Review Article

BASIC PROPERTIES, LIMIT & DERIVATIVE OF THE GREATEST INTEGER FUNCTION

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Abstract

The greatest integer function is a very important function which is extensively used especially in math, accounting and computer science. This paper discusses the definition and various notations used for the greatest integer and fractional part functions simply. Brief description for some basic properties of these functions are also shown. Limit of the function is also introduced here; the limit of the floor function is said to exist at non-integer values while not existing for integer values. The derivative of the function is computed using definition which is also related to the limit and the continuity of the function.

Definition & Notation

The greatest integer function or the floor function is defined as the following: the function *f: R → Z* given by *f(x) = [x] or f(x)=* |_*x*_|, where *[x]* or |_*x*_| denotes the largest integer not exceeding *x* [1]. Another definition is: $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$, and since there is exactly one integer in a half-open interval of length one, for any real x there is a unique integer m [2]. Refer to the plot of the function in figure 1.

The following aternative expression can be used to simplify the previous definition:

$\lfloor x \rfloor = m$ if and only if $m \leq x < m+1$

and $\lfloor x \rfloor = m$ if and only if $x - 1 < m \leq x$.

[2].

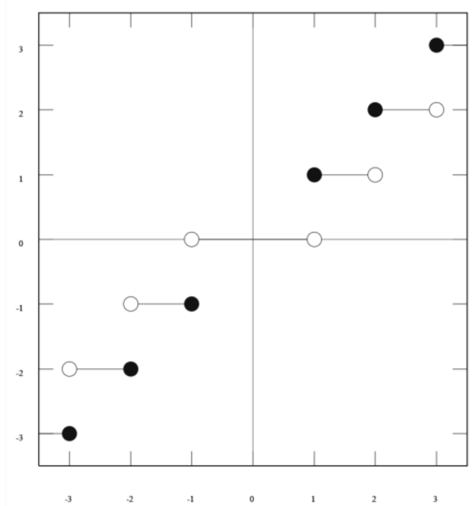
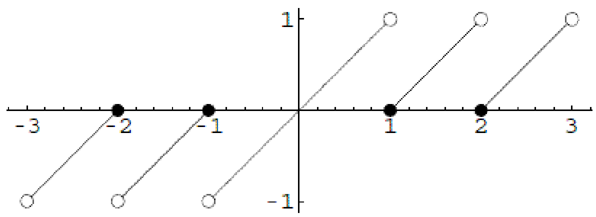


Figure 1: Plot of The Greatest Integer Function

The fractional part is the saw tooth function, denoted by *{x}* for *x* ∈ *R* and defined by the formula: *∀ x, {x}= x - [x], 0≤{x}<1* [2]. Look at table 1.



$f(x) = frac(x)$

Figure 2: Fractional Part (Saw Tooth) Function

In many older and current works, the symbol [x] is used instead of |_x_| [3][4][5]. In fact, this notation harks back to Gauss in his third proof of quadratic reciprocity in 1808 [5]. Some mathematicians prefer |_x_| because it can highlight the difference of ceiling and floor function [6]. However, the [x] symbol will be used throughout this work.

x	[x]	{x}
4	4	0
0.5	0	0.5
0	0	0
-1.7	-2	0.3
-2	-2	0

Table 1: Example of Greatest Integer & Fractional Part Function

Properties of Greatest Integer Function

For all x, x ∈R and n ∈Z [7]:

$\lfloor x \rfloor = x$ if and only if x is an integer.

$\lfloor x \rfloor = n$ if and only if $n \leqslant x < n+1$ if and only if $x - 1 < n \leqslant x$.

$\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1$ and $x - 1 < \lfloor x \rfloor \leqslant x$.

$\lfloor x + n \rfloor = \lfloor x \rfloor + n.$

$$\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x = \lfloor x \rfloor, \\ -\lfloor x \rfloor - 1 & \text{if } x \neq \lfloor x \rfloor. \end{cases}$$

$\lfloor x/n \rfloor = \lfloor \lfloor x \rfloor / n \rfloor$ *if* $n \geq 1$.

$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$. *More generally,*

$$\lfloor nx \rfloor = \sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor .$$

$$\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ -1, & \text{if } x \notin \mathbb{Z} \end{cases}$$

$\{x\} = 0$ if and only if x is an integer.

$\{x + n\} = \{x\}$ for all real numbers x and n, with n an integer.

$$\{x\} + \{-x\} = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ 1, & \text{if } x \notin \mathbb{Z} \end{cases}$$

$0 \leq \left\{ \frac{m}{n} \right\} \leq 1 - \frac{1}{|n|}$ for all integers m and n with $n \neq 0$.

$$\lfloor mx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + \cdots + \left\lfloor x + \frac{m-1}{m} \right\rfloor .$$

$$\sum_{i=1}^{n-1} \left\lfloor \frac{im}{n} \right\rfloor = \frac{1}{2}(m-1)(n-1).$$

$$\left\lfloor \frac{\lfloor x/m \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{mn} \right\rfloor$$

Limit of Greatest Integer Function

Informal Definition: Let f(x) be defined on an open interval about x_o, except at x_o, If f(x) gets arbitrarily close to L for all x sufficiently close to x_o, we say that f approaches the limit L as x approaches x_0, that is $\lim_{x \to x_0} f(x) = L$ [8] . Formally limit can be defined as:

$$\lim_{x \to c} f(x) = L \iff (\forall \varepsilon > 0)(\exists\, \delta > 0)(\forall x \in D)$$

$(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$

A function f(x) has a limit as x approaches x_o if and only if it has left-hand and right-hand limits there and these one-sided limits are equal [8];

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \lim_{x \to x_0^-} f(x) = L \wedge \lim_{x \to x_0^+} f(x) = L .$$

The greatest-integer function *f(x)* = $\lfloor x \rfloor$ has different right-hand and left-hand limits at each integer.

$$\lim_{x \to 3^+} \lfloor x \rfloor = 3 \text{ and } \lim_{x \to 3^-} \lfloor x \rfloor = 2$$

The limit of [x] as *x* approaches an integer *n* from above is *n*, while the limit as *x* approaches *n* from below is *n* - 1. So the greatest integer function has no limit at any integer.

At the same time, the greatest-integer function *f(x)* = *[x]* has the same greatest integer function at every *x* such that *x* is not an integer.

Example: $\lim_{x \to 0.5^+} [x] = \lim_{x \to 0.5^-} [x] = 0$ or $\lim_{x \to 4.5^+} [x] = \lim_{x \to 4.5^-} [x] = -5$

The Derivative of The Greatest Integer Function

Recall the definition of the derivative. If *f(x)* is a function then:

- $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
- $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$
- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Using this definition we can find the greatest integer function:

$$Let\ f'(x) = \frac{d}{dx}[x]$$

$1 -\ Let\ x\ \notin Z, then:$

$$f'(x) = \lim_{h \to 0} \frac{[x+h] - [x]}{h}$$
$$= \lim_{h \to 0} \frac{[x] - [x]}{h}$$
$$= \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

$2^{nd}\ step: the\ value\ of\ h\ has\ been\ neglected$

$assuming\ that\ h\ is\ very\ small\ compared\ to\ x$

$3^{rd}\ step: \frac{0}{h} = 0\ becuase\ h\ is\ a\ non$

$- zero\ value$

$$\therefore f'(x) = 0\ for\ all\ x\ \notin Z$$

$2 -\ Let\ x\ \in Z, then:$

$$f'_+(x) = \lim_{h \to 0^+} \frac{[x+h] - [x]}{h}$$
$$= \lim_{h \to 0^+} \frac{0}{h} = \lim_{h \to 0^+} 0$$
$$= 0$$

$$f'_-(x) = \lim_{h\to 0^-} \frac{[x+h]-[x]}{h}$$

$$= \lim_{h\to 0^+} \frac{x-1-x}{h}$$

$$= \lim_{h\to 0^+} \frac{-1}{h} = \infty$$

$$\because\ f'_-(x) \neq f'_+$$

$$\therefore f'(x)\ does\ not\ exist\ for\ all\ x \in Z$$

Thus the greatest integer function is not differentiable for all integers and is equal to zero for all non-integer values:

$$\frac{d}{dx}[x] = \begin{cases} not\ differentiable\ , x \in Z \\ 0\ , \quad x \notin Z \end{cases}$$

The previous statement can be related to the limit of the greatest integer function through the following:

$Theorem$: $\frac{d}{dx} f(x)\ exists\ if\ and\ only\ if\ f(x)$ $is\ a\ continous\ function$ [8].

The greatest integer function is continuous at any integer *n* from the right only because:

$f(n) = [n] = n$

$and \lim_{x\to n^+} f(x) = n$

$but \lim_{x\to n^-} f(x)$

Hence, $\lim_{x\to n} f(x) \neq f(x)$ and f(x) is not continuous at n from the left. Note that the greatest integer function is continuous from the right and from the left at any noninteger value of *x*.

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