V.—*A Development of a Pfaffian having a Vacant Minor.*

By THOMAS MUIR, LL.D.

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(1) The simplest possible case of the development referred to in the title is one which is seen to follow instantly from the determinant definition of a Pfaffian. Thus, by the said definition,

$$\begin{vmatrix} \cdot & \cdot & a_4 & a_5 & a_6 \\ & \cdot & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{vmatrix} = \sqrt{\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & a_4 & a_5 & a_6 \\ \cdot & \cdot & \cdot & \cdot & b_4 & b_5 & b_6 \\ \cdot & \cdot & \cdot & \cdot & c_4 & c_5 & c_6 \\ -a_4 & -b_4 & -c_4 & \cdot & d_5 & d_6 \\ -a_5 & -b_5 & -c_5 & -d_5 & \cdot & e_6 \\ -a_6 & -b_6 & -c_6 & -d_6 & -e_6 & \cdot \end{vmatrix}},$$

and therefore

$$\begin{aligned} &= \sqrt{\left\{ - \begin{vmatrix} a_4 & a_5 & a_6 \\ b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \end{vmatrix} \cdot \begin{vmatrix} -a_4 & -a_5 & -a_6 \\ -b_4 & -b_5 & -b_6 \\ -c_4 & -c_5 & -c_6 \end{vmatrix} \right\}} \\ &= \sqrt{|a_4 \ b_5 \ c_6|^2}, \\ &= \pm |a_4 \ b_5 \ c_6|; \end{aligned}$$

the result reached being that a Pfaffian with certain zero elements is expressible as a determinant. It should be noted that, as in many instances where this form of definition is used, there is an ambiguity as to sign: it is of less importance to note that here the appropriate sign is $-$.

(2) Another case may readily be established by using either as a definition or as a proved theorem the recurrent law of formation which, in the original notation of JACOBI and CAYLEY, is exemplified by the identity

$$\begin{aligned} [123456] &= 12[3456] - 13[2456] + 14[2356] \\ &\quad - 15[2346] + 16[2345]. \end{aligned}$$

Thus when there is only one zero element and the Pfaffian is of the 3rd order, we have

$$\begin{aligned} \begin{vmatrix} \cdot & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{vmatrix} &= e_6 \begin{vmatrix} \cdot & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{vmatrix} - d_6 \begin{vmatrix} \cdot & a_3 & a_5 \\ & b_3 & b_5 \\ & & c_5 \end{vmatrix} \\ &\quad + c_6 \begin{vmatrix} \cdot & a_4 & a_5 \\ & b_4 & b_5 \\ & & d_5 \end{vmatrix} - b_6 \begin{vmatrix} a_3 & a_4 & a_5 \\ & c_4 & c_5 \\ & & d_5 \end{vmatrix} \\ &\quad + a_6 \begin{vmatrix} b_3 & b_4 & b_5 \\ & c_4 & c_5 \\ & & d_5 \end{vmatrix} \end{aligned}$$

But, by the previous case, the 1st, 2nd, and 3rd terms of this expansion are equal to

$$-e_6|a_3b_4|, \quad +d_6|a_3b_5|, \quad -c_6|a_4b_5|,$$

respectively ; and the remainder

$$\begin{aligned} &= - \begin{vmatrix} a_3b_6 & a_4b_6 & a_5b_6 \\ & c_4 & c_5 \\ & & d_5 \end{vmatrix} + \begin{vmatrix} a_6b_3 & a_6b_4 & a_6b_5 \\ & & c_4 \\ & & c_5 \\ & & d_5 \end{vmatrix}, \\ &= \begin{vmatrix} a_6b_3 - a_3b_6 & a_6b_4 - a_4b_6 & a_6b_5 - a_5b_6 \\ & c_4 & c_5 \\ & & d_5 \end{vmatrix}, \\ &= -|a_3b_6|d_5 + |a_4b_6|c_5 - |a_5b_6|c_4. \end{aligned}$$

Consequently we have the result

$$\begin{vmatrix} a_3 & a_4 & a_5 & a_6 \\ b_3 & b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \\ d_5 & d_6 \\ e_6 \end{vmatrix} = -|a_3b_4|e_6 + |a_3b_5|d_6 - |a_4b_5|c_6 \\ -|a_3b_6|d_5 + |a_4b_6|c_5 - |a_5b_6|c_4,$$

where the first factors on the right are the set of six $(C_{4,2})$ two-lined determinants formable from

$$\begin{vmatrix} a_3 & a_4 & a_5 & a_6 \\ b_3 & b_4 & b_5 & b_6 \end{vmatrix},$$

and their cofactors are the remaining six $(3+2+1)$ elements

$$\begin{vmatrix} c_4 & c_5 & c_6 \\ d_5 & d_6 \\ e_6 \end{vmatrix}$$

of the Pfaffian.

(3) Had the given Pfaffian been of a higher order than the 3rd, it is clear that the cofactors of the two-lined determinants could not have been linear. It will now be seen by considering another case that in general they are themselves Pfaffians, and that consequently in the case just dealt with they have in strictness to be viewed not as elements but as Pfaffian minors of the 1st order. The expansion to which we are leading may thus be described as an aggregate of terms each of which is a product of a determinant and a Pfaffian.

As before we have

$$\begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\ c_4 & c_5 & c_6 & c_7 & c_8 \\ d_5 & d_6 & d_7 & d_8 \\ e_6 & e_7 & e_8 \\ f_7 & f_8 \\ g_8 \end{vmatrix}$$

$$\begin{aligned}
&= g_8 \begin{vmatrix} a_3 & a_4 & a_5 & a_6 \\ b_3 & b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \\ d_5 & d_6 \\ e_6 \end{vmatrix} - f_8 \begin{vmatrix} a_3 & a_4 & a_5 & a_7 \\ b_3 & b_4 & b_5 & b_7 \\ c_4 & c_5 & c_7 \\ d_5 & d_7 \\ e_7 \end{vmatrix} + \dots + e_8 \begin{vmatrix} a_4 & a_5 & a_6 & a_7 \\ b_4 & b_5 & b_6 & b_7 \\ d_5 & d_6 & d_7 \\ e_6 & e_7 \\ f_7 \end{vmatrix} \\
&\quad - b_8 \begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 \\ c_4 & c_5 & c_6 & c_7 \\ d_5 & d_6 & d_7 \\ e_6 & e_7 \\ f_7 \end{vmatrix} + a_8 \begin{vmatrix} b_3 & b_4 & b_5 & b_6 & b_7 \\ c_4 & c_5 & c_6 & c_7 \\ d_5 & d_6 & d_7 \\ e_6 & e_7 \\ f_7 \end{vmatrix},
\end{aligned}$$

where, again, all the terms on the right except the last two can be dealt with by using the preceding case, the result of such use being an expression consisting of 30 (*i.e.*, 5×6) terms of the form

$$-g_8 |a_8 b_4| e_6.$$

On examination, however, it will be found that these can be grouped in sets of three by reason of the fact that each of the ten (*i.e.*, $C_{5,2}$) two-lined determinants appearing in the expression occurs three times. For example, the determinant $|a_3 b_4|$ occurs the first time, as we have just seen, with the cofactor $-g_8 e_6$, the second time with $f_8 e_7$, and the third time with $-e_8 f_7$; its full cofactor thus being

$$-(g_8 e_6 - f_8 e_7 + e_8 f_7) \quad \text{or} \quad - \begin{vmatrix} e_6 & e_7 & e_8 \\ f_7 & f_8 \\ g_8 \end{vmatrix}.$$

By condensation in this way, therefore, there is obtained from the first five terms of the development with which we started an expression consisting of ten terms of the form

$$- |a_3 b_4| \begin{vmatrix} e_6 & e_7 & e_8 \\ f_7 & f_8 \\ g_8 \end{vmatrix}.$$

As for the remaining two terms of the said development, the same reasoning as before gives their aggregate

$$\begin{aligned}
&= \begin{vmatrix} a_8 b_3 - a_3 b_8 & a_8 b_4 - a_4 b_8 & a_8 b_5 - a_5 b_8 & a_8 b_6 - a_6 b_8 & a_8 b_7 - a_7 b_8 \\ c_4 & c_5 & c_6 & c_7 \\ d_5 & d_6 & d_7 \\ e_6 & e_7 \\ f_7 \end{vmatrix}, \\
&= - |a_3 b_8| \begin{vmatrix} d_5 & d_6 & d_7 \\ e_6 & e_7 \\ f_7 \end{vmatrix} + |a_4 b_8| \begin{vmatrix} c_5 & c_6 & c_7 \\ e_6 & e_7 \\ f_7 \end{vmatrix} - \dots - |a_7 b_8| \begin{vmatrix} c_4 & c_5 & c_6 \\ d_5 & d_6 \\ e_6 \end{vmatrix}.
\end{aligned}$$

We thus have as a final result

$$\begin{vmatrix} \cdot & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\ & & c_4 & c_5 & c_6 & c_7 & c_8 \\ & & & d_5 & d_6 & d_7 & d_8 \\ & & & & e_6 & e_7 & e_8 \\ & & & & & f_7 & f_8 \\ & & & & & & g_8 \end{vmatrix} = - |a_3 b_4| \cdot \begin{vmatrix} e_6 & e_7 & e_8 \\ f_7 & f_8 \\ g_8 \end{vmatrix} + \dots - |a_7 b_8| \cdot \begin{vmatrix} c_4 & c_5 & c_6 \\ d_5 & d_6 \\ e_6 \end{vmatrix},$$

where the first factors of the terms on the right are the fifteen (*i.e.*, $C_{5,2}$) two-lined determinants formable from

$$\begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \end{vmatrix}$$

and their cofactors are the fifteen (*i.e.*, $5 + 4 + 3 + 2 + 1$) principal minors of the Pfaffian

$$\begin{vmatrix} c_4 & c_5 & c_6 & c_7 & c_8 \\ & d_5 & d_6 & d_7 & d_8 \\ & & e_6 & e_7 & e_8 \\ & & & f_7 & f_8 \\ & & & & g_8 \end{vmatrix},$$

the first determinant $|a_3 b_4|$ going along with the complementary minor of the first element c_4 of the Pfaffian, the second determinant $|a_3 b_5|$ going along with the complementary minor of the second element c_5 of the Pfaffian, and so on in every case.

(4) For the full investigation of the general theorem thus shadowed forth, neither of the definitions here employed is well suited: what is needed is a definition prescribing the mode of formation of the terms from the elements and the mode of determining the sign of each term—a definition, that is to say, similar to that ordinarily used for a determinant. The following will be found to satisfy these requirements:—

If $n(2n-1)$ elements be each numbered by a pair of integers in order of magnitude, and be arranged in semiquadrate form, thus—

$$\begin{array}{ccccccc} 12 & 13 & 14 & \dots & 1,2n \\ & 23 & 24 & \dots & 2,2n \\ & & & \dots & \\ & & & & 2n-1,2n \end{array}$$

and all possible terms be taken which are products of n elements whose united place-numbers include all the integers from 1 to $2n$, the sign of each term being taken + or — according as the number of inverted-pairs in the series of integers specifying the term is even or odd; then, the function which is the aggregate of these terms is called a Pfaffian of the n^{th} order, and is denoted by the semiquadrate collection of elements bounded by two straight lines, a shorter on the left and a longer on the right.

For example, the Pfaffian of the 2nd order,

$$\begin{vmatrix} 12 & 13 & 14 \\ & 23 & 24 \\ & & 34 \end{vmatrix} = 12.34 - 13.24 + 14.23.$$

(5) Instead of the elements being viewed as forming $2n-1$ rows of $2n-1$, $2n-2, \dots, 1$ elements respectively, and at the same time $2n-1$ columns of $1, 2, \dots, 2n-1$ elements respectively, they may, without alteration of position, be viewed as situated at the intersections of $2n$ frame-lines each containing $2n-1$ elements, the r^{th} frame-line being in every case made up of the r^{th} row and the $(r-1)^{\text{th}}$ column: and from this point of view the two integers used to specify an element are the numbers of the two frame-lines in which the latter is situated. This will be apparent from a glance at the following diagram of frame-lines for a Pfaffian of the 3rd order:—

It follows also that the rule for the formation of the terms is equivalent to a direction that all possible products of n elements are to be taken, no two elements in any product being from the same frame-line.

Thus, if in trying to form a term of the Pfaffian of the 3rd order whose elements are $a, b, c, \dots, m, n, o$, we took, to commence with, the element a in the first row, we should thereby be debarred not only from taking anything else from this row (which is the first frame-line) but also from the second row (which is a part of the second frame-line), because a is an element in both, its place-name being 12. Our choice, consequently, would then be from among the elements left after deletion of these two lines, *i.e.*, from

Here the first three frame-lines, when freed of the portions containing zero elements, constitute a rectangular array of 3 rows and 7 columns, from which thirty-five (*i.e.*, $C_{7,3}$) determinants of the 3rd order are formable; and the initial proposition to be made good is that every one of the 210 terms of these thirty-five determinants is a portion of a term of the given Pfaffian. To do this we have only got to put in mental contiguity for a moment the definitions of a determinant and a Pfaffian: for each determinant term being required to consist of three elements taken from the rectangular array referred to, no two of which must belong to the same row or to the same column, complies with the requirement regarding the first three elements needed to form part of a term of the Pfaffian, *viz.*, that they must be chosen from the first three frame-lines but that at the same time no two of them must belong to the same frame-line. The next proposition is that all the six terms of any particular one of the thirty-five determinants require the same cofactor in order that terms of the Pfaffian may be produced. This is made clear by considering the fact that all of the six terms have their elements taken from the same set of frame-lines, and that the remaining two elements in each case must be chosen from the elements left when the said set has been deleted from the Pfaffian. Thus, if the particular determinant were

$$\begin{vmatrix} 15 & 17 & 18 \\ 25 & 27 & 28 \\ 35 & 37 & 38 \end{vmatrix},$$

each of its terms could only become a term of the Pfaffian by having annexed to it two elements selected from those left when the 1st, 2nd, 3rd, 5th, 7th, 8th frame-lines of the Pfaffian have been deleted—that is to say, from

$$\begin{array}{ccc} 46 & 49 & 4t \\ & 69 & 6t \\ & & 9t. \end{array}$$

We are thus prepared to advance a third proposition derived from the previous two, *viz.*, that, apart from the question of sign, all the eighteen terms of each one of the products of the form

$$\begin{vmatrix} 15 & 17 & 18 \\ 25 & 27 & 28 \\ 35 & 37 & 38 \end{vmatrix} \cdot \begin{vmatrix} 46 & 49 & 4t \\ & 69 & 6t \\ & & 9t \end{vmatrix}$$

are terms of the Pfaffian. This means that 35×18 , *i.e.*, 630, terms of the Pfaffian are accounted for. Now the total number of terms in a Pfaffian of the 5th order is 1.3.5.7.9; the number of these which will vanish when the element 12 vanishes is 1.3.5.7; the additional number which will vanish when 13 vanishes is 1.3.5.7; and the additional number which will vanish when 23 vanishes is 1.3.5.7. It follows, therefore, that the total number of non-vanishing terms in the Pfaffian under discussion ought to be

$$\begin{aligned}
 & 1.3.5.7.9 - 3(1.3.5.7), \\
 \text{i.e.,} \quad & 1.3.5.7(9 - 3), \\
 \text{i.e.,} \quad & 35 \times 18;
 \end{aligned}$$

and this is exactly the number obtained from our development.

(8) The only matters remaining now for consideration are those which concern the *signs* of the terms thus obtained, it being necessary for our purpose (1) to establish the fact that the eighteen signs of any product of the form

$$\left| \begin{array}{ccc} 15 & 17 & 18 \\ 25 & 27 & 28 \\ 35 & 37 & 38 \end{array} \right| \cdot \left| \begin{array}{ccc} 46 & 49 & 4t \\ & 69 & 6t \\ & & 9t \end{array} \right|$$

are either all right or all wrong, and (2) to formulate a rule for distinguishing between products of these two kinds, so that the sign + may be prefixed to the one and - to the other.

Now if no sign precede the product, the sign of any of the eighteen terms is determinable from the sign of the portion of the term which comes from the determinant factor and the sign of the portion which comes from the Pfaffian factor; and the former being dependent upon the number of inverted-pairs in the series of column-numbers specifying the elements of the first part of the term, and the latter upon the number of inverted-pairs in the series of frame-line numbers specifying the elements of the second part of the term, it is clear that if σ be the sum of the said two numbers of inverted pairs the sign of the complete term will be $(-)^{\sigma}$. On the other hand, the sign which it *ought* to bear as a term of the parent Pfaffian is fixed by the number ν of inverted-pairs in the series of integers specifying the frame-lines of *all* the elements composing it. What is wanted, therefore, is a comparison of this number ν with σ ; and, if we can show that $\nu - \sigma$ is constant for all the eighteen terms, it will follow, of course, that the eighteen signs are either all right or all wrong. For example, in the product

$$\left| \begin{array}{ccc} 15 & 17 & 18 \\ 25 & 27 & 28 \\ 35 & 37 & 38 \end{array} \right| \cdot \left| \begin{array}{ccc} 46 & 49 & 4t \\ & 69 & 6t \\ & & 9t \end{array} \right|$$

$-17.25.38$ is a term of the determinant and $-49.6t$ a term of the Pfaffian, the sign - in the one case being fixed by the number of inverted-pairs in 758 and in the other by the number in 496t, whereas the sign of the resulting term $17.25.38.49.6t$ of the Pfaffian is fixed by the number of inverted-pairs in 172538496t.

For ease in making the necessary comparison let us use

$$I(a\beta\gamma\dots)$$

to stand for the number of inversions found in the pairs of integers obtainable by placing each integer in front of those which follow it, and

$$I(a\beta\gamma\dots; a'\beta'\gamma'\dots)$$

for the number of inversions found in the pairs obtainable by placing each integer of the first group in front of each integer of the second group. In the former, $I(a\beta\gamma\dots)$, the order in which we write the integers of the group is all-important; in the latter, $I(a\beta\gamma\dots; a'\beta'\gamma'\dots)$, the order is of no consequence so long as we do not mix the two groups.

With this notation a fundamental proposition regarding inverted-pairs, which bears directly on the subject in hand, can be stated very simply, viz.,

$$I(a\beta\gamma\dots a'\beta'\gamma'\dots) = I(a\beta\gamma\dots) + I(a'\beta'\gamma'\dots) \\ + I(a\beta\gamma\dots; a'\beta'\gamma'\dots),$$

and it is immediately evident therefrom that whatever interchanges may be made in the group $a\beta\gamma\dots$ or in the group $a'\beta'\gamma'\dots$

$$I(a\beta\gamma\dots a'\beta'\gamma'\dots) - I(a\beta\gamma\dots) - I(a'\beta'\gamma'\dots)$$

will remain constant.

As applied to the special groups of integers connected with the Pfaffian term above-mentioned this ensures that

$$I(172538496t) - I(172538) - I(496t)$$

will not alter by reason of any interchanges taking place in the group 172538 or in the group 496t. If in addition the integers 123 in the first group be excluded from interchange, we shall have

$$I(172538) = I(7; 23) + I(5; 2) + I(758), \\ = 3 + I(758),$$

through all interchanges in 758. By substitution it therefore follows that the number

$$I(172538496t) - I(758) - I(496t)$$

will remain constant while interchanges take place in 758 or in 496t; and this is equivalent to saying that the terms of the parent Pfaffian which are obtainable from the product

$$\begin{vmatrix} 15 & 17 & 18 \\ 25 & 27 & 28 \\ 35 & 37 & 38 \end{vmatrix} \cdot \begin{vmatrix} 46 & 49 & 4t \\ & 69 & 6t \\ & & 9t \end{vmatrix}$$

are either all correctly or all incorrectly signed.

If the said constant be even, the sign which ought to precede the product will of course be +, in the other possibility -. As a matter of practice, however, the easiest way of determining the sign to be placed in front of any product is to make the sign such that one of the terms obtainable from the product shall be correctly signed, and for this purpose the facts given in § 6 will be found useful. If the whole expansion be wanted, and the products be arranged in the natural order—that is to say, in such a way that the series of determinant factors shall begin with | 14 25 36 |

and end with $|18\ 29\ 3t|$, the sign to be prefixed to any product is easily known from that of the preceding product.

(9) The number of different forms of this new development which are possible in the case of a Pfaffian of the n^{th} order is of course the number of partitions of the integer n into two integers, the first of the latter corresponding to the order of the determinant factors in the development, and the other to the order of the Pfaffian cofactors. For example, in the case of the Pfaffian of the 5th order we shall have the five developments

$$\begin{aligned}
 (\alpha) & + 12.||\ 3456789t| \quad - \quad 13.||\ 2456789t| \quad + \dots (C_{9,1} \text{ terms}), \\
 (\beta) & - |13\ 24\ |.||\ 56789t| \quad + \quad |13\ 25\ |.||\ 46789t| \quad - \dots (C_{8,2} \text{ terms}), \\
 (\gamma) & - |14\ 25\ 36\ |.||\ 789t| \quad + \quad |14\ 25\ 37\ |.||\ 689t| \quad - \dots (C_{7,3} \text{ terms}), \\
 (\delta) & + |15\ 26\ 37\ 48\ |.9t \quad - \quad |15\ 26\ 37\ 49\ |.8t \quad + \dots (C_{6,4} \text{ terms}), \\
 (\epsilon) & + |16\ 27\ 38\ 49\ 5t| \quad (C_{5,5} \text{ term}),
 \end{aligned}$$

the parent Pfaffian containing no zero elements in the first case, 1 in the second, 1 + 2 in the third, 1 + 2 + 3 in the fourth, and 1 + 2 + 3 + 4 in the fifth.

The single elements in the first development may be viewed as determinants of the 1st order and in the fourth development as Pfaffians of that order.