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"The Strength of the Ideal Column."

By FITZROY TOZER CHAPMAN, B.Sc., Assoc. M. Inst. C.E.

By an ideal column is meant a column which is perfectly straight, of uniform cross section, and composed of homogeneous material. For such a column it is generally conceded that Euler's formula for the limiting stable load:—

$$P = \frac{\pi^2 EI}{l^2}$$

is correct both as to the principles on which it is based and the processes by which it is derived.¹ Yet experiments on columns, though carried out with the utmost care, give results which differ markedly from those which Euler's formula predicts. The obvious reason is that, as soon as the stresses in the column exceed the limit of proportionality between stress and strain, the coefficient of elasticity ceases to be applicable. Under these circumstances various methods have been proposed for the calculation of P, and these fall naturally into two groups, namely:—

(a) Those employing empirical formulas, such as Rankine's, Johnson's, etc., which entirely discard the quantity E and depend on constants so chosen as to fit in with experimental results obtained from columns.

(b) Those which take E as constant, assume a certain eccentricity of loading (e), and calculate the limiting load as that load which produces a certain maximum fibre stress f_m , the values of E, e, and f_m being so chosen as to agree with experimental results.²

Before either of these methods could be applied to calculate columns constructed from a new material it would be necessary to

¹ W. Cain, Trans. Am. Soc. C.E., vol. xxxix (1898), p. 96.

² E.g., J. M. Moncrieff, Trans. Am. Soc. C.E., vol. xlv (1901), p. 334. Prof. T. C. Fidler's method (Minutes of Proceedings Inst. C.E., vol. lxxvi, p. 261) may also be included in this class.

carry out an exhaustive research on a large number of columns of different proportions in order to determine the necessary empirical constants.

In the Author's opinion, however, the quantity E in Euler's formula may be replaced by one of wider significance, which corresponds with a natural property of the material and can be determined from compression tests suitably carried out with an ordinary

testing-machine; the formula will then give results agreeing much more closely with practical experience.

Consider a column of uniform cross section A , and assume that the loading of the column makes no appreciable difference to the value of A . Let the critical load be P , and let the strain corresponding with a compressive stress $f_c = P/A$ be s . Then the length of the loaded but unbent column will be:—

$$l_1 = l_0 (1 - s).$$

Let $ABCD$ in *Figs. 1* be a small portion of the column, the distance between the parallel sections

AB and CD being λ_0 when the stress is zero. When the strut is loaded the length of this portion becomes:—

$$\lambda_1 = \lambda_0 (1 - s).$$

Suppose a small amount of bending occurs, too small to affect the length l_1 , and that the deflection at the centre of the portion $ABCD$ is y , while the radius of curvature at the same point is ρ , as in *Fig. 2*.

The strain will not be uniform now throughout $ABCD$, but at a distance z from the neutral plane will be, say, s_1 , so that

$$\frac{\lambda_0 (s - s_1)}{z} = \frac{\lambda_0 (1 - s)}{\rho}$$

or

$$s - s_1 = \frac{z (1 - s)}{\rho}.$$

If df/ds is the rate of change of stress with respect to strain, as

Figs. 1.

determined from the ordinary stress-strain diagram, the stress at this point will be

$$f = f_c + \int_s^{s_1} \frac{df}{ds} \cdot ds,$$

where

$$f_c = \frac{P}{A}.$$

Now the amount of the bending is very small, hence the variation in stress and strain will be so small that df/ds may be supposed constant at the value corresponding with f_c , and

$$\begin{aligned} f &= f_c + \left(\frac{df}{ds} \right) (s - s_1) \\ &= f_c + \left(\frac{df}{ds} \right) (1 - s) \cdot \frac{z}{\rho}. \end{aligned}$$

Fig. 2.

Putting aside the constant portion of this stress, the bending moment due to the variable part is—

$$M = \frac{I}{\rho} \cdot \left(\frac{df}{ds} \right) (1 - s),$$

I being the moment of inertia of the section.

Writing $M = Py$ and $\frac{1}{\rho} = \frac{d^2y}{dl^2},$

$$\frac{d^2y}{dl^2} = -y \cdot \frac{P}{\left(\frac{df}{ds} \right) (1 - s) \cdot I},$$

whence
$$y = (c_1 \cos l + c_2 \sin l) \sqrt{\frac{P}{\left(\frac{df}{ds}\right)(1-s) I}}$$

Introducing the conditions $y = 0$ when $l = \frac{l_1}{2}$

and
$$y = y_1 \quad , \quad l = 0,$$

the equation becomes

$$y = y_1 \cos l \sqrt{\frac{P}{\left(\frac{df}{ds}\right)(1-s) I}},$$

$$0 = y_1 \cos \frac{l_1}{2} \sqrt{\frac{P}{\left(\frac{df}{ds}\right)(1-s) I}},$$

$$n\pi = l_1 \sqrt{\frac{P}{\left(\frac{df}{ds}\right)(1-s) I}},$$

or

$$P = \frac{n^2 \pi^2 \left(\frac{df}{ds}\right)(1-s) I}{l_1^2}$$

substituting

$$l_1 = l_0 (1-s),$$

$$P = n^2 \pi^2 \cdot \left(\frac{df}{ds}\right) \cdot \frac{1}{1-s} \cdot \frac{I}{l_0^2} \cdot \dots \cdot (1)$$

The factor n , of course, is 1 for free-ended columns and 2 for those with fixed ends.

It will be seen that this is precisely the same as Euler's formula with E replaced by $\left(\frac{df}{ds}\right) \cdot \frac{1}{1-s}$, which is not constant, but is a definite function of f , and may be denoted by the symbol $\mathcal{E}$. The quantity $\frac{1}{1-s}$ differs inappreciably from unity, in general, so that

$\mathcal{E}$ is virtually $\frac{df}{ds}$.

Equation (1) may be written

$$P = \frac{n^2 \pi^2 \left(\frac{df}{ds}\right) \cdot I}{l_1 l_0} \cdot \dots \cdot (2)$$

which agrees with the form given by Professor Cain in the Paper already referred to, if $\left(\frac{df}{ds}\right)$ is replaced by E .

Writing I as $A k^2$ and putting $l_1 = l_0$ equation (2) becomes

$$P = \frac{n^2 \pi^2 E A k^2}{l_0^2} \dots \dots \dots (3)$$

and since

$$\frac{P}{A} = f_c,$$

$$\frac{l_0}{n k} = \pi \sqrt{\frac{E}{f_c}} \dots \dots \dots (4)$$

Fig. 3.

Now the value of $\frac{E}{f_c}$ can be determined from the compression tests, and plotted as a function of f_c ; so also can $\frac{l_0}{n k}$, and from such a curve the critical load of any column can be readily determined.

The curves which are given in *Figs. 3 and 4* have been deduced

from Hodgkinson's tests of cast iron as given by Dr. W. C. Unwin,¹ curve No. 1 of *Fig. 3* being taken directly from that authority. Curve No. 2 of the same Figure shows the relation between ε and

f_c , and is derived from curve No. 1. The full-line curve of *Fig. 4* shows the relation between $\frac{l_0}{nk}$ and f_c , calculated from equation (4).

The dotted curve shows the relation between $\frac{l_0}{nk}$ and f_c calculated by Euler's formula on the assumption that E is constant and equal to 5,800 tons per square inch.

Fig. 4 also contains the whole of the first series of Hodgkinson's

¹ "The Testing of Materials of Construction," pp. 63-65.

results for solid circular cast-iron columns with free and fixed ends.¹ The points marked thus ● refer to round-ended columns for which no measurable bending was observed, while those marked thus ○ refer to similar columns for which bending is recorded. Similarly the points marked ◆ and ◇ refer respectively to flat-ended columns which broke without previous bending, and to those which bent measurably during the test.

The close agreement of the calculated curve with the observed points lends considerable support to the theory outlined above.

The Paper is accompanied by three sheets of diagrams, from which the Figures in the text have been prepared.

¹ Philosophical Transactions of the Royal Society, 1840, p. 431, etc.