Philosophical Magazine Series 5

ISSN: 1941-5982 (Print) 1941-5990 (Online) Journal homepage: http://www.tandfonline.com/loi/tphm16

XIX. On the extra current

Oliver Heaviside

To cite this article: Oliver Heaviside (1876) XIX. On the extra current , Philosophical Magazine Series 5, 2:9, 135-145, DOI: 10.1080/14786447608639176

To link to this article: http://dx.doi.org/10.1080/14786447608639176

Published online: 13 May 2009.

XIX. *On the Extra Current.* By Oliver Heaviside *.

Let a wire possessing uniform electrical properties throughout be of length l, resistance kl, electric capacity cl, and let its coefficient of self-induction be sl. Further, let P and Q denote the two ends of the wire, and x the distance of any point from the end P. Let v be the electric potential at the point x at the time t, and Q the quantity of electricity that has passed that point from the time $t=0$, so that $\frac{dQ}{dt}$ is the current. The differential equation of the potential of the wire may be found from the following two equations:—

$$-\frac{dQ}{dx}=cv, \quad \ldots \ldots \ldots \quad (1)$$

$$-\frac{dv}{dx}=k\frac{dQ}{dt}+s\frac{d^2Q}{dt^2}. \quad \ldots \ldots \quad (2)$$

The first expresses the fact that the quantity of electricity existing on the surface of the wire between sections at x and $x+\delta x$ at any moment is the product of the potential and the capacity of the portion of wire considered. The second expresses that the electromotive force at the point x at any moment is the sum of the electromotive force producing the current $\frac{dQ}{dt}$ and the rate of increase of the momentum of that current. By eliminating Q we obtain

$$\frac{d^2v}{dx^2}=ck\frac{dv}{dt}+cs\frac{d^2v}{dt^2}. \quad \ldots \ldots \quad (3)$$

If $s=0$, the above equation becomes

$$\frac{d^2v}{dx^2}=ck\frac{dv}{dt},$$

the differential equation of the linear flow of heat, or of electricity in a submarine cable, the practical solution of which for a wire of finite length can only be accomplished with the assistance of Fourier's theorem. And if $k=0$, we have

$$\frac{d^2v}{dx^2}=cs\frac{d^2v}{dt^2};$$

which is of the same form as the equation of motion of a vibrating wire, the solution also requiring the use of Fourier's theorem. It is therefore probable that the same method must

* Communicated by the Author.

be adopted to solve the equation under consideration, viz.

$$\frac{d^2v}{dx^2}=ck\frac{dv}{dt}+cs\frac{d^2v}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let the potential of the wire at any moment be

$$v=\mathrm{V}\frac{\sin}{\cos}\frac{i\pi x}{l}.f(t), \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $f(t)$ is a function of t only, and V is constant. From (4) by differentiation,

$$\frac{d^2v}{dx^2}=-\frac{i^2\pi^2}{l^2}v;$$

therefore by (3),

$$cs\frac{d^2v}{dt^2}+ck\frac{dv}{dt}+\frac{i^2\pi^2}{l^2}v=0,$$

the solution of which is

$$v=\epsilon^{-\frac{t}{2\alpha}}\left(\mathrm{A}\epsilon^{\frac{t}{2\alpha}\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}}}+\mathrm{B}\epsilon^{-\frac{t}{2\alpha}\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}}}\right)$$

if $4i^2\pi^2\frac{\alpha}{\beta}<1$, and

$$v=\epsilon^{-\frac{t}{2\alpha}}(\mathrm{A}'\cos+\mathrm{B}'\sin)\frac{t}{2\alpha}\sqrt{4i^2\pi^2\frac{\alpha}{\beta}-1}$$

if $4i^2\pi^2\frac{\alpha}{\beta}>1$. Here A, B, A′ and B′ are constants, and $\alpha=\frac{s}{k}$ and $\beta=ckl^2$, both time-constants. Therefore if the potential when $t=0$ is

$$v=\mathrm{V}\frac{\sin}{\cos}\frac{i\pi x}{l},$$

the potential of the wire at the time t is

$$v=\mathrm{V}\frac{\sin}{\cos}\frac{i\pi x}{l}.\epsilon^{-\frac{t}{2\alpha}}.\left\{\mathrm{A}.\epsilon^{\frac{t}{2\alpha}\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}}}+(1-\mathrm{A})\epsilon^{-\frac{t}{2\alpha}\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}}}\right\}, \quad (5)$$

or

$$v=\mathrm{V}\frac{\sin}{\cos}\frac{i\pi x}{l}.\epsilon^{-\frac{t}{2\alpha}}(\cos+\mathrm{B}'\sin)\frac{t}{2\alpha}\sqrt{4i^2\pi^2\frac{\alpha}{\beta}-1}, \quad . \quad . \quad . \quad (6)$$

according as $4i^2\pi^2\frac{\alpha}{\beta}<\text{or}>1$. The remaining constants A and B′ must be determined from the value of the current at some fixed time. By solving equation (2), where $-\frac{dv}{dx}$ is to be found from (5) and (6), we shall find

$$\frac{dQ}{dt}=C\epsilon^{-\frac{t}{\alpha}}\mp\frac{Vi\pi}{kl}\frac{\cos}{\sin}\frac{i\pi x}{l}\cdot\epsilon^{-\frac{t}{2\alpha}}\left\{\frac{2A\epsilon^{\frac{t}{2\alpha}\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}}}}{1+\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}}}+\frac{2(1-A)\epsilon^{-\frac{t}{2\alpha}\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}}}}{1-\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}}}\right\},$$

or

$$\frac{dQ}{dt}=C\epsilon^{-\frac{t}{\alpha}}\mp\frac{Vi\pi}{kl}\frac{\cos}{\sin}\frac{i\pi x}{l}\cdot\epsilon^{-\frac{t}{2\alpha}}\cdot\frac{\beta}{2\alpha i^2\pi^2}\cdot$$

$$\left\{\left(1-B'\sqrt{4i^2\pi^2\frac{\alpha}{\beta}-1}\right)\cos+\left(B'+\sqrt{4i^2\pi^2\frac{\alpha}{\beta}-1}\right)\sin\right\}\cdot\frac{t}{2\alpha}\sqrt{4i^2\pi^2\frac{\alpha}{\beta}-1},$$

where C is a constant current. Let the initial current be C, then

$$A=\frac{1+\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}}}{2\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}}},\quad B'=\frac{1}{\sqrt{4i^2\pi^2\frac{\alpha}{\beta}-1}};$$

therefore the expressions for the potential and current become

$$v=V\frac{\sin}{\cos}\cdot\frac{i\pi x}{l}\frac{\epsilon^{-\frac{t}{2\alpha}}}{2m}\{(1+m)\epsilon^{\frac{tm}{2\alpha}}-(1-m)\epsilon^{-\frac{tm}{2\alpha}}\},\quad . \quad . \quad . \quad (7)$$

or

$$v=V\frac{\sin}{\cos}\frac{i\pi x}{l}\cdot\epsilon^{-\frac{t}{2\alpha}}\left(\cos+\frac{1}{m'}\sin\right)\frac{tm'}{2\alpha},\quad . \quad . \quad . \quad . \quad . \quad (8)$$

where

$$m=\sqrt{1-4i^2\pi^2\frac{\alpha}{\beta}},\quad m'=\sqrt{4i^2\pi^2\frac{\alpha}{\beta}-1};$$

and

$$\frac{dQ}{dt}=C\epsilon^{-\frac{t}{\alpha}}\mp\frac{Vi\pi}{kl}\frac{\cos}{\sin}\frac{i\pi x}{l}\cdot\frac{\epsilon^{-\frac{t}{2\alpha}}}{m}(\epsilon^{\frac{tm}{2\alpha}}-\epsilon^{-\frac{tm}{2\alpha}}),\quad . \quad . \quad . \quad . \quad (9)$$

or

$$\frac{dQ}{dt}=C\epsilon^{-\frac{t}{\alpha}}\mp\frac{Vi\pi}{kl}\cdot\frac{\cos}{\sin}\frac{i\pi x}{l}\cdot\frac{2\epsilon^{-\frac{t}{2\alpha}}}{m'}\cdot\sin\frac{tm'}{2\alpha}.\quad . \quad . \quad . \quad . \quad (10)$$

In the intermediate case, when $m=m'=0$,

$$v=V\frac{\sin}{\cos}\frac{i\pi x}{l}\epsilon^{-\frac{t}{2\alpha}}\left(1+\frac{t}{2\alpha}\right), \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$\frac{dQ}{dt}=C\epsilon^{-\frac{t}{\alpha}}\mp\frac{Vi\pi}{kl}\frac{\cos}{\sin}\frac{i\pi x}{l}.\epsilon^{-\frac{t}{2\alpha}}.\frac{t}{\alpha}. \quad . \quad . \quad (12)$$

The current $C\epsilon^{-\frac{t}{\alpha}}$ does not influence the potential in any way. The above solutions suppose that the initial current is C, and the initial potential $v=V\frac{\sin}{\cos}\frac{i\pi x}{l}$, and give the potential and current at any time after. When $\sin\frac{i\pi x}{l}$ is taken, the potential at the ends of the wire is always zero; and when $\cos\frac{i\pi x}{l}$ is taken, the current is always zero at the ends.

After this preliminary we can pass to more practical cases. In the first place, let a constant current $\frac{V}{kl}$ be flowing through the wire, caused by a battery of negligible resistance and electromotive force V; and let the potential of the wire be $V\left(1-\frac{x}{l}\right)$, so that it is V at the end P and 0 at the end Q. By Fourier's theorem,

$$V\left(1-\frac{x}{l}\right)=\frac{2V}{\pi}\sum_1^\infty\frac{1}{i}\sin\frac{i\pi x}{l};$$

therefore, if the end P is put to earth at the time $t=0$, the potential at the time t is

$$v=\frac{2V}{\pi}\Sigma\frac{1}{i}\sin\frac{i\pi x}{l}.\epsilon^{-\frac{t}{2\alpha}}.\left\{\frac{1+m_i}{2m_i}.\epsilon^{\frac{tm_i}{2\alpha}}-\frac{1-m_i}{2m_i}.\epsilon^{-\frac{tm_i}{2\alpha}}\right\}$$

$$+\frac{2V}{\pi}\Sigma\frac{1}{i}\sin\frac{i\pi x}{l}.\epsilon^{-\frac{t}{2\alpha}}\left(\cos+\frac{1}{m'_i}\sin\right)\frac{tm'_i}{2\alpha} \quad . \quad (13)$$

by (7) and (8), where the first series includes all integral values of i which make $4i^2\pi^2\frac{\alpha}{\beta}-1$ negative, and the second series all the rest up to $i=\infty$. And by (9) and (10),

$$\frac{dQ}{dt}=\frac{V}{kl}.\epsilon^{-\frac{t}{\alpha}}-\frac{2V}{kl}\Sigma\cos\frac{i\pi x}{l}.\frac{\epsilon^{-\frac{t}{\alpha}}}{m_i}\left(\epsilon^{\frac{tm_i}{2\alpha}}-\epsilon^{-\frac{tm_i}{2\alpha}}\right)$$

$$-\frac{2V}{kl}\Sigma\cos\frac{i\pi x}{l}.\frac{2\epsilon^{-\frac{t}{2\alpha}}}{m'_i}.\sin\frac{tm'_i}{2\alpha} \quad . \quad . \quad (14)$$

expresses the current at time t. If the wire is originally everywhere at potential zero and without current, the potential v' and current $\frac{dQ'}{dt}$ at time t after the end P is raised to potential V, the end Q being to earth, are

$$v' = V\left(1 - \frac{x}{l}\right) - v,$$

$$\frac{dQ'}{dt} = \frac{V}{kl} - \frac{dQ}{dt},$$

where v and $\frac{dQ}{dt}$ have the values given in (13) and (14).

Suppose $4\pi^2\frac{\alpha}{\beta} > 1$, then the first series in (13) and (14) disappear, and we have

$$v = \frac{2V}{\pi} \cdot \epsilon^{-\frac{t}{2\alpha}} \sum_1^\infty \frac{1}{i} \sin\frac{i\pi x}{l}\left(\cos + \frac{1}{m'_i}\sin\right)\frac{tm'_i}{2\alpha}, \qquad (15)$$

$$\frac{dQ}{dt} = \frac{V}{kl} \cdot \epsilon^{-\frac{t}{\alpha}} - \frac{4V}{kl} \cdot \epsilon^{-\frac{t}{2\alpha}} \sum_1^\infty \frac{1}{m'_i} \cos\frac{i\pi x}{l} \cdot \sin\frac{tm'_i}{2\alpha}. \qquad (16)$$

The extra current is exhibited in (16) as consisting of two parts. One, a current $\frac{V}{kl} \cdot \epsilon^{-\frac{t}{\alpha}}$, uniform at all parts of the wire, which dies away without oscillations with a rapidity proportional to $\frac{1}{\alpha}$. This current is due entirely to the momentum of the original current $\frac{V}{kl}$. The other part,

$$-\frac{4V}{kl} \cdot \epsilon^{-\frac{t}{2\alpha}} \sum_1^\infty \cdot \frac{1}{m'_i} \cos\frac{i\pi x}{l} \cdot \sin\frac{tm'_i}{2\alpha},$$

is due entirely to the original charge of the wire, and consists at any point x of an infinite series of currents alternately positive and negative, which die away with only half the rapidity. These oscillations are of greatest intensity at the end P, and least at the end Q. They are insensible both when $\frac{\alpha}{\beta}$ is very small and when it is very large. In the former case only the higher terms in (13) and (14) are periodic with respect to the time; and in the latter case they become very rapid and weak in the same proportion. But when the time-constants α and β are not very different, the oscillations are of considerable strength, and may become observable by proper means. Suppose $\frac{\alpha}{\beta}$ of such magnitude that $\sqrt{4i^2\pi^2\frac{\alpha}{\beta}-1}$ is appreciably $=2i\pi\sqrt{\frac{\alpha}{\beta}}$, then the time of a complete oscillation, including a positive and a negative current at any point, is nearly $2\sqrt{\alpha\beta}$, so that there are $\sqrt{\frac{\alpha}{\beta}}$ complete oscillations in the time 2α. The strength of these oscillations is proportional to $\sqrt{\frac{\beta}{\alpha}}$; so that the larger $\frac{\alpha}{\beta}$ the weaker the oscillations, they being at the same time more rapid in the same proportion.

The time-integral of the extra current is

$$\frac{\mathrm{V}\alpha}{kl}-\frac{\mathrm{V}cl}{2}\left(\frac{x^2}{l^2}-\frac{2x}{l}+\frac{2}{3}\right),$$

where the first part is the same at all points, and is due entirely to the momentum of the initial current. The second part is the excess of the positive over the negative currents due to the initial charge, and is twice as great at the end P as at Q. This is the same when $s=0$, or when there is no self-induction.

The work done in the wire by the extra current is

$$\int_0^\infty kl\left(\frac{d\mathrm{Q}}{dt}\right)^2 dt,$$

when $\frac{d\mathrm{Q}}{dt}$ is the same at all points, and

$$\int_0^l\int_0^\infty k\left(\frac{d\mathrm{Q}}{dt}\right)^2 dx\, dt$$

when $\frac{d\mathrm{Q}}{dt}$ varies with x. Hence the amount of work done by the first part of the current in equation (16) is $\frac{sl}{2}\times\left(\frac{\mathrm{V}}{kl}\right)^2$, and by the second part $\frac{\mathrm{V}^2cl}{6}$, which was the energy of the initial charge $=\frac{1}{2}\int_0^l \mathrm{V}c\left(1-\frac{x}{l}\right)^2 dx$.

As another example, suppose that before the time $t=0$ a uniform current $\frac{\mathrm{V}}{kl}$ existed in the wire, with potential $v=\mathrm{V}\left(1-\frac{x}{l}\right)$, and that at the time $t=0$ both ends of the wire are instantaneously and simultaneously insulated without allowing a spark to pass. Then we have $\frac{dQ}{dt}=0$ at P and Q. Let us first consider v and $\frac{dQ}{dt}$ resulting from the initial charge, supposing $4\pi^2\frac{\alpha}{\beta}>1$. By Fourier's theorem,

$$V\left(1-\frac{x}{l}\right)=\frac{V}{2}+\frac{2V}{\pi^2}\sum_1^\infty\frac{1-\cos i\pi}{i^2}\cos\frac{i\pi x}{l},$$

where $\frac{V}{2}$ is the final potential. Therefore, by (8), that part of the potential due to the initial charge is

$$\frac{V}{2}+\frac{2V}{\pi^2}\epsilon^{-\frac{t}{2\alpha}}\sum_1^\infty\frac{1-\cos i\pi}{i^2}\cdot\cos\frac{i\pi x}{l}\left(\cos+\frac{1}{m'_i}\sin\right)\frac{tm'_i}{2\alpha}; \quad (17)$$

and by (10) that part of the current due to the initial charge is

$$\frac{2V}{\pi kl}\epsilon^{-\frac{t}{2\alpha}}\sum_1^\infty\frac{1-\cos i\pi}{i}\cdot\sin\frac{i\pi x}{l}\cdot\frac{2}{m'_i}\cdot\sin\frac{tm'_i}{2\alpha}. \quad . \quad (18)$$

To find the potential and current due to the initial current, we have

$$\frac{V}{kl}=\frac{2V}{\pi kl}\sum_1^\infty\frac{1-\cos i\pi}{i}\sin\frac{i\pi x}{l};$$

therefore

$$\frac{dQ}{dt}=\frac{2V}{\pi kl}\cdot\epsilon^{-\frac{t}{2\alpha}}\sum_1^\infty\frac{1-\cos i\pi}{i}\sin\frac{i\pi x}{l}\left(A_i\cos+B_i\sin\right)\frac{tm'_i}{2\alpha},$$

and

$$v=V'+\frac{2V}{\pi^2}\sum_1^\infty\frac{1-\cos i\pi}{i^2}\cos\frac{i\pi x}{l}\epsilon^{-\frac{t}{2\alpha}}\left(\frac{A_i}{2}\cos+\frac{B_i}{2}\sin\right.$$
$$\left.+\frac{B_i m'_i}{2}\cos-\frac{A_i m'_i}{2}\sin\right)\frac{tm'_i}{2\alpha},$$

where V', A_i, and B_i are constants. The conditions to determine them are that $v=0$ when $t=0$ and when $t=\infty$. Also $\frac{dQ}{dt}=\frac{V}{kl}$ when $t=0$. Therefore

$$V'=0,\quad A_i=1,\text{ and } B_i=-\frac{1}{m'_i}.$$

Thus

$$\frac{dQ}{dt}=\frac{2V}{\pi kl}\cdot\epsilon^{-\frac{t}{2\alpha}}\sum_1^\infty\frac{1-\cos i\pi}{i}\sin\frac{i\pi x}{l}\cdot\left(\cos-\frac{1}{m'_i}\sin\right)\frac{tm'_i}{2\alpha}, \quad (19)$$

$$v=-4V\cdot\frac{\alpha}{\beta}\cdot\epsilon^{-\frac{t}{2\alpha}}\sum\frac{1-\cos i\pi}{m'_i}\cos\frac{i\pi x}{l}\cdot\sin\frac{tm'_i}{2\alpha}. \quad . \quad . \quad (20)$$

The actual potential is the sum of (17) and (20), and the actual current the sum of (18) and (19). When $\frac{\alpha}{\beta}$ is large the initial charge may be neglected altogether. Considering only the potential and current due to the initial current, we find that the current in the wire consists of a series of decreasing waves in opposite directions, causing corresponding changes in the potential of the wire. At the first moment after disconnexion the potential of the end Q becomes positive $=\mathrm{V}\sqrt{\frac{\alpha}{\beta}}$ nearly, and the end P negative to an equal extent. Provided this electromotive force suddenly developed is not sufficiently great to cause a spark, this state of things is rapidly reversed, the end P becoming positive and the end Q negative, which is followed by another reversal, and so on till the energy of the initial current is all used up against the resistance of the wire.

It is obvious that the simplicity (?) of the above formulæ must be considerably departed from in all practical cases that occur, as in the above c and s are assumed to be the same for every unit of length of the wire, which cannot be true, except perhaps in a coiled submarine cable. But we may be sure that, in virtue of that property of the electric current which Professor Maxwell terms its "electromagnetic momentum," whenever any sudden change of current or of charge takes place in a circuit possessing an appreciable amount of self-induction, the new state of equilibrium is arrived at through a series of oscillations in the strength of the current which may be noticeable under certain circumstances. It is naturally difficult to observe such oscillations with a galvanometer; but some telegraph-instruments show them very distinctly. For instance, there is Wheatstone's "alphabetical indicator." The pointer of this instrument is moved one letter forward round a dial by every current passing through it, provided the currents are alternately positive and negative. Now if an insulated straight wire a few miles in length is suddenly raised to a high potential by means of a single current of very short duration from a magneto-electric machine, and then immediately discharged to earth through the coils of an "indicator," the pointer does not merely move one step forward, as it would if the discharge consisted of a single current, but several steps, indicating a succession of reverse currents. The same thing occurs when a condenser of small capacity is first charged to a high potential and then discharged through the instrument. Expressed in popular language, what happens is as follows. The first discharge-current is first retarded by the self-induction of the coils, and then, acquiring momentum, carries to earth a greater quantity of electricity than the line or condenser originally contained, thus reversing the potential of the line. Hence a reverse current follows to restore the equilibrium, which in its turn carries to the line more than enough electricity to supply the deficiency; hence another current from line to earth; and so on, till the currents are too weak to produce any observable effect.

By supposing that the current at any moment is of the same strength in all parts of the coil, the theory of these alternating currents when a charged condenser is discharged through the coil is much simplified. Let Q_0 be the initial charge and V the initial potential of the condenser, whose capacity is c; and let R be the resistance and L the coefficient of self-induction of the coil. Then, if Q is the charge and v the potential of the condenser at the time t, the current in the coil is

$$\frac{dQ}{dt}=-c\frac{dv}{dt},$$

and

$$v=R\frac{dQ}{dt}+L\frac{d^2Q}{dt^2},$$

since v is the difference of potential between the ends of the coil. Therefore

$$cL\frac{d^2v}{dt^2}+cR\frac{dv}{dt}+v=0,$$

the solution of which satisfying the conditions $v=V$ when $t=0$ (and $\frac{dQ}{dt}=0$ when $t=0$) is

$$v=\frac{V\epsilon^{-\frac{t}{2a}}}{2\sqrt{1-4\frac{\alpha}{\beta}}}\left\{\left(1+\sqrt{1-4\frac{\alpha}{\beta}}\right)\epsilon^{\frac{t}{2a}\sqrt{1-4\frac{\alpha}{\beta}}}-\left(1-\sqrt{1-4\frac{\alpha}{\beta}}\right)\epsilon^{-\frac{t}{2\alpha}\sqrt{1-4\frac{\alpha}{\beta}}}\right\},$$

or

$$v=V\epsilon^{-\frac{t}{2\alpha}}\left(\cos+\frac{1}{\sqrt{4\frac{\alpha}{\beta}-1}}\sin\right)\frac{t}{2\alpha}\sqrt{4\frac{\alpha}{\beta}-1},$$

according as $1-4\frac{\alpha}{\beta}$ is + or −. And the current in the coil is

$$\frac{dQ}{dt}=\frac{V}{R}\cdot\frac{\epsilon^{-\frac{t}{2\alpha}}}{\sqrt{1-4\frac{\alpha}{\beta}}}\left(\epsilon^{\frac{t}{2a}\sqrt{1-4\frac{\alpha}{\beta}}}-\epsilon^{-\frac{t}{2\alpha}\sqrt{1-4\frac{\alpha}{\beta}}}\right),$$

or

$$\frac{dQ}{dt}=\frac{2V}{R}\cdot\frac{\epsilon^{-\frac{t}{2\alpha}}}{\sqrt{4\frac{\alpha}{\beta}-1}}\sin\frac{t}{2\alpha}\sqrt{4\frac{\alpha}{\beta}-1},$$

where $\alpha=\frac{L}{R}$ and $\beta=cR$. In the first case, when $1>4\frac{\alpha}{\beta}$, the potential and current are never reversed; but in the second case, when $4\frac{\alpha}{\beta}>1$, they are reversed an infinite number of times, the successive charges of the condenser decreasing in geometrical proportion. The current changes sign when t is any multiple of $\frac{2\alpha\pi}{\sqrt{4\frac{\alpha}{\beta}-1}}$, and has its maxima or minima values when

$$\cos\frac{t}{2\alpha}\sqrt{4\frac{\alpha}{\beta}-1}=\frac{1}{2}\sqrt{\frac{\beta}{\alpha}}.$$

The quantity Q' of electricity conveyed in the first current is

$$Q'=Q_0(1+\epsilon^{-\frac{\pi}{\sqrt{4\frac{\alpha}{\beta}-1}}}),$$

where Q_0 is the initial charge of the condenser. As $\frac{\alpha}{\beta}$ is increased Q' approaches $2Q_0$ as its limit; *i. e.* when the resistance of the coil is reduced, or its magnetic capacity increased, the quantity of electricity conveyed by any current increases till it is nearly double the charge of the condenser at the commencement of that current, and the oscillations are more slowly diminished. The amount of energy expended by the first current is

$$\frac{V^2c}{2}\left(1-\epsilon^{-\frac{2\pi}{\sqrt{4\frac{\alpha}{\beta}-1}}}\right),$$

where $\frac{V^2c}{2}$ is the energy of the original charge Q_0, which becomes indefinitely small as $\frac{\alpha}{\beta}$ increases. The integral current, irrespective of sign, is

$$\frac{Q_0}{1-\epsilon^{-\frac{\pi}{\sqrt{4\frac{\alpha}{\beta}-1}}}},$$

which increases indefinitely with $\frac{\alpha}{\beta}$. From the number of oscillations in a given time, L may be determined in terms of R and c; for if the current is reversed n times per second, then

$$L=\frac{1}{2c\pi^2n^2}(1+\sqrt{1-c^2R^2\pi^2n^2}).$$

Electrical vibrations due to induction occur under various circumstances. For example, the "false discharge" from a submarine cable; the oscillatory phenomena described by M. Blaserna and others; and Mr. Edison's "ætheric-force" experiments.

XX. *Proceedings of Learned Societies.*

ROYAL SOCIETY.

[Continued from p. 71.]

Feb. 10, 1876.—Dr. J. Dalton Hooker, C.B., President, in the Chair.

THE following papers were read:—

"On Repulsion resulting from Radiation."—Part III. By William Crookes, F.R.S. &c.

This paper contains an account of experiments on the action of radiation on bodies the surfaces of which have their radiating and absorbing powers modified by various coatings. The difference between a white and a lampblacked surface in this respect was at first not very decided; and experiments have been instituted with the object of clearing up some anomalies observed in the actions. Two pith disks, one white and the other black, were suspended on a light arm in a glass bulb by means of a fine silk fibre; after perfect exhaustion the white and black disks were found to be equally repelled by heat of low intensity, such as from the fingers, warm water, &c. A copper ball was then tried at gradually increasing temperatures. Up to 250° C. it repelled both equally, above that the black was more repelled than the white, and at a full red heat the repulsion of the black disk was very energetic. A lighted candle acts with more energy than the red-hot copper.

The presence of even a small quantity of aqueous vapour in the exhausted apparatus almost, if not quite, neutralizes the more energetic action which luminous rays appear to exert on a blackened surface.

After describing several different modifications and some new forms of apparatus devised to facilitate experiment, the author gives a drawing of an instrument which enables him to get quantitative measurements of the amount of incident light falling on it. It consists of a flat bar of pith, half black and half white, suspended horizontally in a bulb by means of a long silk fibre. A small magnet and reflecting-mirror are fastened to the pit; hand a controlling magnet is fastened outside so that it can slide up and down the tube, and thus increase or diminish sensitiveness. The whole is completely exhausted, and then enclosed in a box