoid under its own Attraction.

By M. J. M. Hill, M.A., Sc.D., Professor of Mathematics at University College, London. Communicated in Abstract, June 11th, 1891. Received January 27th, 1892.

1. In regard to this case of Fluid Motion, it was shown by Dirichlet that the particles which, at any instant, lie on an ellipsoid concentric, similar, and similarly situated to the external free surface, always do so; or, one family of surfaces, which always contain the same particles, is a family of ellipsoids which move so as always to be similar and similarly situated to the boundary.

The following additional results regarding this case of Fluid Motion were recently communicated to me by Mr. A. E. H. Love.

2. The particles which, at any instant, lie on a tangent plane to one of the ellipsoids concentric, similar, and similarly situated to the free surface, always lie on a tangent plane to this moving ellipsoid, and the same particle is always at the point of contact.

3. The enveloping cylinders of the ellipsoids mentioned above, whose generators are parallel to the vortex lines, always contain the same particles. Hence there are two families of surfaces, viz., the ellipsoids and the enveloping cylinders, which move so as to contain the same particles, and are such that one particular member of one always touches one particular member of the other.

4. The planes which are conjugate to the directions of the vortex lines with regard to the ellipsoids always contain the same particles.

I.—The object of this note is to express the component velocities in Clebsch's forms, viz.:

$$\frac{\partial \chi}{\partial x} + \lambda \frac{\partial \psi}{\partial x}, \quad \frac{\partial \chi}{\partial y} + \lambda \frac{\partial \psi}{\partial y}, \quad \frac{\partial \chi}{\partial z} + \lambda \frac{\partial \psi}{\partial z},$$

where $\lambda = \text{const.}$, $\psi = \text{const.}$ are vortex sheets; and to determine χ , λ , ψ as far as possible.

The following equations are taken from Basset's *Hydrodynamics*, Vol. II., pages 96, 97.

$$\omega_1 = \Omega_1 + \xi, \quad \omega_2 = \Omega_2 + \eta, \quad \omega_3 = \Omega_3 + \zeta.$$

The kinematical condition at the surface is

$$\frac{\partial F}{\partial t} + U \frac{\partial F}{\partial x} + V \frac{\partial F}{\partial y} + W \frac{\partial F}{\partial z} = 0,$$

where

$$F = (x/a)^2 + (y/b)^2 + (z/c)^2 - 1 = 0,$$

$$U = u + \omega_3 y - \omega_2 z,$$

$$V = v + \omega_1 z - \omega_3 x,$$

$$W = w + \omega_2 x - \omega_1 y;$$

$$u = \frac{\dot{a}}{a} x + \frac{\omega_3 (a^2 - b^2) - 2a^2 \xi}{a^2 + b^2} y + \frac{\omega_2 (c^2 - a^2) + 2a^2 \eta}{c^2 + a^2} z,$$

$$v = \frac{\dot{b}}{b} y + \frac{\omega_1 (b^2 - c^2) - 2b^2 \xi}{b^2 + c^2} z + \frac{\omega_3 (a^2 - b^2) + 2b^2 \zeta}{a^2 + b^2} x,$$

$$w = \frac{\dot{c}}{c} z + \frac{\omega_2 (c^2 - a^2) - 2c^2 \eta}{c^2 + a^2} x + \frac{\omega_1 (b^2 - c^2) + 2c^2 \xi}{b^2 + c^2} y.$$

The condition of continuity requires that

$$\dot{a}/a + \dot{b}/b + \dot{c}/c = 0.$$

Also the following equations hold:—

$$\frac{d}{dt} (\xi/a) - \frac{2ab\Omega_2}{a^2 + b^2} (\eta/b) + \frac{2ca\Omega_3}{c^2 + a^2} (\zeta/c) = 0,$$

$$\frac{d}{dt} (\eta/b) - \frac{2bc\Omega_1}{b^2 + c^2} (\zeta/c) + \frac{2ab\Omega_3}{a^2 + b^2} (\xi/a) = 0,$$

$$\frac{d}{dt} (\zeta/c) - \frac{2ca\Omega_2}{c^2 + a^2} (\xi/a) + \frac{2bc\Omega_1}{b^2 + c^2} (\eta/b) = 0.$$

Then I find, by means of a somewhat complicated analysis, the following values of χ , λ , ψ :—

$$\begin{aligned}\chi &= \frac{1}{2} \left[\frac{a}{a} x^2 + \frac{b}{b} y^2 + \frac{c}{c} z^2 \right. \\ &\quad \left. + \frac{2\Omega_1 (b^2 - c^2)}{b^2 + c^2} yz + \frac{2\Omega_2 (c^2 - a^2)}{c^2 + a^2} zx + \frac{2\Omega_3 (a^2 - b^2)}{a^2 + b^2} xy \right] \\ &\quad + \text{an arbitrary function of the time;} \\ \lambda &= \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} \right) - \left(\frac{x\xi}{a^2} + \frac{y\eta}{b^2} + \frac{z\zeta}{c^2} \right)^2; \\ \psi &= - \frac{abc}{\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} \right)^{\frac{1}{2}}} \tan^{-1} \left\{ \frac{\frac{z}{c} \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} \right) - \frac{\zeta}{c} \left(\frac{x\xi}{a^2} + \frac{y\eta}{b^2} + \frac{z\zeta}{c^2} \right)}{\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} \right)^{\frac{1}{2}} \left(\frac{x}{a} \frac{\eta}{b} - \frac{y}{b} \frac{\xi}{a} \right)} \right\} \\ &\quad + abc \int \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} \right)^{-1} \left(\frac{2bc\Omega_1}{b^2 + c^2} \frac{\xi}{a} + \frac{2ca\Omega_2}{c^2 + a^2} \frac{\eta}{b} \right) dt.\end{aligned}$$

It is interesting to observe

(a) that the surfaces $\psi = \text{const.}$ are a family of planes;

(β) that χ is the velocity-potential of the motion due to the motions of the enveloping case (supposed to be annihilated after the motion is set up), which have been described by Professor Greenhill in the *Proceedings of the Cambridge Philosophical Society*, Vol. IV., p. 4, so that

$$\lambda \frac{\partial \psi}{\partial x} = z\eta - y\zeta, \quad \lambda \frac{\partial \psi}{\partial y} = x\zeta - z\xi, \quad \lambda \frac{\partial \psi}{\partial z} = y\xi - x\eta.$$

The arbitrary function of the time is determined when the pressure is known at a given instant.

The values of χ , λ , ψ are not fully known until ξ , η , ζ have been determined as functions of the time.

This has been accomplished in the cases mentioned in Basset's *Hydrodynamics*, Vol. II., Chap. xv., and also

(a) when the ellipsoid becomes an elliptic cylinder, the solution being due to Lipschitz (see *Crelle-Borchardt*, LXXVIII., pp. 245-272); and

(β) when $\Omega_1 = \Omega_2 = \Omega_3 = 0$, and $\xi = \eta = 0$, but ζ is a function of the time, the solution being due to Padova (see *Annali della R. Scuola Normale Superiore di Pisa—Scienze Fisiche e Matematiche*, Vol. I., Pisa, 1871).

II.—It does not seem worth while to give the analysis by which the values of χ , λ , ψ were originally obtained, as it is so much simpler to verify them as follows:—

$$\text{Putting} \quad \frac{D}{dt} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y} + W \frac{\partial}{\partial z};$$

then, since

$$U = \frac{\dot{ax}}{a} + \frac{2a^2(\omega_1 - \zeta)}{a^2 + b^2} y - \frac{2a^2(\omega_1 - \eta)}{c^2 + a^2} z,$$

it follows that

$$\frac{Dx}{dt} = U = \frac{\dot{a}}{a} x + \frac{2a^2\Omega_1}{a^2 + b^2} y - \frac{2a^2\Omega_1}{c^2 + a^2} z;$$

$$\text{therefo} \quad \frac{D}{dt} (x/a) - \frac{2ab\Omega_1}{a^2 + b^2} (y/b) + \frac{2ac\Omega_1}{c^2 + a^2} (z/c) = 0,$$

with two similar equations.

$$\begin{aligned} \text{Now, putting} \quad N_1 &= 2bc\Omega_1/(b^2 + c^2), \\ N_2 &= 2ca\Omega_2/(c^2 + a^2), \\ N_3 &= 2ab\Omega_3/(a^2 + b^2), \end{aligned}$$

the above equations are

$$\frac{D}{dt} (x/a) - N_1 (y/b) + N_2 (z/c) = 0 \dots\dots\dots (A),$$

and two similar equations.

It may be observed that ξ/a , η/b , ζ/c , also satisfy the equations (A). Equations of the form (A) have been discussed by Riemann (see his *Mathematische Werke*, pp. 175–176).

They possess the property that, if X_1 , Y_1 , Z_1 and X_2 , Y_2 , Z_2 be two systems of integrals (not necessarily distinct), then

$$\frac{D}{dt} (X_1 X_2 + Y_1 Y_2 + Z_1 Z_2) = 0.$$

Further

$$\frac{D}{dt} (Y_1 Z_2 - Y_2 Z_1) - N_1 (Z_1 X_2 - Z_2 X_1) + N_2 (X_1 Y_2 - X_2 Y_1) = 0,$$

and two similar equations.

Hence $Y_1Z_2 - Y_2Z_1$, $Z_1X_2 - Z_2X_1$, $X_1Y_2 - X_2Y_1$ are also integrals of the equations (A).

Hence (1) $\frac{D\lambda}{dt} = 0$; for,

$$\begin{aligned} x/a, \quad y/b, \quad z/c, \\ \xi/a, \quad \eta/b, \quad \zeta/c \end{aligned}$$

being two systems of integrals of equations (A),

$$\frac{D}{dt} [(x/a)^2 + (y/b)^2 + (z/c)^2] = 0,$$

$$\frac{D}{dt} [(\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2] = 0,$$

$$\frac{D}{dt} [x\xi/a^2 + (y\eta/b^2) + (z\zeta/c^2)] = 0.$$

$$\begin{aligned} \text{Now } \lambda = [(x/a)^2 + (y/b)^2 + (z/c)^2] [(\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2] \\ - [x\xi/a^2 + (y\eta/b^2) + (z\zeta/c^2)]^2. \end{aligned}$$

$$\text{Hence } \frac{D\lambda}{dt} = 0.$$

$$\begin{aligned} (2) \text{ Putting } q_1 &= (y\zeta - z\eta)/(bc), \\ q_2 &= (x\zeta - x\zeta)/(ca), \\ q_3 &= (x\eta - y\xi)/(ab), \end{aligned}$$

it follows that q_1, q_2, q_3 are integrals of the equations (A).

The following algebraic identity is useful:—

$$\begin{aligned} [(\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2] q_3^2 + [q_3\xi/a - q_1\eta/b]^2 \\ = [(\xi/a)^2 + (\eta/b)^2] (q_1^2 + q_2^2 + q_3^2). \end{aligned}$$

$$(3) \quad \frac{D\psi}{dt} = 0;$$

$$\begin{aligned} \text{for } \psi &= abc \int (N_1\xi/a + N_2\eta/b) \{(\xi/a)^2 + (\eta/b)^2\}^{-1} dt \\ &- abc [(\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2]^{-\frac{1}{2}} \\ &\times \tan^{-1} \{ (q_3\xi/a - q_1\eta/b) / [q_3 [(\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2]^{\frac{1}{2}}] \}. \end{aligned}$$

$$\text{Putting } Q_1 = q_3\eta/b - q_2\zeta/c,$$

and using two similar expressions for Q_2 , Q_3 , then Q_1 , Q_2 , Q_3 are integrals of the equations (A); therefore

$$\frac{DQ_3}{dt} = N_2 Q_1 - N_1 Q_2.$$

Also
$$\frac{Dq_3}{dt} = N_2 q_1 - N_1 q_2.$$

Then
$$\frac{D\psi}{dt} = abc (N_1 \xi/a + N_2 \eta/b) [(\xi/a)^2 + (\eta/b)^2]^{-1}$$

$$- abc \frac{q_3 (N_2 Q_1 - N_1 Q_2) - Q_3 (N_2 q_1 - N_1 q_2)}{q_3^2 \{(\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2\} + (q_3 \xi/a - q_1 \eta/b)^2}.$$

The denominator of the last fraction is $(q_1^2 + q_2^2 + q_3^2) \{(\xi/a)^2 + (\eta/b)^2\}$.

The numerator is

$$\begin{aligned} &= q_3 (N_2 q_3 \eta/b - N_2 q_3 \zeta/c - N_1 q_1 \zeta/c + N_1 q_3 \xi/a) \\ &\quad - q_3 (N_2 q_1 - N_1 q_2) \xi/a + q_1 (N_2 q_1 - N_1 q_2) \eta/b \\ &= N_1 (-q_1 q_3 \zeta/c + q_2^2 \xi/a + q_3^2 \xi/a - q_1 q_2 \eta/b) \\ &\quad + N_2 (q_2^2 \eta/b - q_2 q_3 \zeta/c - q_1 q_2 \xi/a + q_1^2 \eta/b) \\ &= N_1 [(q_1^2 + q_2^2 + q_3^2) \xi/a - q_1 (q_1 \xi/a + q_2 \eta/b + q_3 \zeta/c)] \\ &\quad + N_2 [(q_1^2 + q_2^2 + q_3^2) \eta/b - q_2 (q_1 \xi/a + q_2 \eta/b + q_3 \zeta/c)] \\ &= (q_1^2 + q_2^2 + q_3^2) (N_1 \xi/a + N_2 \eta/b); \end{aligned}$$

because $q_1 \xi/a + q_2 \eta/b + q_3 \zeta/c = 0$.

Substituting these values, it follows that

$$\frac{D\psi}{dt} = 0.$$

4. The expressions for the component velocities may be verified.

$$u = \dot{a}x/a + \frac{\omega_3 (a^2 - b^2) - 2a^2 \zeta}{a^2 + b^2} y + \frac{\omega_3 (c^2 - a^2) + 2a^2 \eta}{c^2 + a^2} z,$$

$$\frac{\partial X}{\partial x} = \dot{a}x/a + \frac{\Omega_3 (a^2 - b^2)}{a^2 + b^2} y + \frac{\Omega_3 (c^2 - a^2)}{c^2 + a^2} z;$$

therefore

$$\begin{aligned}
 u - \frac{\partial \chi}{\partial x} &= \frac{y}{a^2 + b^2} \{ (a^2 - b^2)(\omega_3 - \Omega_3) - 2a^2 \zeta \} \\
 &\quad + \frac{z}{c^2 + a^2} \{ (c^2 - a^2)(\omega_3 - \Omega_3) + 2a^2 \eta \} \\
 &= z\eta - y\zeta, \\
 \frac{\partial \psi}{\partial x} &= -abc \left[\{ (\xi/a)^2 + (\eta/b)^2 \} (y\zeta - z\eta) / abc \right] \\
 &\quad \div [q_1^2 \{ (\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2 \} + \{ q_2 \xi/a - q_1 \eta/b \}^2] \\
 &= - \left[\{ (\xi/a)^2 + (\eta/b)^2 \} (y\zeta - z\eta) \right] / [(q_1^2 + q_2^2 + q_3^2) \{ (\xi/a)^2 + (\eta/b)^2 \}] \\
 &= (z\eta - y\zeta) / (q_1^2 + q_2^2 + q_3^2) \\
 &= (z\eta - y\zeta) / \lambda;
 \end{aligned}$$

therefore
$$u - \frac{\partial \chi}{\partial x} = \lambda \frac{\partial \psi}{\partial x}.$$

Hence
$$u = \frac{\partial \chi}{\partial x} + \lambda \frac{\partial \psi}{\partial x},$$

and two similar equations.

III. To obtain demonstrations of the results which stand at the head of this note, the properties of the equations (A) must be employed.

(i.) Observing that these equations are satisfied by x/a , y/b , z/c , it follows that

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = \text{const.}$$

(This is Dirichlet's theorem.)

(ii.) If x_1 , y_1 , z_1 be the coordinates of a particle on the surface

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = 0,$$

and if x , y , z be the coordinates of a particle on the tangent plane to this surface, then x/a , y/b , z/c , and also x_1/a , y_1/b , z_1/c satisfy the equations (A); therefore

$$\frac{D}{dt} [xx_1/a^2 + yy_1/b^2 + zz_1/c^2] = 0.$$

Hence the particle x, y, z always remains on the tangent plane to the surface

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = O.$$

This is theorem (2).

(iii.) The cylinder, whose generators are parallel to the vortex lines, enveloping the ellipsoid

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = O,$$

$$\text{is } [(\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2] [(x/a)^2 + (y/b)^2 + (z/c)^2 - O] \\ - [x\xi/a^2 + y\eta/b^2 + z\zeta/c^2]^2 = 0,$$

$$\text{i.e., } \lambda - O [(\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2] = 0.$$

$$\text{But } \frac{D\lambda}{dt} = 0, \quad \frac{D}{dt} [(\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2] = 0;$$

therefore

$$\frac{D}{dt} [\{ (\xi/a)^2 + (\eta/b)^2 + (\zeta/c)^2 \} \{ (x/a)^2 + (y/b)^2 + (z/c)^2 - O \} \\ - \{ x\xi/a^2 + y\eta/b^2 + z\zeta/c^2 \}^2] = 0.$$

Hence the cylinders, whose generators are parallel to the vortex lines, and which envelope the ellipsoids

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = \text{const.},$$

contain the same particles of fluid.

This is theorem (3).

(iv.) The planes conjugate to the direction of the vortex lines are

$$x\xi/a^2 + y\eta/b^2 + z\zeta/c^2 = \text{const.}$$

$$\text{But } \frac{D}{dt} [x\xi/a^2 + y\eta/b^2 + z\zeta/c^2] = 0.$$

Hence these planes contain the same particles of fluid.

This is theorem (4).