XXV.—On the Hydrodynamical Theory of Seiches. By Professor Chrystal.

WITH A BIBLIOGRAPHICAL SKETCH.

(Read June 19, 1905. Issued separately July 3, 1905.)

PART I.

GENERAL SUMMARY.*

§ 1. The variations of the surface-level of lakes due to the direct action of wind and rain, and the smaller disturbances caused by surface waves, of small or moderate length, due to the action of the wind and the movement of boats and animals, must have been familiar phenomena at all times. The first accurately recorded observation, that lake-levels are subject to a rhythmic variation, similar in some respects to the ocean tides, seems to have been made at Geneva in 1730 by FATIO DE DUILLIER, a well-known Swiss engineer. Owing to the peculiar configuration of the Geneva end of Lake Léman, these variations occasionally reach a magnitude of 5 or even 6 feet; and DUILLIER mentions that they were known in his time by the local name of "Seiches," which has now been applied to rhythmic alterations of the level of lakes in general.

From DUILLIER's time onwards various observations and speculations regarding the seiches of Lake Léman are recorded. It seems to have been J. P. E. VAUCHER, Pastor, and Professor successively of Botany and Church History at Geneva, who, in a memoir written between 1802 and 1804, and published in the memoirs of the Physical Society of Geneva in 1833, first pointed out that seiches are not confined to Léman, but are to be found more or less in all lakes; that they may be of all degrees of amplitude up to 5 feet; and may occur at all seasons of the year, although their occurrence seems to be affected by the state of the atmosphere. He also pointed out that the amplitude of the seiches in Léman increases towards its western end; and that the seiches at its eastern end are not more marked than those observed in other lakes.

These and other early observations of seiches are mentioned by FOREL in his great monograph on the Lake of Geneva, vol. ii. p. 50. In particular, he cites one observed at Kenmore on Loch Tay in 1784, which lasted several hours, and is said to have had a period of seven minutes and a maximum amplitude of nearly 5 feet.

A still earlier example is given in the *Scots Magazine* for 1755, p. 593, from which it appears that seiches were caused in several of the lakes of Scotland by the great earthquake of Lisbon on 1st November 1755.

As the source is not easily accessible to everyone, an extract may be printed here :—

* For the convenience of those who are more interested in the observation of seiches than in the purely mathematical theory, I have separated the mathematics, so far as possible, from the general statement of the conclusions arrived at and the suggestions of further problems to be solved by experiment or observation.

"On the first of November last, Loch Lomond, all of a sudden, and without the least gust of wind, rose against its banks with great rapidity, and immediately retiring, in about five minutes subsided as low, in appearance, as ever it used to be in the greatest drought of summer. In about five minutes after, it returned again, as high and with as great rapidity as before. The agitation continued in the same manner, from half an hour past nine till fifteen minutes after ten in the morning; the waters taking five minutes to subside and as many to rise again. From ten to eleven the agitation was not so great; and every rise was somewhat less than the immediately preceding one; but taking the same time, viz., five minutes to flow and five to ebb, as before. About eleven the agitation ceased. The height the waters rose was measured immediately after and was found to be 2 feet 6 inches (76^{cm}) perpendicular. The same day, at the same hour, Loch Lung and Loch Keatrin were agitated in much the same manner; and we are informed from Inverness, that the agitation in Loch Ness was so violent as to threaten destruction to some houses built on the sides of it."

From this clear description there can be no doubt that the phenomenon observed was a longitudinal seiche of Loch Lomond of exceptional amplitude, having a period of ten minutes, probably the trinodal or quadrinodal seiche of that lake. The longest dimension of both Lomond and Ness is nearly in a straight line with the centre of disturbance at Lisbon, and a plurinodal seiche is the result we should expect. The greater disturbance in Loch Ness may be due to the fact that one of the seiche periods of that lake is about nine minutes.

But our really accurate knowledge of the phenomena of seiches dates from the commencement of FOREL's own observations at the harbour of Morges, on the Lake of Geneva, in 1869. He may with justice be called the Faraday of seiches. He worked at first with a small portable apparatus (plemyrameter), and later (1876) with a self-registering limnograph installed at Morges, and a portable limnograph of simpler construction. In 1877 PLANTAMOUR established a magnificent self-registering limnograph at his villa at Sécheron, near Geneva, which has been in continuous operation since. In 1879 SARASIN devised his portable limnograph, with which observations were made at Tour de Peilz, Chillon, Rolle, and various other stations on Léman, and also upon other Swiss lakes. In 1880 the French Government engineers also installed a fixed limnograph at Thonon, with which observations have been made under the superintendence of DELEBECQUE, DU BOYS, and LAURIOL.

During the last twenty years a large number of enthusiastic observers have followed the lead given by FOREL and his fellow-countrymen; and we have now a great mass of information regarding the seiches in lakes in various parts of the world,* from the 15-hour seiches observed by HENRY in Lake Erie, which is 396 km. long, to the seiches of 14 seconds, observed by ENDRÖS in a small pond whose length was only 111 m.

§ 2. The accurate theoretical discussion and co-ordination of the results has scarcely

* See the extension of FOREL's bibliography appended to this paper.

kept pace with their accumulation. In the main, the original theory of FOREL has been clearly established, viz., that a seiche is a standing oscillation of a lake, usually in the direction of its longest dimension, but occasionally transverse. In a motion of this kind every particle of the lake oscillates synchronously with every other, the periods and phases being the same for all; and the orbits similar (in fact, rectilinear), but of different dimensions, and not similarly situated. Taking, for simplicity, a longitudinal seiche in a lake of uniform breadth and rectangular section, but varying depth, the horizontal and vertical displacements of any particle on the surface originally at a distance x from a fixed point of reference would be given by $\xi = \phi_n(x) \sin n(t - \tau)$, $\zeta = \chi_n(x) \sin n(t - \tau)$; where t is the time measured from any fixed epoch, τ an arbitrary constant determining the phase of the oscillation, and $T = 2\pi/n$ is the period of the oscillation.

For a lake of given configuration, an infinite number of different values of n (but not any value) are admissible, say $n_1, n_2, n_3, \dots$; and the functions $\phi_n(x)$ and $\chi_n(x)$ are determined when n is given.

For any given value of n , say n_ν , the function $\chi_{n_\nu}(x)$ vanishes for ν different values of x . At these points, which are called *nodes*, the level of the surface is unaltered by the seiche. Corresponding to $\nu = 1, 2, 3$, etc., we have uninodal, binodal, trinodal, etc., seiches. Any number of these may coexist; and the total seiche displacement is obtained by adding these. When only one of these harmonic components is present we shall call the seiche *pure*.

For a number of values of x , intermediate between the nodal values, $\phi_n(x)$ vanishes, and there is no horizontal motion of the surface particles. These points are called *ventral points*. Four times the distance between a node and the next ventral point is called the wave length. Obviously the wave length is not in general the same at all points of the lake. When the wave length is large compared with the depth, which is always the case in a seiche, the wave is spoken of as a *long wave*; and the hydrodynamical theory in that case admits, as is well known,* of considerable simplification.

§ 3. When the depth of the lake is constant, the theory of long waves leads to the well-known result

$$\begin{aligned}\xi &= A_\nu \sin \frac{\nu\pi x}{l} \sin \frac{\nu\pi \sqrt{(gd)} }{l} (t - \tau), \\ \zeta &= -\frac{ld\nu\pi}{l} A_\nu \cos \frac{\nu\pi x}{l} \sin \frac{\nu\pi \sqrt{(gd)} }{l} (t - \tau) ;\end{aligned}$$

where l is the length of the lake, d its depth, x the initial distance of the surface particle in question from one end, all measured in feet, and $g = 32.2 : t$ denotes the time, and A_ν and τ are arbitrary constants (amplitude and epoch).

It follows that in a lake of uniform depth the period of the uninodal seiche is $2l/\sqrt{(gd)}$; and the periods of the uninodal, binodal, trinodal, etc. seiches are proportional to the terms of the harmonic series $1, \frac{1}{2}, \frac{1}{3}, \dots$. Also the uninode

* See AIRY, art. "Tides and Waves," §§ 187 *et seq.*, *Encyclopædia Metropolitana*, 1848.

is given by $x_1 = l/2$; the binodes by $x_1 = l/4$, $x_2 = 3l/4$; the trinodes by $x_1 = l/6$, $x_2 = 3l/6$, $x_3 = 5l/6$; and so on.

In this case the wave length for each pure seiche is the same at all parts of the lake; the ventral points are midway between the nodes; and the uninode, middle trinode, middle quinquinode, etc. are all at the middle of the lake. In fact, the periods, nodes, and ventral points follow the same law as the periods, nodes, and ventral points of the fundamental and over tones of an organ pipe open at both ends.

§ 4. When, however, the depth of the lake varies, this acoustic analogy is in some important respects misleading. The theory of long waves applied to a longitudinal seiche in a lake of uniform breadth and rectangular cross section, but varying depth, leads to the following among other general results.

§ 5. In any given lake, seiches of all degrees of nodality, *i.e.* uninodal, binodal, trinodal, etc., are possible; and any actual seiche is either one of these or a superposition of several of them. Perhaps the most commonly occurring case is what FOREL calls a dicrote seiche, whose components are uninodal and binodal.

§ 6. The periods of the series of pure seiches are not in general proportional to the terms of the harmonic series $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$. The ratios of the periods are in general incommensurable; in general, not even algebraic numbers, although in certain special cases the periods are inversely proportional to the square roots of integral numbers. Thus, for a lake of symmetric complete parabolic longitudinal section, we have $T_\nu = \pi l / \sqrt{\nu(\nu+1)gh}$; so that the ratios are

$$T_1 : T_2 : T_3 : \dots = 1/\sqrt{1 \times 2} : 1/\sqrt{2 \times 3} : 1/\sqrt{3 \times 4} : \dots$$

Indeed, it follows from a result* which I obtained for lakes whose longitudinal section is part of the quartic curve $z = h(1 - x^2/a^2)^2$ that concave lakes can be imagined in which $T_1, T_2, T_3, \dots$ all approach as nearly to equality as we please. Hence, for example, it may very well happen that it is the trinodal, and not the binodal seiche whose period is half the period of the uninodal,—a result wholly in contradiction with the acoustic analogy suggested by the consideration of lakes of uniform depth.

§ 7. As this is a matter which seems to have caused some perplexity, it may be well to give some numerical illustrations. Let us take a quartic lake of the kind discussed in the paper to which I have referred. The period of the ν -nodal seiche is given by $T_\nu = 2\pi l / \gamma \sqrt{gd(4\nu^2\pi^2/k^2 + 1)}$, where γ and k depend on the configuration of the lake. Suppose that $T_3/T_1 = 1/2$, then we find that we must have $4\pi^2/k^2 = 3/5$. It then follows that $T_2/T_1 = .686$. For a lake of this kind we should therefore have $T_1 : T_2 : T_3 = 1 : .686 : .5$. In a communication to the Société Vaudoise des Sciences Naturelles, FOREL† mentions that three periods have been determined from observations on the Lake of Constance, *viz.*, $T_1 = 55.8$, $T_2 = 39.1$, $T_3 = 28.1$, which give $T_1 : T_2 : T_3 = 1 : .701 : .504$. It would appear, therefore, that the Lake of Constance

* See *Proc. Roy. Soc. Edin.*, vol. xxv. p. 328, Mar. 20, 1905.

† *Ect. Bull.*, vol. xl. 149, Feb. 3, 1904.

behaves not very differently from a quartic lake of the kind supposed. Of course it by no means follows that the normal curve* of the Lake of Constance is even approximately like the quartic curve supposed.† The point I wish to make is that, from the hydrodynamical point of view, there is nothing surprising in the relation of its periods. Farther investigation of the phases of the seiche at different parts of the lake will probably show that the *seiche à la quinte*, as FOREL calls it, for which $T=39'$, is really the binodal seiche; and that the seiche for which $T=28\cdot1'$ is a trinodal.

§ 8. The formula for the periods of a quartic lake given above may be written $T_\nu = \rho / \sqrt{(\nu^2 + \epsilon)}$, where $\rho = kl/\gamma \sqrt{(gd)}$, $\epsilon = \pm k^2/4\pi^2$, the minus sign corresponding to convex lakes. As this formula applies to quartic lakes of every variety, concave or convex, symmetric or asymmetric, it may be conjectured that it will give rough approximations at least to the periods of any actual lake of fairly regular configuration. The constants could not be determined *à priori* without discussing the bathymetric data for each lake. The ratios of the periods, however, depend only on the single constant ϵ . We have, in fact, $T_\nu/T_1 = \sqrt{\{(1+\epsilon)/(\nu^2+\epsilon)\}}$. In general the two longest periods are best known. If we take these as given, the equation $T_2^2/T_1^2 = (1+\epsilon)/(4+\epsilon)$ gives the value of ϵ ; and then $T_\nu = T_1 \sqrt{\{(1+\epsilon)/(\nu^2+\epsilon)\}} = \rho / \sqrt{(\nu^2+\epsilon)}$, where $\rho = T_1 \sqrt{(1+\epsilon)}$.‡ For very large values of ν , we have approximately $T_\nu = \rho/\nu$; in other words, the periods of the seiches of higher nodality approximate to a harmonic series. By means of these formulæ—which I shall call the *Quartic approximation*—the table on p. 604 has been calculated for a number of lakes whose periods are fairly well determined. I have gone in most cases as far as $T=4\cdot7'$, which is the period found by FOREL for the longest progressive surface waves ever observed on Léman, calculated for infinite depth of water.

As a control, I have added at the end of the table, under A, B, C, D, the periods for a complete symmetric rectilinear, semirectilinear, complete parabolic, and semiparabolic lake respectively, as calculated by the quartic approximation, and as calculated by the accurate formulæ of §§ 27, 34, 49, and 51 below.

* See § 12 below.

† Nevertheless it is curious to pursue this numerical case a little further. Referring to my paper already quoted, and calculating k as above, we get $k=8\cdot112$. If we assume the longitudinal section to be symmetrical, then $\gamma=2 \tanh(k/4)$; and we have $\gamma=1\cdot932$. Hence $r=\{1-(\gamma/2)^2\}^2 d$, gives, if we put $d=252^m$ (the maximum depth of Constance), $r=1\cdot1^m$. If then we take a symmetric quartic lake having the same length as Constance, viz., 65^m , the same maximum depth, and end depths of $1\cdot1^m$, we find $T_1=\pi 65 \times 10^6/966 \sqrt{\{981 \times 25200 \times 8 \div 5\}}=56\cdot0$. Hence $T_1=56\cdot0$, $T_2=38\cdot4$, $T_3=28'$. The agreement with the observed periods of Constance is curiously close, and is, no doubt, partly accidental. It will be of great interest to work out the normal curve for Constance, and calculate the periods by a rigorous application of the theory, as has been done by Mr WEDDERBURN and myself for Treig and Earn.

‡ It is interesting to notice that in the case of a concave lake $\rho/\sqrt{\epsilon}$ is the period of the "anomalous seiche." See *Proc. R.S.E.*, xxv. (1905), p. 645.

Quartic Approximations to Lake Periods.

Lake.		T ₁	T ₂	T ₃	T ₄	T ₅	T ₆	T ₇	T ₈	T ₉	T ₁₀	T ₁₁	T ₁₂	T ₁₃	T ₁₄	T ₁₅	T ₁₆	T ₁₇	T ₁₈	T ₁₉	T ₂₀
Constance* ε=1.89 ρ=94.92	Calc.	...	...	28.76	22.44	18.30	15.42	13.31	11.69	10.43	9.40	8.56	7.86	7.26	6.75	6.30	5.91	5.57	5.26	4.98	4.74
	Obs.	55.80	39.10	28.10	...	...	15.00	...	...	...	...	...	...	...	...	...	...	...	...	...	...
Garra ε=1.84 ρ=72.33	Calc.	...	...	22.02	17.17	14.00	11.79	10.17	8.94	7.97	7.19	6.54	6.01	5.55	5.16	4.82	4.52	...	...	...	...
	Obs.	43.00	30.00	23.00	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
Madisce ε=4.16 ρ=12.24	Calc.	...	...	13.76	10.43	8.38	7.00	6.01	5.26	4.68	...	...	...	...	...	...	...	...	...	...	...
	Obs.	35.50	20.10	13.7	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
Neuchâtel† ε=4.13 ρ=112.4	Calc.	...	...	31.01	25.04	20.82	17.74	15.41	13.61	12.18	11.01	10.05	9.23	8.54	7.94	7.12	6.97	6.56	6.20	5.88	5.60
	Obs.	49.6	39.4	...	24.20	20.60	...	...	...	...	...	...	9.04	8.3	7.7	...	...	...	...	...	5.7
Starnberg ε=991 ρ=35.2	Calc.	...	...	11.15	8.55	6.92	5.80	4.99	4.37	...	...	...	...	...	...	...	...	...	...	...	...
	Obs.	24.98	15.78	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
Geneva ε=-0.871 ρ=70.23	Calc.	...	...	23.52	17.61	14.09	11.72	10.04	8.79	7.80	7.03	6.38	5.86	5.41	5.00	4.68	...	...	...	...	...
	Obs.	73.50	35.50	...	20.7	...	...	10.3	...	...	7	...	...	...	...	...	...	...	...	...	...
Ness ε=-0.738 ρ=80.32	Calc.	...	...	10.14	7.60	6.07	5.05	4.33	...	...	...	...	...	...	...	...	...	...	...	...	...
	Obs.	31.5	15.3	8.8	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
A ε=9497 ρ=139.6	Quart.	...	...	44.27	33.92	27.41	22.97	19.76	17.33	15.43	13.90	...	...	...	...	...	...	...	...	...	...
	Acc.	100	62.76	43.57	34.28	27.79	23.65	20.40	18.05	16.09	14.60	...	...	...	...	...	...	...	...	...	...
B ε=2753 ρ=112.9	Quart.	...	...	37.08	27.90	22.46	18.75	16.09	14.09	12.53	11.28	...	...	...	...	...	...	...	...	...	...
	Acc.	100	54.62	37.67	28.76	23.27	19.54	16.84	14.79	13.19	11.90	...	...	...	...	...	...	...	...	...	...
C ε=5000 ρ=122.5	Quart.	...	...	39.74	30.15	24.25	20.27	17.41	15.25	13.57	12.22	...	...	...	...	...	...	...	...	...	...
	Acc.	100	57.74	40.82	31.62	25.81	21.82	18.90	16.67	14.91	13.48	...	...	...	...	...	...	...	...	...	...
D ε=2857 ρ=113.4	Quart.	...	...	37.21	28.10	22.55	18.83	16.15	14.14	12.58	11.32	...	...	...	...	...	...	...	...	...	...
	Acc.	100	54.77	37.80	28.87	23.35	19.61	16.90	14.85	13.24	11.95	...	...	...	...	...	...	...	...	...	...

* Also Calc. T₂₃=4.13', T₂₄=3.96', Obs. T=1.00'.† Also Calc. T₂₅=4.3', Obs. 4.3'; Calc. T₂₈=2.9', Obs. 1.1' (due to steamers).510m, the values of T for progressive waves at depths of 10m. and 50m. are 24.6" and 51.7" as compared with T₁₀₀=45".

Authorities for the various lakes are given in Part III.

For Neuchâtel T₁₅₀=75'. For a wave-length of

This table must, of course, be used with great caution. In the first place, as we increase the number of the nodes we trench more and more upon the limits of the hypothesis that the waves are "long." Then it cannot possibly represent the case of convex (*i.e.* in practice concavo-convex lakes) so accurately as the case of wholly concave lakes. It will be noticed, for example, that in Geneva and Ness, which are similar in their seiche-character, there are seiches of 20' and 8·8', probably trinodal or quadri-nodal, which do not fit into the scheme suggested. As the nodality increases, the periods become more and more nearly equal, and therefore more difficult to distinguish, either by a rough calculation, or, for that matter, by observation, from one another, and from progressive surface waves of purely local significance which are not seiches at all.

Subject to these qualifications, the table is very interesting and suggestive. It shows the greater variety of possible periods in some lakes as compared with others. It shows that there is nothing surprising, from the hydrodynamical point of view, in the fact that the three longest periods for Constance are 55·8', 39·1', and 28·1'. The *seiche à la quinte* of which FOREL speaks in the cases of Constance, Garda, and Starnberg is in all probability simply the binodal seiche; and the seiche whose period is approximately half the longest period is a trinodal. Such questions cannot be finally settled until the phases of the seiches have been determined by simultaneous limnographic observations at different parts of the lake, as has been done by ENDROS in his admirable investigation of the seiches of the Chiemsee, *Seespiegelschwankungen beobachtet am Chiemsee*, Traunstein, 1903.

I have included in the table as longitudinal seiches some which have been held by observers to be transversal. This I have done for two reasons. In the first place, the absolute identification of these by means of phase observations has not in all cases been satisfactorily accomplished.* Again, it is possible that a transversal seiche might coexist with a longitudinal one of nearly the same period throughout a considerable part of the lake. The interference of these at the ventral points of the longitudinal seiche would produce the phenomenon of seiche beats† at various points along the shore. At the ends of the lake, which are ventral points for all the different pure seiches, all these seiches interfere. It follows, equally from observation and from the analogy of the vibrating string of varying density, presently to be mentioned, that the average amplitudes of the seiches that occur in nature diminish rapidly as their nodality increases. Hence the chief features of the limnographic trace at the shallow ends of a lake will in general be the periodic configuration due to the interference of the uninodal and binodal seiches; the effect of the others will merely be to produce an embroidery on the main outline. Also, since the periods become more nearly equal as the nodality increases, this embroidery will have an

* See FOREL, *Le Léman*, t. ii. p. 148.

† Since this was written I have noticed that ENDROS, in his able analysis of the seiches of the Chiemsee, cites examples of variations in the phases and amplitudes of nearly pure seiches, which he regards as due to the interference of seiches of the same period differing in phase. He suggests, with great probability, that such seiches are generated by a common but intermittent cause of disturbance.

irregular character, due to the beats of seiches not very different in period. The ends of a lake are therefore the worst places for distinguishing seiches of higher nodality. If a lake had three periods 10', 11', 12', the proper place to observe at, in order to establish clearly the seiche corresponding to the first or last of these periods, would be at a node of the seiche whose period is 11'. Thus FOREL's argument, that the 10' seiche of Léman is a transversal and not a plurinodal longitudinal one, because it is not detected at Geneva, is by no means conclusive. I may also add, although I have little theoretical or experimental ground as yet for the opinion, that it seems to me unlikely that any transversal seiche would be so stable, especially in a lake of the shape of Léman, as the beautiful records of FOREL's limnograph seem to indicate. Nevertheless, great weight must be attached to the inclination of so sagacious an observer, who has all the data before him.

The table also raises many interesting subjects of inquiry. Why, for example, are no seiches observed in Constance of the periods 22·4' and 18·3'? Is this an accident, due to the position at which the limnograph was placed; or are these seiches unstable, owing to irregularities of the lake-bottom near one or more of the corresponding nodes?

§ 9. In a purely concave lake the ratio of the uninodal period to the binodal period is less than a half. In a purely convex lake, if such a thing could be found in nature, the corresponding ratio would be greater than a half. In lakes which are neither purely concave nor purely convex the value of T_2/T_1 will be greater or less than a half according as the concavity or convexity predominates.

§ 10. In the case of parabolic and quartic lakes the rule given by DU BOIS for calculating the periods, viz., ${}_aT_v = (2/v) \int dl / \sqrt{gh}$, where h is the depth corresponding to the element dl of the line of maximum depth, gives too high a value for purely concave and too low a value for purely convex lakes; but it gives in many cases a good first approximation to the periods. This approximation is better for concavo-convex lakes than for purely concave or purely convex lakes; and for purely concave or purely convex lakes, the approximation is better the higher the nodality of the seiche. For a purely concave symmetric parabolic lake DU BOIS' rule would be considerably out; in fact, for such a case ${}_aT_1/T_1 = 1·414$. It may also err greatly in cases where there are great variations of the breadth of the lake, as the method of applying the formula takes no account of such peculiarities.*

§ 11. In a lake of varying depth the uninode is not in general in the middle of the lake, and the uninode, middle trinode, middle quinquinode, etc. are not coincident. Also the ventral points are not midway between the nodes; and the wave length varies from node to node. Thus, for example, in a symmetric parabolic lake the uninode is of course in the middle, but the binodes are displaced towards the shallow ends. It results from the calculations *à priori*, made by myself and Mr WEDDERBURN, that in

* Dr ENDRÖS has found a striking example in the uninodal seiche of the Waginger See, of which he was good enough to tell me by letter.

Earn, which is asymmetric, the uninode is very near the middle; but in Treig this is not so nearly the case: in neither is the trinode coincident with the uninode. In both these lakes the deep binode is not far from the point of greatest depth; in both the shallow binode is nearer the end than a fourth of the length of the lake: but, whereas in Earn the binodes are on opposite sides of the deepest point, in Treig they are on the same side. In neither lake are the binodes equidistant from the uninode. It remains to be seen how far these results of theory will agree with observation.

§ 12. When the breadth and the form of the transverse section of a lake vary as well as the depth, provided these variations are not too abrupt, it can be submitted to calculation by introducing two new variables, viz., σ , which is the product of the area of the transverse section by the breadth of this section at the surface; and v , which is the area of the surface of the lake between the trace on the surface of the transverse section corresponding to σ , and any other similar line chosen for reference. In order to submit the lake to calculation, its line of maximum depth is taken and laid out straight: and practically the lake is treated as if it were a lake of uniform breadth and rectangular cross section, whose longitudinal section is the curve, the abscissa and ordinate of any point on which are v and σ respectively. This curve I call the *normal curve* of the lake. If we may judge by our results for Treig and Earn, these assumptions are sufficiently correct for ordinary concave lakes at least.

§ 13. In my calculations no account is taken of the dissipative forces which damp the seiche oscillation. In some cases the damping is hardly sensible during the period for which a seiche is observed to be pure, or even simply dicrote. FOREL quotes an observation of PLANTAMOUR'S, in which a pure uninodal seiche in Léman, whose maximum double amplitude was about 169^{mm}, lasted for seven and a half days, and consisted of 148 oscillations. The mean double amplitude of 20 oscillations was at first 167^{mm}, and at the 140th, 80^{mm}. It was finally disturbed by the appearance of a binodal component, which turned it into a dicrote seiche; otherwise FOREL calculates that it might have lasted two days more.* In other cases the damping of some of the pure seiches seems to be considerable, owing probably to the fact that the lake is, so to speak, not well tuned for particular periods. This is seen by studying the form of the limnograph trace, by the elegant method suggested by SORET.† A serious difficulty is thus raised if we attempt to determine the seiche periods from the dicrote trace; and it is not unlikely that some of the divergence between the results of observations at different times on the same lake may be due to this cause. Calculation and observation alike have led me to the conclusion that the best way to determine with final accuracy the seiche constants for a lake is to deduce approximately the periods and the positions of its uninode and binode beforehand

* It is just possible that this seiche may have been maintained by some continuing but partly intermittent external cause. The limnogram given by FOREL seems to show traces of the interference to which ENDRÖS has called attention.

† *Arch. de Sc. Phys. et Nat. Genève*, 3^{me} Pér. t. iii. p. 1, 1880. The method has been elaborated and used with great effect by ENDRÖS, *l.c.*, p. 15.

by calculation, from the contours of the lake basin, or by using rough preliminary observations, if such are available; and then to deduce the uninodal and binodal periods from limnographic traces obtained at the binode and uninode respectively, where these seiches are very nearly pure.*

§ 14. It will be obvious that a seiche, properly so called, differs essentially from an ocean tide. The origin of a seiche, and the absolute and relative magnitudes of the pure seiches of which it is composed, no doubt depend on external circumstances; but the periods and the positions of the nodes of the component seiches depend merely on the configuration of the lake basin, and on the surface-level of the water at the time. In a tide, on the other hand, the periods are dependent on external disturbing agencies, chiefly the sun and moon. In the language of physicists, a seiche is a *free oscillation*; a tide a *forced oscillation*.

It is by no means the case that the limnograph traces always indicate a seiche, properly so called.

True and unmistakable seiches are generally seen in comparatively calm weather, or in the lull after storm. At the commencement of a seiche the trace is often irregular; and during storm, oscillations of great amplitude are found, the intervals between the successive maxima of which are not equal. This inequality might, it is true, be due to a jumble of different seiche components, only some of which survive in the subsequent steady oscillation. In the traces for Lake Erie, which does not show the ordinary seiche phenomena with remarkable clearness, but, on the other hand, is remarkable for oscillations of enormous amplitude, associated with the passage of tornadoes, I have noticed indications of the retardation and acceleration of phase which characterise the initial stages of the action of an intermittent external disturbing force, whose period differs from any seiche-period of the lake. The same thing is, I think, visible in the trace of PLANTAMOUR's limnograph during the cyclone of 20th February 1879, which is reproduced by FOREL on p. 194 of his monograph on Léman. Seiches of this kind I propose, for distinction, to call *forced seiches*, and I propose to discuss them farther in a subsequent communication, dealing with the external causes of seiche phenomena.

§ 15. In the following purely mathematical part of this memoir my purpose is twofold:—first, to establish generally, and also by means of special instances, the leading principles stated above; secondly, to furnish formulæ and methods which can be applied in the investigation *à priori* of the periods and nodes of a lake whose length is considerably greater than its breadth, and which does not present excessive or abrupt variations in the configuration of its basin. I believe that in many such cases a sufficiently close approximation can be obtained by finding the normal curve of the lake, and replacing this curve by a combination of parabolas, straight lines, or simple quartic curves. Thus, for example, as will be seen in a paper by Mr E.

* This was written before I had access to the monograph of ENDRÖS, whose plan of observation for the Chiemsee is in many ways a model.

MACLAGAN-WEDDERBURN and myself, presently to be submitted to the Society, good results for Lochs Earn and Treig have been obtained by using two parabolas.

§ 16. In the course of the mathematical investigation, an abstract of which was published in the *Proceedings* of the Society in October last, I have had occasion to introduce certain new functions, which I have called the Seiche-cosine, Seiche-sine, Seiche-cotangent, and the Lake-function. My reason for venturing to give special names to these functions is, that I believe that some of them, especially the Seiche-cosine and the Seiche-sine, which are independent synectic integrals of a special case of the hyper-geometric equation, will be found interesting theoretically, and very useful in applications. Preoccupation with the physical side of the present problem has prevented me from following the pure mathematics which it suggests. Happily, this part of the question has been taken up by a younger mathematician, Dr HALM, who, in a paper which will be submitted to the Society along with the present communication, has obtained a number of interesting results, which tend to justify my belief that the seiche functions are not unlikely to occupy a permanent place in applied mathematics. He has also given tables for calculating $C(c, 1)$ and $S(c, 1)$, and graphs of these functions.

§ 17. I have said that the acoustic analogy with an organ pipe is unsatisfactory in the general case. There is, however, an acoustic analogy which is perfect. Consider a stretched string, fixed at both ends, whose length in conveniently chosen units represents the median line of the lake, which we take to be of uniform breadth and rectangular cross section, but of variable depth, $h(x)$, at any distance x along the median line. Suppose the string so constructed that its density at x is $1/h(x)$; and let $u = P(x) \sin nt$ be the transverse displacement in any fundamental (normal) mode of vibration at time t of the point which was originally at x . Then, if the tension of the string be properly adjusted, we shall have

$$\xi h(x) = u, \quad \zeta = -du/dx,$$

ξ and ζ denoting the horizontal and vertical displacements of the seiche. The motion of the string, therefore, exactly represents the seiche movement, the transverse displacement of the string corresponding to the horizontal displacement of the seiche; and the gradient of the curve formed by the string at any moment to the vertical displacement of the seiche.

It will be noticed that *nodes of the string correspond to ventral points of the lake*; * and the ventral points on the string to nodes on the lake. This analogy is most useful, both in explanation and in suggesting methods of calculation. To a uniform string with its binodal, trinodal, quadrinodal, etc. modes of vibration, corresponds a lake of uniform depth with its biventral, triventral, quadriventral, etc., *i.e.* uninodal, binodal, trinodal, quadrinodal, etc. seiches.

Starting with a uniform string, let us increase its density by adding a small

* There seems to have been a good deal of confusion and some false analogy in this respect.

weight at the middle. The result is to lengthen the periods of the binodal, quadrinodal, etc. modes, but to leave unaltered the periods of the trinodal, quinquinodal, etc. modes. Hence, if we start with a lake of uniform depth, and diminish its depth a little in the middle, the result will be to lengthen the periods of the uninodal, trinodal, etc. seiches, and to leave the periods of the binodal, quadrinodal, etc. seiches unaltered. One consequence of this is, that in the modified lake the period of the binodal seiche is less than half the period of the uninodal. This accords with what we have said before, since the lake is now convex. Similarly, by diminishing the density in the middle of the string, we get an explanation of the corresponding peculiarity of concave lakes.

A slight modification of MELDE's experiment will give us another illustration of the use of this acoustic analogy. Fix a tuning-fork with its prongs horizontal, so that it vibrates in a vertical plane. If we stretch a fine string vertically upwards from a point of attachment on one of the prongs, then, with the proper length and tension, when the fork is excited it will vibrate steadily with a beautifully sharp node in the middle, its lower ventral point being at a distance of about one-fourth of the length from the lower end. If a very small pellet of beeswax be fixed on the string at a distance of about one-eighth of its length from the lower end, it will be found, after slightly adjusting the tension, if necessary, so as to re-establish full resonance, that the string will vibrate with one node as before; but that node and also the lower ventral point will be nearer the lower end than before.* Hence, if we start with a lake of uniform depth, the nodes of the binodal seiche will be each at a distance of one-fourth of the length from its respective end; but if we make a shallow at one end, say about one-eighth of the length of the lake from that end, the corresponding node will be displaced towards the shallow. A depression of the lake bottom would, of course, have the opposite effect.

In a similar way, we can see that an irregularity of the lake bottom will seriously disturb the seiches that have nodes there, while it may affect very little those seiches which have ventral points there. This suggests a very natural explanation why, of the many theoretically possible seiches for any given lake, we may find traces of only a limited number on the limnogram. To this point I shall return when we come to consider special lakes in detail, and the causes which may excite the seiches observed in them.

A very important practical conclusion suggested by the acoustic analogy is that, while the boundary conditions at the end of the lake may seriously affect seiches of higher nodality, they have comparatively little influence on the seiches of lower nodality, in particular on the uninodal and binodal seiches.

§ 18. So far as I am aware, the only serious attempt hitherto made to submit seiche phenomena in detail to mathematical calculation is contained in an interesting paper

* With a Koenig's fork (256 U_{t_2}), 140 cm. of No. 30 thread, which weighs .00058 gm. per cm., and a tension of 45 gm., 2.5 mg. of wax produced a displacement of the ventral point exceeding 8 cm.

by DU BOYS, entitled "Essai Théorique sur les Seiches" (*Arch. d. Sc. Phys. et Nat. Genève*, t. xxv., June 15, 1891). In this paper he reproduces the well-known theory for a lake of uniform depth and section. Then, by a confessedly inexact application of the theory of progressive waves ("onde solitaire de translation"), he arrives at the formula $T_r = (2v) \int dl \sqrt{gh}$, to which we have already referred. DU BOYS' theory gives, for reasons explained elsewhere,* a good approximation in certain cases for T_1 , and also for the position of the uninode, but it gives no account of the incommensurability of the seiche periods (it would give $T_1 : T_2 : T_3 : \dots = 1 : 1/2 : 1/3 : \dots$ in all cases): nor does it lead to a correct determination of the binodes. Moreover, no good reason can be given for the fact that in order to get a good approximation for T_1 we must integrate along the line of greatest depth, instead of using KELLAND's formula † for the wave velocity, and putting $T_1 = 2 \int dl \sqrt{b/ag}$, where b is the surface breadth, and a the area of the cross section. Nevertheless DU BOYS' work was a notable initial step in a difficult investigation.

§ 19. In an address, an abstract of which was published in *Il Nuovo Cimento*, 4. vol. viii. p. 270, 1898, VOLTERRA, at the instigation of FOREL, pointed out the great interest of the phenomena of seiches from a physico-mathematical point of view; and promised to develop a mathematical theory in a separate paper. It is, however, impossible to gather the exact nature of his views from the brief abstract referred to; and his promised memoir has not, so far as I can learn, yet appeared.

At a meeting of the Royal Society of Edinburgh on 16th February 1903, I gave, at the request of Sir JOHN MURRAY, and chiefly for the sake of his Lake Surveyors, a similar address, in which I stated most of the general principles above enunciated on the strength of approximative calculations, which I have since replaced by the more rigorous and more effective methods of the present paper. During this address, Mr E. MACLAGAN-WEDDERBURN demonstrated the uninodal and binodal standing waves in a long rectangular trough by means of the following elegant apparatus devised by himself.

C P is a long rectangular trough, in which there is a fixed partition, D, pierced at the bottom, dividing C P into a small and a large compartment, with a communication at the bottom of moderate size. E is a movable partition, by means of which the longer part of the trough can be shortened at will. A strong spiral spring, S, suitably supported and weighted with a heavy ball, B, to give sufficient inertia, is used to make a plunger execute a vertical simple harmonic motion, of period T, in the compartment C D. So long as D E is not adjusted so that the period of the standing wave in that part of the trough is T or a multiple of T, the forced oscillations in D E attain no particular magnitude; but when E is so adjusted that the periods are very nearly equal, or the one double the other, a large standing wave is set up, which can be maintained for any length of time by giving properly timed impulses to B. In this

* See *Proc. R.S.E.*, vol. xxv. p. 646; also § 49 below.

† *Trans. Roy. Soc. Edin.*, xiv. p. 524, 1839.

way Mr WEDDERBURN was able to verify with very fair accuracy MERIAN'S law for the uninodal and binodal periods in a trough of uniform depth: to show that for a trough of the same length and same maximum depth, T_1 is greater when the bottom is caused to slope by inserting a wedge * inside it, or, what comes to the same thing, that when T_1 is unaltered, CE must be made shorter; and that then $T_2 > \frac{1}{2}T_1$. The displacement of the uninode towards the shallow end and the greater amplitude of the wave there were also readily demonstrated. The foregoing experiments were made with water 8.18 cm. deep, in a trough of length CE = 39.8 cm.; and the periods were $T_1 = .96''$,

FIG. 1.

$T_2 = .52''$. With a depth of 4.1 cm. and CE = 30.7 cm. it was still possible to maintain a uninodal seiche of period .97'', which agrees very well with the formula $T = 2l/\sqrt{gd}$ applicable to a long wave.

FOREL based his theory of seiches in part upon an interesting series of experiments of this kind made in 1870.†

* To incline the trough, keeping the volume of water the same, is not the same thing.

† ENDRÖS commenced his investigation of the complicated seiche-phenomena in the Chiemsee with a series of experiments on the oscillation of mercury in a vessel imitating the configuration of the lake, the results of which, on the whole, were in remarkable agreement with his subsequent observations, *loc. cit.*, p. 8.

PART II.

MATHEMATICAL THEORY.

GENERAL THEORY OF A SMALL LONGITUDINAL SEICHE IN A LAKE OF VARYING DEPTH AND CROSS SECTION.

§ 20. Let O X be a longitudinal axis in the undisturbed surface of the lake. Observation seems to show that this axis should be as nearly as possible in the average direction of the channel of greatest depth. Take O Z vertical, and O Y horizontal and perpendicular to O X.

Consider any cross section at a distance O P = x from the origin. Let the area of this section be $A(x)$, and its breadth at the surface $b(x)$. Take a section parallel to $A(x)$ at a distance dx from $A(x)$. The volume of this slice (S) will be, to the first order of small quantities, $A(x)dx$.

Suppose that, after a time, t , the slice, S, has moved into a new position, so that the distance of its posterior face from O is now $x + \xi$. Then the breadth of S in its new position will be $dx(1 + \partial\xi/\partial x)$; and the part of its volume below the normal level of the lake will be $A(x + \xi)dx(1 + \partial\xi/\partial x)$.

If we suppose the rise in level of the slice to be the same throughout, say ζ , which involves the assumption that there is no flow parallel to O Y, and that all the water particles in the same transverse vertical plane have the same velocity parallel to the plane Z O X, then we may take the increment of the slice owing to the rise of the water above the original level to be $b(x)\zeta dx(1 + \partial\xi/\partial x)$. In so doing we neglect the effect of the shelving of the shore; so that our calculation would certainly not apply in cases where the seiche causes a large horizontal displacement of the high-water mark.

With these assumptions the equation of continuity is

$$A(x)dx = \{A(x + \xi) + b(x)\zeta\}dx(1 + \partial\xi/\partial x);$$

that is,

$$\zeta b(x) = A(x)/(1 + \partial\xi/\partial x) - A(x + \xi) \quad . \quad . \quad . \quad . \quad (1).$$

Since the amplitude of the seiche is small, we neglect vertical and consider only horizontal acceleration. The difference between the pressures on the two sides of the slice in its disturbed position will therefore be simply that due to the difference of level at its two ends, viz., $g\rho d\zeta$ per unit of area, ρ being the density of the liquid.* The equation of motion for S, regarded as a whole, is therefore

$$dx(1 + \partial\xi/\partial x)\rho \frac{\partial^2 \xi}{\partial t^2} = -g\rho \frac{\partial \zeta}{\partial x} dx;$$

that is

$$\frac{\partial^2 \xi}{\partial t^2} = -g \frac{\partial \zeta}{\partial x} \left(1 + \frac{\partial \xi}{\partial x}\right) \quad . \quad . \quad . \quad . \quad (2).$$

* In order that these assumptions may be justified, the square of the ratio of the depth to the wave length must be negligible at every part of the lake. See Lamb's *Hydrodynamics* (1895), § 169.

The amplitudes being small, we shall neglect quantities of the order of $\xi^2, (\partial\xi/\partial x)^2$. The above equation then becomes

$$\begin{aligned} b(x)\zeta &= A(x)\left(1 - \frac{\partial\xi}{\partial x}\right) - A(x) - \xi \frac{\partial A(x)}{\partial x}, \\ &= -A(x) \frac{\partial\xi}{\partial x} - \xi \frac{\partial A(x)}{\partial x}; \end{aligned}$$

or

$$\zeta = -\frac{1}{b(x)} \frac{\partial}{\partial x} \{A(x)\xi\} \quad (3);$$

and

$$A(x) \frac{\partial^2 \xi}{\partial t^2} = -gA(x) \frac{\partial \zeta}{\partial x} \quad (4).$$

If we substitute the value of ζ in (4), we get

$$A(x) \frac{\partial^2 \xi}{\partial t^2} = gA(x) \frac{\partial}{\partial x} \left[\frac{1}{b(x)} \frac{\partial}{\partial x} \{A(x)\xi\} \right].$$

Now this last equation may be written

$$\frac{\partial^2 \{A(x)\xi\}}{\partial t^2} = gA(x)b(x) \cdot \frac{1}{b(x)} \frac{\partial}{\partial x} \left[\frac{1}{b(x)} \frac{\partial}{\partial x} \{A(x)\xi\} \right] \quad (5).$$

If we determine new variables u and v by the equations

$$u = A(x)\xi, \quad v = \int dx b(x) \quad (6),$$

then (5) may be written

$$\frac{\partial^2 u}{\partial t^2} = gA(x)b(x) \frac{\partial^2 u}{\partial v^2}, \quad = g\sigma(v) \frac{\partial^2 u}{\partial v^2} \quad (7),$$

where x is to be determined as a function of v by the second equation of (6); and $\sigma(v) = A(x)b(x)$. Also (3) becomes

$$\zeta = -\frac{\partial u}{\partial v} \quad (8).$$

The curve whose ordinate and abscissa are σ and v we shall call the *normal curve* of the lake.

Since a seiche is a standing oscillation, ξ , and therefore u , is a periodic function of the time. We may suppose this periodic function analysed into simple harmonic terms, and write

$$u = \Sigma P \sin n(t - \tau) \quad (9),$$

where P is a function of v alone and τ is constant. The values of n admissible depend on the circumstances of each case: but, in order that (9) may satisfy (7) we must have

$$-n^2 P = g\sigma(v) \frac{\partial^2 P}{\partial v^2}.$$

The mathematical theory of a seiche of small amplitude depends therefore essentially on the differential equation

$$\frac{d^2 P}{dv^2} + \frac{n^2}{g\sigma(v)} P = 0 \quad (10),$$

where x is determined in terms of v by the equation $v = \int dx b(x)$; and $\sigma(v) = A(x)b(x)$.

As (10) is simply the canonical form of the linear differential equation of the second order, a variety of cases can be devised in which the seiche problem can be solved in finite terms; and in any case where $A(x)$ and $b(x)$ are given slowly varying functions, approximate solutions can be found, with more or less labour.

§ 21. In all the seiche problems considered in this paper we have at the ends of the lake either $A(x) = 0$ or else $\xi = 0$; that is $u = 0$.

Moreover, the equation (7) may be regarded as the equation of motion of a vertical string vibrating in one plane, the ratio of whose tension to the longitudinal density is $g\sigma(v)$, v being the distance of any point P of the string from one end when the whole is at rest. The variable u denotes the lateral displacement of the point P at time t ; and, in view of the conditions $u = 0$ at both ends of the lake, we may suppose both ends of the string fixed. We can then deduce the seiche displacements from the motion of the string by the equations

$$\xi = u/A(x), \quad \zeta = -\partial u/\partial v.$$

It will be observed that the nodes of the string correspond to ventral points of the seiche, and *vice versa*; and it appears that we could, by experimenting with a string loaded so that its density is inversely proportional to the product of the area and surface breadth of the cross section of a lake, roughly determine in the laboratory the periods and nodes of the pure seiches that might occur in the lake.

It follows from STURM'S Oscillation theorem* that in any given lake seiches are possible which have

$$2, 3, 4, \dots, \nu+1, \dots$$

ventral points and

$$1, 2, 3, \dots, \nu, \dots$$

nodes respectively.

In other words, pure seiches of all degrees of nodality are possible; and the most general seiche disturbance is a sum of such pure seiches with arbitrary amplitudes and phases. We regard the ends of the lake as ventral points because u always vanishes there, although in most cases the horizontal displacement does not vanish, as it should do at a ventral point properly so called.

The identification of the seiche problem with the theory of a vibrating string is not only very instructive from the physical point of view, but is very helpful mathematically. For example, when we have worked out the periods and nodes of a seiche for any simple configuration approximately fitting a given lake, we can correct for the divergence of the actual lake from the assumed mathematical form by means of the beautiful method described by LORD RAYLEIGH in his *Theory of Sound*, vol. i. § 90.

* See RAYLEIGH'S *Sound* (1877), vol. i. § 142.

CASE OF A LAKE OF CONSTANT BREADTH, RECTANGULAR SECTION,
BUT VARYING DEPTH.

§ 22. Let $b(x) = b$, $A(x) = bh(x)$, where b is the constant breadth, and $h(x)$ the varying depth.

Then the equations (7) and (8) may be written

$$\frac{\partial^2 u}{\partial t^2} = gh(x) \frac{\partial^2 u}{\partial x^2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (7'),$$

$$\zeta = -\frac{\partial u}{\partial x} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (8');$$

where $u = \xi h(x)$.

In the case of a stationary oscillation, we shall have

$$\xi h(x) = u = P \sin n(t - \tau) \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9')$$

where

$$\frac{\partial^2 P}{\partial x^2} + \frac{n^2}{gh(x)} P = 0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (10').$$

It will be useful in this case to give the displacement ζ_1 for a point P in the water not lying on the surface. If the co-ordinates of P be (x, y, z) before and $(x + \xi, y, z + \zeta_1)$ after disturbance, ξ and ζ denoting as before the displacements of the point $(x, y, 0)$ vertically over (x, y, z) , then, applying the equation of continuity to a small parallelopiped reaching from (x, y, z) to the surface and having sides parallel to the axes, we get, supposing z measured downwards, ζ and ζ_1 upwards, $z dx dy = (z - \zeta_1 + \zeta) dx (+ \partial \xi / \partial x) dy$. Therefore, to the order of approximation contemplated, we have

$$-\zeta_1 = -\zeta - z \frac{\partial \xi}{\partial x}$$

that is

$$\begin{aligned} -\zeta_1 &= \frac{\partial}{\partial x} (h(x) \xi) - z \frac{\partial \xi}{\partial x} \\ &= h'(x) \xi + (h(x) - z) \frac{\partial \xi}{\partial x} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (11). \end{aligned}$$

It follows that

$$\begin{aligned} -\frac{\zeta_1}{\xi} &= h'(x) + \frac{h(x)(h(x) - z)}{P} \frac{d}{dx} \left(\frac{P}{h(x)} \right) \\ &= \frac{zh'(x)}{h(x)} + \frac{h(x) - z}{P} \frac{dP}{dx} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (12). \end{aligned}$$

Hence as we pass vertically upwards we find all the particles of water oscillating in rectilinear orbits the inclinations of which vary gradually from tangency with the bottom to tangency with the free surface of the lake.

§ 23. If we compare equations (7), (8), (9), (10) with (7'), (8'), (9'), (10'), it is at once obvious that mathematically there is no difference between the general case and the special one where the breadth of the lake is constant, its cross section rectangular, and the depth alone varies. We can pass from the one case to the other by obvious changes in the meanings of the variables and constants involved. We shall therefore in future confine ourselves to the more special case, of which it is easier to form a clear mental picture.

GENERAL SOLUTION FOR A PARABOLIC LONGITUDINAL SECTION.

INTRODUCTION OF THE SEICHE FUNCTIONS.

§ 24. Consider the equation

$$\frac{d^2 P}{dv^2} + \frac{c}{1 - \lambda v^2} P = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Assuming $y = a_0 + a_1 v + a_2 v^2 + \dots$, we find in the usual way

$$\begin{aligned} ca_0 + 1.2a_2 &= 0, & ca_1 + 2.3a_3 &= 0, \\ (c - 1.2\lambda)a_2 &= 3.4a_4 = 0, & (c - 2.3\lambda)a_3 &+ 4.5a_5 = 0, \\ & \cdot & \cdot & \cdot \\ \{c - (n-3)(n-2)\lambda\}a_{n-2} &+ (n-1)na_n = 0. \end{aligned}$$

Hence we have

$$\begin{aligned} P &= A \left\{ 1 - \frac{c}{1.2}v^2 + \frac{c(c-1.2\lambda)}{1.2 \times 3.4}v^4 + \frac{c(c-1.2\lambda)(c-3.4\lambda)}{1.2 \times 3.4 \times 5.6}v^6 + \dots \right\} \\ &+ B \sqrt{c} \left\{ v - \frac{c}{2.3}v^3 + \frac{c(c-2.3\lambda)}{2.3 \times 4.5}v^5 + \frac{c(c-2.3\lambda)(c-4.5\lambda)}{2.3 \times 4.5 \times 6.7}v^7 + \dots \right\} \quad (14); \end{aligned}$$

where A and B are arbitrary constants. The series in the brackets are obviously convergent if $|v| < 1/\sqrt{|\lambda|}$; and divergent if $|v| > 1/\sqrt{|\lambda|}$. They are also convergent if $v = \pm 1/\sqrt{\lambda}$; for the general term, of say the first series, may be written

$$\pm u_n = \left(1 - \frac{c/\lambda}{1.2}\right) \left(1 - \frac{c/\lambda}{3.4}\right) \dots \dots \left(1 - \frac{c/\lambda}{(2n-1)2n}\right) \frac{c}{(2n+1)(2n+2)};$$

and, since the infinite product

$$\left(1 - \frac{c/\lambda}{1.2}\right) \left(1 - \frac{c/\lambda}{3.4}\right) \dots \dots$$

is convergent, u_n is ultimately of the same order as $1/n^2$.

If we introduce the notation

$$\begin{aligned} C(c, \lambda, v) &= 1 - \frac{c}{1.2}v^2 + \frac{c(c-1.2\lambda)}{1.2 \times 3.4}v^4 - \dots \dots \dots \\ S(c, \lambda, v) &= \sqrt{cv} \left\{ 1 - \frac{c}{2.3}v^2 + \frac{c(c-2.3\lambda)}{2.3 \times 4.5}v^4 - \dots \dots \dots \right\}, \end{aligned}$$

we see that the functions C and S have certain properties in common with the circular functions. For example, we have

$$\begin{aligned} C(c, \lambda, -v) &= C(c, \lambda, v); \\ S(c, \lambda, -v) &= -S(c, \lambda, v); \\ C(c, \lambda, 0) &= 1, \quad S(c, \lambda, 0) = 0. \end{aligned}$$

Also

$$C(c, 0, v) = \cos(\sqrt{cv}), \quad S(c, 0, v) = \sin(\sqrt{cv});$$

so that the cosine and sine are particular cases of the functions just defined.

From EULER's identity, viz.

$$\begin{aligned} (1-u_1)(1-u_2) \dots (1-u_n) \\ = 1 - u_1 + u_2(u_1-1) - u_3(u_1-1)(u_2-1) + \dots \\ \dots (-)^n u_n(u_1-1)(u_2-1) \dots (u_{n-1}-1), \end{aligned}$$

we see that

$$\begin{aligned} C(c, \lambda, 1/\sqrt{\lambda}) &= \left(1 - \frac{c/\lambda}{1.2}\right) \left(1 - \frac{c/\lambda}{3.4}\right) \dots \text{ad } \infty; \\ S(c, \lambda, 1/\sqrt{\lambda}) &= \frac{\sqrt{c}}{\sqrt{\lambda}} \left(1 - \frac{c/\lambda}{2.3}\right) \left(1 - \frac{c/\lambda}{4.5}\right) \dots \text{ad } \infty. \end{aligned}$$

In our applications we shall for the most part put $\lambda = 1$, or $\lambda = -1$. Then we may omit the argument λ and write

$$C(c, v) = 1 - \frac{c}{1.2}v^2 + \frac{c(c-1.2)}{1.2 \times 3.4}v^4 - \dots \quad (15);$$

$$S(c, v) = v - \frac{cv^3}{2.3} + \frac{c(c-2.3)}{2.3 \times 4.5}v^5 - \dots \quad (16).$$

We shall call $C(c, v)$ and $S(c, v)$ the Seiche-cosine and the Seiche-sine respectively. Also

$$\mathfrak{C}(c, v) = 1 - \frac{cv^2}{1.2} + \frac{c(c+1.2)}{1.2 \times 3.4}v^4 - \dots \quad (17);$$

$$\mathfrak{S}(c, v) = v - \frac{cv^3}{2.3} + \frac{c(c+2.3)}{2.3 \times 4.5}v^5 - \dots \quad (18).$$

These may be called the hyperbolic seiche-cosine and the hyperbolic seiche-sine.

When c has one of the integral values $1.2, 3.4, \dots$, $C(c, v)$ reduces to a rational integral function of v ; and when c has one of the integral values $2.3, 4.5, \dots$, the like happens to $S(c, v)$.

The same holds for $\mathfrak{C}(c, v)$ and $\mathfrak{S}(c, v)$ with regard to the negative integral values $-1.2, -3.4, \dots$, and $-2.3, -4.5, \dots$; but this is of little interest in the seiche problem, for which the values of c must be positive.

By a well-known property of the solutions of a linear equation of the second order, we must have $C(c, v) S'(c, v) - C'(c, v) S(c, v) = \text{constant}$, where the dash denotes differentiation with respect to v . In the present case this constant is easily seen to be unity; and we have

$$C(c, v) S'(c, v) - C'(c, v) S(c, v) = 1 \quad (19);$$

$$\mathfrak{C}(c, v) \mathfrak{S}'(c, v) - \mathfrak{C}'(c, v) \mathfrak{S}(c, v) = 1 \quad (20).$$

These are the analogues of the relation $\cos^2 \theta + \sin^2 \theta = 1$ for the circular functions; and they are very useful in seiche calculations. We might also define a seiche-tangent, seiche-cotangent, seiche-secant, and seiche-cosecant. We shall only have occasion to use the seiche-cotangent, viz., $C(c, v)/S(c, v)$, which we shall denote by $K(c, v)$. These functions have many curious properties more or less analogous to those of the circular functions: e.g. $K'(c, v) = -1/S^2(c, v)$, but it is needless to encumber the present paper with details of this description.

§ 25. We have now, of course,

$$C(c, 1) = \left(1 - \frac{c}{1.2}\right) \left(1 - \frac{c}{3.4}\right) \dots \dots \dots ad \infty \quad . \quad . \quad . \quad (21);$$

$$S(c, 1) = \left(1 - \frac{c}{2.3}\right) \left(1 - \frac{c}{4.5}\right) \dots \dots \dots ad \infty \quad . \quad . \quad . \quad (22);$$

$$\mathfrak{S}(c, i) = \left(1 + \frac{c}{1.2}\right) \left(1 + \frac{c}{3.4}\right) \dots \dots \dots ad \infty \quad . \quad . \quad . \quad (23);$$

$$\mathfrak{S}(c, i) = i \left(1 + \frac{c}{2.3}\right) \left(1 + \frac{c}{4.5}\right) \dots \dots \dots ad \infty \quad . \quad . \quad . \quad (24).$$

We shall have frequent occasion to use $C(c, 1)$ and $S(c, 1)$; and it is convenient for purposes of calculation to express them in terms of the gamma function.

We have

$$\begin{aligned} C(c, 1) &= \left(1 - \frac{c}{2}\right) \prod_{n=1}^{\infty} \frac{4n^2 + 6n + 2 - c}{(2n+1)(2n+2)}, \\ &= \left(1 - \frac{c}{2}\right) \prod \frac{\{n + \frac{1}{4}(3+\alpha)\} \{n + \frac{1}{4}(3-\alpha)\}}{(n+\frac{1}{2})(n+1)}, \end{aligned}$$

where $\alpha = \sqrt{4c+1}$.

Since $\frac{1}{4}(3+\alpha) + \frac{1}{4}(3-\alpha) = \frac{1}{2} + 1$, it follows by a well-known theorem* that

$$C(c, 1) = \left(1 - \frac{c}{2}\right) \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) \Gamma(1) / \left(\frac{3+\alpha}{4}\right) \left(\frac{3-\alpha}{4}\right) \Gamma\left(\frac{3+\alpha}{4}\right) \Gamma\left(\frac{3-\alpha}{4}\right).$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(1) = 1$, and $\alpha^2 = 4c+1$, we get finally

$$C(c, 1) = \pi^{1/2} / \Gamma\left(\frac{3+\alpha}{4}\right) \Gamma\left(\frac{3-\alpha}{4}\right) \quad . \quad . \quad . \quad (25);$$

and in exactly the same way

$$S(c, 1) = \pi^{1/2} / 2 \Gamma\left(\frac{5+\alpha}{4}\right) \Gamma\left(\frac{5-\alpha}{4}\right) \quad . \quad . \quad . \quad (26).$$

It follows that

$$K(c, 1) = 2 \Gamma\left(\frac{5+\alpha}{4}\right) \Gamma\left(\frac{5-\alpha}{4}\right) / \Gamma\left(\frac{3+\alpha}{4}\right) \Gamma\left(\frac{3-\alpha}{4}\right) \quad . \quad . \quad . \quad (27),$$

a formula which has been much used by Mr WEDDERBURN and myself in the numerical calculation of seiche periods. If we recollect that

$$\begin{aligned} \Gamma\left(\frac{1+\alpha}{4}\right) \Gamma\left(\frac{3-\alpha}{4}\right) &= \pi / \sin\left(\frac{1+\alpha}{4} \pi\right), \\ \Gamma\left(\frac{1-\alpha}{4}\right) \Gamma\left(\frac{3+\alpha}{4}\right) &= \pi / \sin\left(\frac{1-\alpha}{4} \pi\right), \end{aligned}$$

we can put (27) into the form

$$K(c, 1) = -\frac{c}{4\pi^2} \cos\left(\frac{\alpha}{2} \pi\right) \Gamma^2\left(\frac{1+\alpha}{4}\right) \Gamma^2\left(\frac{1-\alpha}{4}\right) \quad . \quad . \quad . \quad (28)$$

which is useful for determining the sign of $K(c, 1)$.

* See *Whittaker's Modern Analysis* (1902), § 96.

§ 26. By STURM'S Oscillation theorem, applied to the solutions of the differential equation (13), we see that for any real value of v not exceeding unity the equations

$$C(c, v) = 0, \quad S(c, v) = 0, \quad . \quad . \quad . \quad . \quad (29).$$

$$\mathfrak{C}(c, v) = 0, \quad \mathfrak{S}(c, v) = 0, \quad . \quad . \quad . \quad . \quad (30),$$

have each an infinite number of real roots, and that the roots of each equation of either pair separate the roots of the other equation of the same pair.

In particular, the roots of

$$C(c, 1) = 0, \quad \text{are } c = 1.2, 3.4, \dots;$$

of

$$S(c, 1) = 0, \quad \text{are } c = 2.3, 4.5, \dots;$$

unfortunately the roots of

$$\mathfrak{C}(c, 1) = 0, \quad \mathfrak{S}(c, 1) = 0$$

are not commensurable; and, owing to the slow convergence of the series involved, they are very difficult to calculate directly. By a very laborious calculation, I find:— for the smallest root of $\mathfrak{C}(c, 1) = 0$, $c = 2.77 \dots$; and for the smallest root of $\mathfrak{S}(c, 1) = 0$, $c = 12.34 \dots$. As these figures agree with the approximations given by Dr HALM in the paper on the seiche functions above referred to, and with calculations which Dr BURGESS, Professor GIBSON, and Mr HORSBURGH have been kind enough to make for me, probably they are correct. It would, however, be hopeless to calculate the higher roots, or even these two to greater accuracy, by direct use of the series as it stands in (17) and (18).

SEICHES IN A CONCAVE SYMMETRIC COMPLETE, PARABOLIC LAKE.

$$h(x) = h \times (1 - x^2/a^2).$$

FIG. 2.

§ 27. The equation for determining P is, by § 22,

$$\frac{\partial^2 P}{\partial x^2} + \frac{n^2}{gh(1 - x^2/a^2)} P = 0;$$

or, if $w = x/a$,

$$(1 - w^2) \frac{d^2 P}{dw^2} + \frac{n^2 a^2}{gh} P = 0,$$

say,

$$(1 - w^2) \frac{d^2 P}{dw^2} + c P = 0, \quad . \quad . \quad . \quad . \quad (31),$$

where

$$c = n^2 a^2 / gh, \quad . \quad . \quad . \quad . \quad (32).$$

We have therefore

$$\xi h(1-w^2) = u = \{A C(c, w) + B S(c, w)\} \sin n(t - \tau),$$

where A and B are arbitrary constants.

Also

$$\begin{aligned} \zeta &= -\frac{\partial u}{\partial c} = -\frac{1}{a} \frac{du}{dw}, \\ &= -\frac{1}{a} \left\{ A C'(c, w) + B S'(c, w) \right\} \sin n(t - \tau); \end{aligned}$$

where the dashes denote differentiation with respect to w .

Since ξ is finite, we have $u = 0$, when $w = \pm 1$. Therefore, since $C(c, -1) = C(c, 1)$, and $S(c, -1) = -S(c, 1)$, the following boundary conditions must be satisfied:

$$\begin{aligned} A C(c, 1) + B S(c, 1) &= 0, \\ A C'(c, 1) - B S'(c, 1) &= 0. \end{aligned}$$

These are equivalent to

$$A C(c, 1) = 0, \quad B S(c, 1) = 0.$$

Now we see from the relation

$$C(c, w)S'(c, w) - C'(c, w)S(c, w) = 1,$$

that $C(c, 1)$ and $S(c, 1)$ cannot vanish simultaneously. Therefore either

$$B = 0, \quad C(c, 1) = 0;$$

or

$$A = 0, \quad S(c, 1) = 0.$$

As we have seen, the roots of $C(c, 1) = 0$ are $c_1 = 1.2, c_3 = 3.4, \dots$. $c_{2s-1} = (2s-1)2s, \dots$; and the roots of $S(c, 1) = 0$, $c_2 = 2.3, c_4 = 4.5, \dots$, $c_{2s} = 2s(2s+1), \dots$. Hence we have the two sets of solutions

$$\left. \begin{aligned} \xi &= \frac{A}{h} \frac{C(c_{2s-1}, w) \sin n_{2s-1}(t - \tau)}{1 - w^2}; \\ \zeta &= -\frac{A}{a} \frac{C'(c_{2s-1}, w) \sin n_{2s-1}(t - \tau)}{1 - w^2} \end{aligned} \right\} \quad (33).$$

Here $C(c_{2s-1}, w)$ is a polynomial of degree $2s$; and $C'(c_{2s-1}, w)$ a polynomial of degree $2s-1$.

Also

$$\left. \begin{aligned} \xi &= \frac{B}{h} \frac{S(c_{2s}, w) \sin n_{2s}(t - \tau)}{1 - w^2}; \\ \zeta &= -\frac{B}{a} \frac{S'(c_{2s}, w) \sin n_{2s}(t - \tau)}{1 - w^2} \end{aligned} \right\} \quad (34),*$$

where $S(c_{2s}, w)$ and $S'(c_{2s}, w)$ are polynomials of degrees $2s+1$ and $2s$ respectively.

* Since the abstract of this paper was published, I have discovered that the solution for the particular case of a symmetric parabolic basin was given by LAMB in the new edition of his *Hydrodynamics* (1895), § 182. He arrives at his result by means of LEGENDRE'S function, which is closely allied to the seiche functions.

In either case, if T_ν be the period of the ν -nodal seiche, we have

$$\begin{aligned} T_\nu &= 2\pi/n_\nu = 2\pi a/\sqrt{(c_\nu gh)}, \\ &= \pi l/\sqrt{\{\nu(\nu+1)gh\}}, \end{aligned} \quad (35),$$

if l denote the whole length of the lake.

UNINODAL SEICHE.

§ 28.

$$\begin{aligned} c_1 &= 1.2, & T_1 &= \pi l/\sqrt{(2gh)} \\ C(c_1, w) &\equiv 1 - w^2, & C'(c_1, w) &= -2w. \\ \xi &= \frac{A}{h} \sin n_2(t - \tau), & \zeta &= \frac{2Ax}{a^2} \sin n_1(t - \tau). \end{aligned}$$

One node

$$x/a = 0.$$

In this case the amplitude of the horizontal displacement is constant; and the free surface is a plane which oscillates about the line of the uninode.

If $\bar{\zeta}$ be the maximum rise of the water at the end of the lake above the undisturbed level, then $\bar{\zeta} = 2A/a = 4A/l$. Hence $A = l\bar{\zeta}/4$. Hence the maximum horizontal displacement of a water particle from its mean position is $\bar{\xi} = l\bar{\zeta}/4h$; and the maximum velocity of the horizontal stream is $nl\bar{\zeta}/4h = \pi l\bar{\zeta}/2hT_1$. For example, if Loch Ness were a symmetric parabolic lake, every inch of maximum vertical seiche at one end would give over 40 inches of maximum horizontal displacement; and a maximum horizontal stream velocity of over 8 inches per minute.

BINODAL SEICHE.

§ 29.

$$\begin{aligned} c_2 &= 2.3, & T_2 &= \pi l/\sqrt{(6gh)}, \\ S(c_2, w) &= w - w^3, & S'(c_2, w) &= 1 - 3w^2, \\ \xi &= \frac{Bx}{ah} \sin n_2(t - \tau), & \zeta &= \frac{B(3x^2 - a^2)}{a^3} \sin n_2(t - \tau). \end{aligned}$$

Two nodes at

$$x/a = \pm \frac{1}{3} \sqrt{3} = \pm .5774 \dots$$

The amplitude of ξ increases uniformly from the centre to the ends of the lake; and the free surface is parabolic,

$$T_2/T_1 = \sqrt{2}/\sqrt{6} = \frac{1}{3} \sqrt{3} = .5774 \dots$$

Hence the period of the binodal seiche is greater than half the period of the uninodal seiche.

Also the nodes are more than half way from the middle of the lake towards the ends; i.e. they are displaced towards the shallows.

If $\bar{\xi}$ and $\bar{\zeta}$ be the maximum horizontal and vertical displacements at the end of the lake, we find $\bar{\xi}/\bar{\zeta} = l/4h$, as before, also $\bar{\xi}/\bar{\zeta} = 2\pi\bar{\xi}/T_2\bar{\zeta}$. For Loch Ness we get

$\bar{\xi}/\bar{\zeta} = 40 \dots$, $\bar{\xi}/\bar{\zeta} = 16.4 \dots$. As a binodal seiche of $3\frac{1}{2}$ in amplitude has been observed, maximum horizontal displacements of about 12 feet and stream velocities of about 5 feet per minute may occur near the end of the loch; these would be reduced in the ratio of 57 : 1 at the two nodes of the seiche. As the centre of the loch is a ventral segment, the horizontal displacement vanishes there at all times, and the value of $\bar{\zeta}$ is half its value at the ends of the loch.

TRINODAL SEICHE.

§ 30.

$$c_3 = 3.4 ; \quad T_3 = \pi l / \sqrt{(12gh)}.$$

$$\begin{aligned} C(c_3, w) &= 1 - 6w^2 + 5w^4, \\ &= (1 - w^2)(1 - 5w^2). \end{aligned}$$

$$C'(c_3, w) = -12w + 20w^3.$$

$$\xi = \frac{A}{h\alpha^2}(\alpha^2 - 5x^2) \sin n_3(t - \tau),$$

$$\zeta = \frac{A}{\alpha^4}(12\alpha^2x - 20x^3) \sin n_3(t - \tau).$$

Three nodes :— $x/\alpha = 0$, and $x/\alpha = \pm\sqrt{3}/\sqrt{5} = \pm.7746 \dots$

Four ventral points :— $x/\alpha = \pm.4472$, ± 1 .

$$T_3/T_1 = \sqrt{2}/\sqrt{12} = \frac{1}{6}\sqrt{6} = .4082 \dots$$

QUADRINODAL SEICHE.

§ 31.

$$c_4 = 4.5 ; \quad T_5 = \pi l / \sqrt{(20gh)}.$$

$$\begin{aligned} S(c_4, w) &= w - \frac{10}{3}w^3 + \frac{7}{3}w^5, \\ &= \frac{1}{3}(3w - 10w^3 + 7w^5), \end{aligned}$$

$$= \frac{1}{3}w(1 - w^2)(3 - 7w^2);$$

$$S'(c_4, w) = \frac{1}{3}(3 - 30w^2 + 35w^4).$$

$$\xi = \frac{B}{h\alpha^3}x(3\alpha^2 - 7x^2) \sin n_4(t - \tau);$$

$$\zeta = \frac{B}{\alpha^5}(-3\alpha^4 + 30\alpha^2x^2 - 35x^4) \sin n_4(t - \tau).$$

Four nodes :— $x/\alpha = \pm.3400$, $\pm.8621$.

Five ventral points :— $x/\alpha = 0$, $\pm.6546$, ± 1 .

$$T_4/T_1 = \frac{1}{10}\sqrt{10} = .3162.$$

QUINQUINODAL SEICHE.

§ 32.

$$\begin{aligned}
c_5 &= 5.6, & T_5 &= \pi l / \sqrt{(30gh)}. \\
C(c_5, w) &= 1 - 15w^2 + 35w^4 - 21w^6 = (1 - w^2)(1 - 14w^2 + 21w^4); \\
C'(c_5, w) &= -30w + 140w^3 - 126w^5. \\
\xi &= \frac{A}{ha^4}(a^4 - 14a^2x^2 + 21x^4) \sin n_5(t - \tau); \\
\zeta &= \frac{A}{a^6}(30a^4x - 140a^2x^3 + 126x^5) \sin n_5(t - \tau).
\end{aligned}$$

Five nodes :— $x = 0$, $x/a = \pm .5384$, $\pm .9062$.

Six ventral points :— $x/a = \pm .2853$, $\pm .7650$, ± 1 .

$$T_5/T_1 = \frac{1}{15} \sqrt{15} = .2582.$$

§ 33. The following conspectus of the numerical distances from the centre of the nodes and ventral points will give a clear idea of the shortening of the wave length towards the ends of the lake :—

I. BINODAL.

	x/a	$\frac{1}{4} \lambda$
V	.0000	.5774
N	.5774	.4226
V	1.0000	

II. TRINODAL.

	x/a	$\frac{1}{4} \lambda$
N	.0000	.4472
V	.4472	.3274
N	.7746	.2254
V	1.0000	

III. QUADRINODAL.

	x/a	$\frac{1}{4} \lambda$
V	.0000	.3400
N	.3400	.3146
V	.6546	.2075
N	.8621	.1379
V	1.0000	

IV. QUINQUINODAL.

	x/a	$\frac{1}{4} \lambda$
N	.0000	.2853
V	.2853	.2531
N	.5384	.2266
V	.7650	.1412
N	.9062	.0938
V	1.0000	

It will thus be seen that the wave length near the centre is greater, and near the end is less than it would be in a lake of the same length but of uniform depth. All the nodes and ventral points which are not central are displaced towards the shallows.

As we see from § 8 that the amplitudes of the various pure seiches at the end of the lake is of special interest, the following table may be given, in which R denotes the

ratio for any seiche of the amplitude at the end of the lake to the amplitude at the ventral point at or nearest to the centre :—

ν	2	3	4	5
R	2.00	2.24	2.66	2.88

SEICHES IN A CONCAVE SEMIPARABOLIC LAKE.

§ 34. Since all the pure seiches in a symmetric parabolic lake have ventral points at the ends, and the seiches of even nodality have also a ventral point at the centre, where there is no horizontal displacement, we could build a wall across the middle of the lake without disturbing these seiches. It follows that the pure seiches of a semi-

FIG. 3.

parabolic lake have the same periods as the seiches of even nodality in a complete parabolic lake of double the length. The nodes and ventral segments will also be the same as in one of the halves of the complete parabolic lake.

If, therefore, T'_ν be the period of the ν -nodal seiche in a semiparabolic lake of length l and maximum depth h , we shall have

$$T'_\nu = 2\pi l / \sqrt{\{2\nu(2\nu + 1)gh\}} \quad . \quad . \quad . \quad . \quad . \quad (36).$$

If T_ν have the meaning of § 27, we find

$$T'_\nu / T_\nu = \sqrt{\{(\nu + 1)/(\nu + \frac{1}{2})\}}.$$

Hence every period of a semiparabolic lake is longer than the corresponding period of a complete parabolic lake of the same length and the same maximum depth; but the ratio of the periods comes nearer unity the higher the nodality.

The nodes and ventral points for the uninodal and binodal, etc. seiches will be given by Tables I. and III. of § 33, provided we remember that x is now measured from the deeper end of the lake, and no longer from the middle, and that a is now the whole length of the lake, and not half the length as before.

§ 35. The results for parabolic and semiparabolic lakes are of great use in forming rough estimates of the constants and periods, either for experimental purposes, or in order to get first approximations to the roots of the transcendental equations which in

general determine these constants. We assume, as in general probable, that any concave lake whose form is not unusual will be intermediate in character between a complete parabolic and a semiparabolic lake. It follows that the periods, nodes, and ventral points will be intermediate; and it is found in practice that in many cases a good first approximation is obtained by taking the arithmetic mean between the two extreme cases. As an example, we should expect that the distances of the uninode and deep binode from the deeper end would lie between $\cdot 5$ and $\cdot 58$, and between $\cdot 78$ and $\cdot 87$ respectively.

In this connection we have found the following table of the ratios of the periods useful:—

	T_2/T_1	T_3/T_1	T_4/T_1	T_5/T_1	T_6/T_1	T_7/T_1	T_8/T_1	T_9/T_1
Parabolic Lake	$\cdot 577$	$\cdot 408$	$\cdot 316$	$\cdot 258$	$\cdot 218$	$\cdot 189$	$\cdot 167$	$\cdot 149$
Semiparabolic Lake	$\cdot 548$	$\cdot 378$	$\cdot 289$	$\cdot 234$	$\cdot 196$	$\cdot 169$	$\cdot 1485$	$\cdot 134$

SEICHES IN A TRUNCATED PARABOLIC LAKE.

§ 36. By means of the seiche functions we can readily find the solution for a parabolic lake which is bounded by vertical cross walls at distances $x = p$, $x = q$ from

FIG. 4.

its deepest point. The formulæ are, in our previous notation,

$$\xi h(1 - w^2) = \frac{A}{S(c, p/a)} \{S(c, p/a)C(c, w) - C(c, p/a)S(c, w)\} \sin n(t - \tau);$$

$$\zeta = -\frac{A}{aS(c, p/a)} \{S(c, p/a)C'(c, w) - C(c, p/a)S'(c, w)\} \sin n(t - \tau).$$

And the values of c are given by the period equation

$$C(c, p/a)S(c, q/a) - S(c, p/a)C(c, q/a) = 0.$$

In the case of a symmetric lake $q = -p$; and the period equation reduces to

$$C(c, p/a)S(c, p/a) = 0.$$

If, further, $p = a$, we get

$$C(c, 1)S(c, 1) = 0;$$

and return to the case of a complete parabolic lake already discussed.

SEICHES IN A CONVEX SYMMETRIC PARABOLIC LAKE.

$$h(x) = h \times (1 + x^2/a^2).$$

FIG. 5.

§ 37. The equation for P is

$$\frac{d^2 P}{dx^2} + \frac{n^2}{gh(1+x^2/a^2)} P = 0;$$

that is, if $w = x/a$,

$$(1+w^2) \frac{d^2 P}{dw^2} + cP = 0, \quad (37),$$

where

$$c = n^2 a^2 / gh, \quad (38).$$

We have therefore

$$\begin{aligned} \xi h(1+w^2) &= u = \{A \mathfrak{G}(c, w) + B \mathfrak{Z}(c, w)\} \sin nt; \\ \zeta &= -\frac{1}{a} \{A \mathfrak{G}'(c, w) + B \mathfrak{Z}'(c, w)\} \sin nt. \end{aligned}$$

Since ξ must vanish at the vertical ends corresponding to $w = \pm 1$, we have, although for a different reason, the same boundary conditions as before, viz. —

$$\begin{aligned} A \mathfrak{G}(c, w) + B \mathfrak{Z}(c, w) &= 0, \\ A \mathfrak{G}(c, w) - B \mathfrak{Z}(c, w) &= 0. \end{aligned}$$

Hence, as before, we arrive at the two sets of solutions

$$\left. \begin{aligned} \xi &= \frac{A}{h} \frac{\mathfrak{G}(\epsilon_{2s-1}, w) \sin n_{2s-1} t}{1+w^2}, \\ \zeta &= -\frac{A}{a} \frac{\mathfrak{G}'(\epsilon_{2s-1}, w) \sin n_{2s-1} t}{1+w^2}; \end{aligned} \right\} \quad (39);$$

and

$$\left. \begin{aligned} \xi &= \frac{B}{h} \frac{\mathfrak{Z}(\epsilon_{2s}, w) \sin n_{2s} t}{1+w^2}, \\ \zeta &= -\frac{B}{a} \frac{\mathfrak{Z}'(\epsilon_{2s}, w) \sin n_{2s} t}{1+w^2}; \end{aligned} \right\} \quad (40);$$

where $\epsilon_1, \epsilon_3, \dots, \epsilon_{2s-1}, \dots$ are the roots of the equation

$$\mathfrak{G}(c, 1) = 0;$$

and $\epsilon_2, \epsilon_4, \dots, \epsilon_{2s}, \dots$ are the roots of

$$\mathfrak{Z}(c, 1) = 0.$$

$\mathfrak{G}(\epsilon_{2s-1}, w)$ and $\mathfrak{Z}(\epsilon_{2s}, w)$ are, of course, no longer polynomials, but transcendental functions of w .

UNINODAL AND BINODAL SEICHES.

§ 38. Since, as has already been mentioned, $c_1 = 2.77 \dots$ and $c_2 = 12.34 \dots$, if $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the periods of the uninodal and binodal seiches respectively, we have

$$\mathfrak{T}_2/\mathfrak{T}_1 = \sqrt{(2.77/12.34)} = .474.$$

The distances of the binodes from the centre of the lake are given approximately by

$$x/a = .472.$$

SEICHES IN A CONCAVE ASYMMETRIC BIPARABOLIC LAKE.

FIG. 6.

§ 39. Let the equations for the portions O A and A'O be $h(x) = h \times (1 - x^2/\alpha^2)$; and $h(x) = h \times (1 - x^2/\alpha'^2)$. Then, if $w = x/\alpha$, $w' = x/\alpha'$; $c = n^2\alpha^2/gh$, $c' = n^2\alpha'^2/gh$, we have for the two portions

$$\begin{aligned} \xi h(1 - w^2) &= \{A C(c, w) + B S(c, w)\} \sin nt, \\ \zeta &= -\frac{1}{\alpha} \{A C'(c, w) + B S'(c, w)\} \sin nt; \end{aligned}$$

and

$$\begin{aligned} \xi' h(1 - w'^2) &= \{A' C(c', w') + B' S(c', w')\} \sin nt, \\ \zeta' &= -\frac{1}{\alpha'} \{A' C'(c', w') + B' S'(c', w')\} \sin nt. \end{aligned}$$

The boundary conditions at A and A' give

$$\begin{aligned} A C(c, 1) + B S(c, 1) &= 0; \\ A' C(c', 1) - B' S(c', 1) &= 0. \end{aligned}$$

The conditions $\xi = \xi'$, $\zeta = \zeta'$ at O give obviously

$$\begin{aligned} A &= A'; \\ B/\alpha &= B'/\alpha'. \end{aligned}$$

From these we deduce

$$\left. \begin{aligned} \alpha' C(c, 1) S(c', 1) + \alpha C(c', 1) S(c, 1) &= 0, \\ \alpha' K(c, 1) + \alpha K(c', 1) &= 0, \end{aligned} \right\} \quad (41),$$

which is the equation that enables us to calculate c or c' , if we remember that $c/c' = \alpha^2/\alpha'^2$.

If we put $\alpha'^2 c = \alpha^2 c' = n^2 c^2 \alpha'^2 / gh = z$, the equation (41) may be written

$$\alpha \left(1 - \frac{z}{1.2\alpha^2}\right) \left(1 - \frac{z}{3.4\alpha^2}\right) \cdots \left(1 - \frac{z}{2.3\alpha'^2}\right) \left(1 - \frac{z}{4.5\alpha'^2}\right) \cdots \\ + \alpha' \left(1 - \frac{z}{1.2\alpha'^2}\right) \left(1 - \frac{z}{3.4\alpha'^2}\right) \cdots \left(1 - \frac{z}{2.3\alpha^2}\right) \left(1 - \frac{z}{4.5\alpha^2}\right) \cdots = 0. \quad (42).$$

Then the period of the ν -nodal seiche is given by

$$T_\nu = 2\pi/n_\nu = 2\pi\alpha'/\sqrt{(z_\nu gh)}; \quad (43);$$

where z_ν is the corresponding root of (42).

For some purposes it is convenient to put $\alpha, \alpha' = \rho$. The equation (41) then becomes

$$(1/\rho)K(\rho^2 c', 1) + K(c', 1) = 0; \quad (44);$$

and we have

$$\left. \begin{aligned} T_\nu &= 2\pi\alpha'/\sqrt{(c'_\nu gh)} \\ &= 2\pi l/(1+\rho)\sqrt{(c'_\nu gh)}, \\ &= 2\pi l/(\sqrt{c_\nu} + \sqrt{c'_\nu})\sqrt{(gh)}. \end{aligned} \right\} \quad (45).$$

The equations for the seiche displacement in the two portions of the lake may now be written

$$\left. \begin{aligned} \xi h(1-w^2) &= \frac{A}{S(c, 1)} \{S(c, 1)C(c, w) - C(c, 1)S(c, w)\} \sin nt; \\ \zeta &= -\frac{A}{aS(c, 1)} \{S(c, 1)C'(c, w) - C(c, 1)S'(c, w)\} \sin nt; \end{aligned} \right\} \quad (46);$$

$$\left. \begin{aligned} \xi' h(1-w'^2) &= \frac{A}{S(c', 1)} \{S(c', 1)C(c', w') + C(c', 1)S(c', w')\} \sin nt; \\ \zeta' &= -\frac{A}{aS(c', 1)} \{S(c', 1)C'(c', w') + C(c', 1)S'(c', w')\} \sin nt. \end{aligned} \right\} \quad (47).$$

It is obvious from (44) that when ρ is given the value of c' is determined. Now ρ is the ratio of the distances of the deepest section of the lake from the ends. Hence, if this ratio remain unaltered, we see that T_ν is proportional directly to the length of the lake, and inversely to the square root of its maximum depth.*

In particular, it follows that, if the basins of two lakes be geometrically similar, the seiche periods are directly proportional to the square roots of the linear dimensions; a result obvious by NEWTON's principle of dynamic similarity.

A graphic picture of the solution of the equation (44) may be obtained as follows:—

If we trace the curves whose equations are $y = K(c', 1)$, $\rho y = -K(\rho^2 c', 1)$, c' being the common abscissa, then the values of c' corresponding to the intersection of these curves are the roots of (44). The latter of these curves is deducible from the former by diminishing all the abscissæ in the ratio $1 : \rho^2$ and all the ordinates in the ratio $1 : \rho$, and then taking the image of this deformed curve in the axis of c' . It is thus easy to see by drawing a schematic diagram that the effect of increasing ρ is to diminish c' .

The period depends on the value of $(1+\rho)\sqrt{c'}$ or $\sqrt{c} + \sqrt{c'}$; and the effect of the increase of ρ upon this is not so easy to trace by direct analysis. Since, however, the

* In the general case h is the maximum value of the product of the area of a cross section by its surface breadth; and a and a' the areas of the lake surface between the corresponding section and the ends.

shifting of the deepest point of the biparabolic lake without alteration of the length or maximum depth does not alter the whole volume of water, general dynamical considerations regarding energy would lead us to expect that increase of ρ would lengthen all the seiche periods; and, in point of fact, in the semiparabolic lake, which may be regarded as the limiting case of a biparabolic lake when $\rho = \infty$, all the periods are greater than in the complete parabolic lake, which corresponds to $\rho = 1$.

SEICHES IN AN UNSYMMETRICAL LAKE WITH ONE SHALLOW AND TWO MAXIMUM DEPTHS.

FIG. 7.

§ 40. A good approximation to the form in many cases that occur in nature can be obtained by piecing together six parabolæ, so as to form a continuous curve.

Let s be the minimum; and h, h' the two maximum depths. D and D' the points of inflexion (the depths at which cannot be arbitrarily assigned). Let $AB = a_1, BD = b, DO = d, D'O' = d', B'D' = b', A'B' = a_1'$; then, for the continuity of the curve of longitudinal section at D and D' we have the following conditions, the laws of depth being $h(x) = h \times (1 - x^2/a_1^2)$ for AB , $h(x) = h \times (1 - x^2/a_2^2)$ for BD , $h(x) = s \times (1 + x^2/a_3^2)$ for OD , $h(x) = s \times (1 + x^2/a_3'^2)$ for OD' , etc. :—

$$\begin{aligned} hb/a_2^2 - sd/a_3^2 &= 0, & hb^2/a_2^2 + sd^2/a_3^2 &= h - s; \\ h'b'/a_2'^2 - sd'/a_3'^2 &= 0, & h'b'^2/a_2'^2 + sd'^2/a_3'^2 &= h' - s. \end{aligned}$$

These lead to

$$\left. \begin{aligned} a_2^2 &= hb(d+b)/(h-s), & a_3^2 &= sd(d+b)/(h-s); \\ a_2'^2 &= h'b'(d'+b')/(h'-s), & a_3'^2 &= sd'(d'+b')/(h'-s). \end{aligned} \right\} \quad (48).$$

With the exception of a_2, a_3, a_2', a_3' , and the depths at D and D' , the other quantities may be arbitrarily chosen.

If now

$$\begin{aligned} v_1 &= x/a_1, & v_2 &= x/a_2, & v_3 &= x/a_3; \\ v_1' &= x/a_1', & v_2' &= x/a_2', & v_3' &= x/a_3'; \\ w_2 &= b/a_2, & w_3 &= d/a_3; \\ w_2' &= b'/a_2', & w_3' &= d'/a_3'; \\ c_1 &= n^2 a_1^2 / gh, & c_2 &= n^2 a_2^2 / gh, & c_3 &= n^2 a_3^2 / gh; \end{aligned}$$

then we have for the various sections A, B , etc.

$$\begin{aligned} \xi h(1 - v_1^2) &= \{A_1 C(c_1, v_1) + B_1 S(c_1, v_1)\} \sin n(t - \tau), \\ \zeta &= -\frac{1}{a_1} \{A_1 C'(c_1, v_1) + B_1 S'(c_1, v_1)\} \sin n(t - \tau), \end{aligned}$$

etc.

The origin for x being in each case the vertex of the corresponding parabola.

The boundary conditions are then as follows:—

$$\begin{aligned} A_1 C(c_1, 1) + B_1 S(c_1, 1) &= 0; \\ A_1 &= A_2, B_1/\alpha_1 = B_2/\alpha_2; \\ A_2 C(c_2, w_2) + B_2 S(c_2, w_2) &= A_3 \mathfrak{G}(c_3, w_3) + B_3 \mathfrak{Z}(c_3, w_3), \\ -A_2 C'(c_2, w_2) + B_2 S'(c_2, w_2) &= \frac{\alpha_2}{\alpha_3} \left\{ A_3 \mathfrak{G}'(c_3, w_3) + B_3 \mathfrak{Z}'(c_3, w_3) \right\}; \\ A_2 &= A_3', B_2/\alpha_2 = B_3'/\alpha_3'; \\ A_2' C(c_2', w_2') + B_2' S(c_2', w_2') &= A_3' \mathfrak{G}(c_3', w_3') + B_3' \mathfrak{Z}(c_3', w_3'), \\ A_2' C'(c_2', w_2') + B_2' S'(c_2', w_2') &= \frac{\alpha_2'}{\alpha_3'} \left\{ -A_3' \mathfrak{G}'(c_3', w_3') + B_3' \mathfrak{Z}'(c_3', w_3') \right\}; \\ A_1' &= A_2', B_1'/\alpha_1' = B_2'/\alpha_2'; \\ A_1' C(c_1', 1) + B_1' S(c_1', 1) &= 0. \end{aligned}$$

From these, since $C(c_2, w_2)S'(c_2, w_2) - C'(c_2, w_2)S(c_2, w_2) = 1$, we derive

$$\begin{aligned} \alpha_2 A_2 C(c_1, 1) + \alpha_1 B_2 S(c_1, 1) &= 0, \\ A_2 &= \left\{ S'(c_2, w_2) \mathfrak{G}(c_3, w_3) + \frac{\alpha_2}{\alpha_3} S(c_2, w_2) \mathfrak{G}'(c_3, w_3) \right\} A_3 \\ &+ \left\{ S'(c_2, w_2) \mathfrak{Z}(c_3, w_3) + \frac{\alpha_2}{\alpha_3} S(c_2, w_2) \mathfrak{Z}'(c_3, w_3) \right\} B_3, \\ &= \lambda A_3 + \mu B_3 \text{ (say).} \\ B_2 &= \left\{ C'(c_2, w_2) \mathfrak{G}(c_3, w_3) + \frac{\alpha_2}{\alpha_3} C(c_2, w_2) \mathfrak{G}'(c_3, w_3) \right\} A_3 \\ &+ \left\{ C'(c_2, w_2) \mathfrak{Z}(c_3, w_3) + \frac{\alpha_2}{\alpha_3} C(c_2, w_2) \mathfrak{Z}'(c_3, w_3) \right\} B_3 \\ &= \nu A_3 + \rho B_3 \text{ (say).} \end{aligned}$$

Hence, using the first of these last equations, we have

$$\alpha_2 C(c_1, 1) \{\lambda A_3 + \mu B_3\} + \alpha_1 S(c_1, 1) \{\nu A_3 + \rho B_3\} = 0,$$

that is,

$$\begin{aligned} \{ \alpha_2 \lambda C(c_1, 1) + \alpha_1 \nu S(c_1, 1) \} A_3 \\ + \{ \alpha_2 \mu C(c_1, 1) + \alpha_1 \rho S(c_1, 1) \} B_3 = 0. \end{aligned}$$

Again,

$$\begin{aligned} \alpha_2' A_2' C(c_1', 1) + \alpha_1' B_2' S(c_1', 1) &= 0 \\ A_2' &= \left\{ S'(c_2', w_2') \mathfrak{G}(c_3', w_3') + \frac{\alpha_2'}{\alpha_3'} S(c_2', w_2') \mathfrak{G}'(c_3', w_3') \right\} A_3' \\ &- \left\{ S'(c_2', w_2') \mathfrak{Z}(c_3', w_3') + \frac{\alpha_2'}{\alpha_3'} S(c_2', w_2') \mathfrak{Z}'(c_3', w_3') \right\} B_3', \\ &= \lambda' A_3' - \mu' B_3' \text{ (say).} \\ B_2' &= - \left\{ C'(c_2', w_2') \mathfrak{G}(c_3', w_3') + \frac{\alpha_2'}{\alpha_3'} C(c_2', w_2') \mathfrak{G}'(c_3', w_3') \right\} A_3' \\ &+ \left\{ C'(c_2', w_2') \mathfrak{Z}(c_3', w_3') + \frac{\alpha_2'}{\alpha_3'} C(c_2', w_2') \mathfrak{Z}'(c_3', w_3') \right\} B_3' \\ &= -\nu' A_3' + \rho' B_3' \text{ (say).} \end{aligned}$$

Hence

$$\alpha_2' C(c_1', 1) \{\lambda' A_3' - \mu' B_3'\} - \alpha_1' S(c_1', 1) \{-\nu' A_3' + \rho' B_3'\} = 0$$

that is,

$$\{\alpha_2' \lambda' C(c_1', 1) + \alpha_1' \nu' S(c_1', 1)\} A_3' - \{\alpha_2' \mu' C(c_1', 1) + \alpha_1' \rho' S(c_1', 1)\} B_3' = 0.$$

Since $A_3' = A_3$, $B_3' = \alpha_3' B_3 / \alpha_3$, the last equation may be written

$$\alpha_3 \{\alpha_2' \lambda' C(c_1', 1) + \alpha_1' \nu' S(c_1', 1)\} A_3 - \alpha_3' \{\alpha_2' \mu' C(c_1', 1) + \alpha_1' \rho' S(c_1', 1)\} B_3 = 0.$$

Eliminating A_3 and B_3 , we get finally

$$\begin{aligned} & \alpha_3' \{ \alpha_2 \lambda C(c_1, 1) + \alpha_1 \nu S(c_1, 1) \} \{ \alpha_2' \mu' C(c_1', 1) + \alpha_1' \rho' S(c_1', 1) \} \\ & + \alpha_3 \{ \alpha_2' \lambda' C(c_1', 1) + \alpha_1' \nu' S(c_1', 1) \} \{ \alpha_2 \mu C(c_1, 1) + \alpha_1 \rho S(c_1, 1) \} = 0; \end{aligned} \quad (49);$$

which is the period equation for the lake. When the lake is symmetrical, that is, when $\alpha_1 = \alpha_1'$, $\alpha_2 = \alpha_2'$, etc., this equation simplifies, and breaks up into the two following:—

$$\begin{aligned} & \alpha_2 C(c_1, 1) \{ \alpha_3 S'(c_2, w_2) \xi(c_3, w_3) + \alpha_2 S(c_2, w_2) \xi'(c_3, w_3) \} \\ & + \alpha_1 S(c_1, 1) \{ \alpha_3 C'(c_2, w_2) \xi(c_3, w_3) + \alpha_2 C(c_2, w_2) \xi'(c_3, w_3) \} = 0, \\ \text{and} \quad & \alpha_2 C(c_1, 1) \{ \alpha_3 S'(c_2, w_2) \xi(c_3, w_3) + \alpha_2 S(c_2, w_2) \xi'(c_3, w_3) \} \\ & + \alpha_1 S(c_1, 1) \{ \alpha_3 C'(c_2, w_2) \xi(c_3, w_3) + \alpha_2 C(c_2, w_2) \xi'(c_3, w_3) \} = 0. \end{aligned} \quad (49').$$

ALTERNATIVE SOLUTION FOR PARABOLIC LAKES.

INTRODUCTION OF THE LAKE FUNCTION.

§ 41. For certain purposes a modification of the solution for parabolic lakes is convenient. This is obtained by shifting the origin to the positive end, and, for convenience, halving the scale of the new variable; that is, we put $w = 1 - 2z$. The equation

$$(1 - w^2) \frac{d^2 P}{dw^2} + cP = 0$$

then becomes

$$(z - z^2) \frac{d^2 P}{dz^2} + cP = 0 \quad (50).$$

If we attempt to solve this by the assumption

$$P = A_0 + A_1 z + A_2 z^2 + \dots,$$

we find in the usual way that we must have

$$\begin{aligned} A_0 &= 0, \\ cA_1 + 1.2A_2 &= 0, \\ (c - 1.2)A_2 + 2.3A_3 &= 0, \\ (c - 2.3)A_3 + 3.4A_4 &= 0, \\ &\dots \\ (c - \overline{n-2} . n - 1)A_{n-1} + \overline{n-1} . nA_n &= 0. \end{aligned}$$

Therefore $A_2 = -A_1 c / 1.2$;

$$A_n = (-1)^{n-1} \frac{c(c-1.2)(c-2.3) \dots (c-\overline{n-2} . n - 1)}{1^2 . 2^2 \dots (n-1)^2 . n} A_1.$$

That is to say, we find

$$P = A_1 \left(z - \frac{c}{1^2} \frac{z^2}{2} + \frac{c(c-1.2)}{1^2 \cdot 2^2} \frac{z^3}{3} - \frac{c(c-1.2)(c-2.3)}{1^2 \cdot 2^2 \cdot 3^2} \frac{z^4}{4} + \dots \right),$$

where the series within the brackets is obviously convergent for all real values of z between -1 and $+1$, both included.

We have thus obtained only one synectic integral of (50). A second integral may be derived in the usual way, but it is not synectic. It has, in fact, an essential singularity at $z=0$; and may be expressed in the form $P \log z + \psi(z)$, where $\psi(z)$ is a power series. It is needless to state the actual result here, as it is of no use for our present purpose.

Dropping the multiplicative constant, we write

$$L(c, z) = z - \frac{c}{1^2} \frac{z^2}{2} + \frac{c(c-1.2)}{1^2 2^2} \frac{z^3}{3} - \frac{c(c-1.2)(c-2.3)}{1^2 2^2 3^2} \frac{z^4}{4} + \dots \quad (51);$$

or

$$L(c, z) = z - c \frac{z^2}{1.2} - c(1-c/1.2) \frac{z^3}{2.3} - c(1-c/1.2)(1-c/2.3) \frac{z^4}{3.4} - \dots \quad (51').$$

The function thus defined will be called the *Lake Function*. It is obvious that

$$P = L(c, z)$$

is that synectic integral of the equation (50) which vanishes when $z=0$, i.e. when $w=1$; and it will be valid for $1 \geq z \geq 0$, that is, for $-1 \leq w \leq +1$, in short, throughout the whole length of the complete parabolic lake.

In particular, we have

$$L(c, 1) = 1 - \frac{c}{1.2} - \left(1 - \frac{c}{1.2}\right) \frac{c}{2.3} - \left(1 - \frac{c}{1.2}\right) \left(1 - \frac{c}{2.3}\right) \frac{c}{3.4} - \dots$$

or

$$L(c, 1) = \left(1 - \frac{c}{1.2}\right) \left(1 - \frac{c}{2.3}\right) \left(1 - \frac{c}{3.4}\right) \dots \quad (52),$$

by EULER's Identity.

Hence, again, we get

$$\begin{aligned} L(c, 1) &= C(c, 1)S(c, 1); \\ &= -\frac{2}{c\pi} \cos \left\{ \frac{\pi}{2} \sqrt{4c+1} \right\}; \end{aligned} \quad (53),$$

if we use the Γ -expressions for $C(c, 1)$ and $S(c, 1)$ given in § 25.

Since $L(c, z)$ is a synectic integral function of the general equation for P , which has the two synectic integral functions $C(c, w)$ and $S(c, w)$, that is, $C(c, 1-2z)$ and $S(c, 1-2z)$, it must be possible to determine two constants A and B , so that

$$L(c, z) \equiv A C(c, 1-2z) + B S(c, 1-2z).$$

Thus, in particular, we must have, for $z=0$ and $z=1$,

$$\begin{aligned} A C(c, 1) + B S(c, 1) &= 0; \\ A C(c, 1) - B S(c, 1) &= L(c, 1) = C(c, 1)S(c, 1). \end{aligned}$$

Whence it appears that $A = \frac{1}{2}S(c, 1)$, and $B = -\frac{1}{2}C(c, 1)$. Therefore

$$\begin{aligned} 2L(c, z) &\equiv S(c, 1)C(c, 1-2z) - C(c, 1)S(c, 1-2z); \\ \text{or} \quad 2L\left(c, \frac{1-w}{2}\right) &\equiv S(c, 1)C(c, w) - C(c, 1)S(c, w). \end{aligned} \quad (54).$$

Also, if the dash on $L(c, z)$ denote differentiation with respect to z ,

$$\left. \begin{aligned} -L'(c, z) &\equiv S(c, 1)C'(c, 1-2z) - C(c, 1)S'(c, 1-2z) \\ \text{or} \quad -L'\left(c, \frac{1-w}{2}\right) &\equiv S(c, 1)C'(c, w) - C(c, 1)S'(c, w) \end{aligned} \right\} \quad (55),$$

we may also notice the particular relations

$$L(c, \frac{1}{2}) = \frac{1}{2}S(c, 1), \quad L'(c, \frac{1}{2}) = C(c, 1). \quad (56).$$

§ 42.

THE L-SOLUTION FOR A COMPLETE PARABOLIC LAKE

is obviously

$$\left. \begin{aligned} \xi h(1-w^2) &= AL(c, z) \sin n(t-\tau); \\ \zeta &= \frac{A}{a} L'(c, z) \sin n(t-\tau). \end{aligned} \right\} \quad (57).$$

The values of c are the roots of the period equation

$$\left. \begin{aligned} L(c, 1) &= 0, \\ c_\nu &= \nu(\nu+1); \end{aligned} \right\} \quad (58)$$

and

$$T_\nu = 2\pi a / \sqrt{\{c\nu(\nu+1)gh\}},$$

as before.

The values of z corresponding to the nodes of the ν -nodal seiche are the roots of the equation

$$L'(c_\nu, z) = 0. \quad (59).$$

THE L-SOLUTION FOR A BIPARABOLIC LAKE.

§ 43. For the part between $w = 0$ and $w = +1$,

$$\left. \begin{aligned} \xi h(1-w^2) &= \frac{A}{L(c, \frac{1}{2})} L(c, z) \sin n(t-\tau); \\ \zeta &= \frac{A}{2aL(c, \frac{1}{2})} L'(c, z) \sin n(t-\tau); \end{aligned} \right\} \quad (60);$$

where $z = \frac{1}{2}(1-w)$.

For the part between $w' = 0$ and $w' = -1$,

$$\left. \begin{aligned} \xi h(1-w'^2) &= \frac{A}{L(c', \frac{1}{2})} L(c', z') \sin n(t-\tau); \\ \zeta &= -\frac{A}{2a'L(c', \frac{1}{2})} L'(c', z') \sin n(t-\tau); \end{aligned} \right\} \quad (61);$$

where $z' = \frac{1}{2}(1+w')$.

The period equation is now

$$aL(c, \frac{1}{2})/L'(c, \frac{1}{2}) + a'L(c', \frac{1}{2})/L'(c', \frac{1}{2}) = 0. \quad (62);$$

where

$$c = n^2 a^2 / gh, \quad c' = n^2 a'^2 / gh. \quad (63);$$

and

$$T_\nu = 2\pi/n_\nu = 2\pi a / \sqrt{cgh} = 2\pi a' / \sqrt{c'gh}. \quad (64).$$

The nodes in the two parts are given by

$$L'(c_v, z) = 0 \quad \text{and} \quad L'(c'_v, z) = 0 \quad \bullet \quad \bullet \quad \bullet \quad (65).$$

It may, of course, happen that one of the two equations (65) has no real root.

RATIOS OF THE SEICHE AMPLITUDES AT VARIOUS PARTS OF A BIPARABOLIC LAKE.

§ 44. It is convenient for purposes of calculation to tabulate the following formulæ : where ζ_0 , ζ_A , ζ_A' , ζ_z denote the amplitudes of a seiche at the deepest part, the positive and negative end, and any point z respectively ; and $\alpha = \sqrt{4c + 1}$.

$$\zeta_A/\zeta_0 = \Gamma(c, 1) = \Gamma\left(\frac{3+\alpha}{4}\right)\Gamma\left(\frac{3-\alpha}{4}\right) / \sqrt{\pi} = \Gamma(c, \frac{1}{2}). \quad (66)$$

$$\zeta_A/\zeta_0=1, U(c', 1)=\Gamma\left(\frac{3}{4}+\frac{a'}{4}\right)\Gamma\left(\frac{3}{4}-\frac{a'}{4}\right)/\sqrt{\pi}=1/\Gamma'(c', \frac{1}{2}). \quad (67)$$

[illegible]

$$\begin{aligned}\zeta_z/\zeta_0 &= L'(c, z)/L'(c, \tfrac{1}{2}), \\ &= L'(c, z)\Gamma\left(\tfrac{3+a}{4}\right)\Gamma\left(\tfrac{3-a}{4}\right)/\sqrt{\pi}, \\ &= L'(c, z)(\zeta/\zeta_0),\end{aligned}\quad (69)$$

All irrespective of algebraic sign.

Owing to the want of a simple companion fundamental integral, the Lake Function is not convenient when the parabolic lake is truncated. In this respect it has the same defect as the LEGENDRE and BESSEL Functions. Its practical advantage is that it gives highly convergent series at points where the series for $C(c, w)$ and $S(c, w)$ converge slowly. Unhappily, the corresponding function for a convex lake has an imaginary argument.

SEICHES IN RECTILINEAR LAKES.

§ 45. If we take the origin of x at a point where the depth is h , then the law of depth will be $h(x) = h \times (1 - x/a)$, where a is a constant, positive or negative according as the lake bottom slopes upwards or downwards in the direction in which x increases.

We have, therefore, with the previous notation

$$\left. \begin{aligned} \xi_h(1-x_i a) &= u = P \sin n(t - \tau), \\ \zeta &= -\frac{\partial u}{\partial x}; \end{aligned} \right\} \quad (70)$$

where P is determined by

$$\frac{d^2P}{dx^2} + \frac{n^2P}{qh(1-x/a)} = 0 \quad (71).$$

If we transform (71) by putting $w = 2na\sqrt{(1-x/a)/\sqrt{(gh)}}$, and $P = Rw$, it becomes

$$\frac{d^2R}{dw^2} + \frac{1}{w} \frac{dR}{dw} + \left(1 - \frac{1}{w^2}\right)R = 0, \quad (72).$$

which is a particular case of the BESSEL Equation.*

If $J_n(w)$ and $Y_n(w)$ denote the BESSEL and NEUMANN Functions, as defined in GRAY and MATHEWS' treatise, the general solution of (72) is

$$R = A J_1(w) + B Y_1(w).$$

Hence, with a slight adaptation of the constants, we find

$$\xi w = \{A J_1(w) + B Y_1(w)\} \sin n(t - \tau); \quad (73);$$

$$\zeta = \frac{2a}{h} \left\{ \frac{A}{w} \frac{d}{dw} (w J_1(w)) + \frac{B}{w} \frac{d}{dw} (w Y_1(w)) \right\} \sin n(t - \tau).$$

Now, by one of the fundamental properties of $J_1(w)$ and $Y_1(w)$, we have

$$\frac{1}{w} \frac{d}{dw} (w J_1(w)) = J_0(w), \quad \frac{1}{w} \frac{d}{dw} (w Y_1(w)) = Y_0(w).$$

Hence

$$\zeta = \frac{2a}{h} \{A J_0(w) + B Y_0(w)\} \sin n(t - \tau). \quad (74).$$

From (73) and (74) solutions for the following particular cases are readily obtained.

RECTILINEAR LAKE WITH TWO SLOPES TRUNCATED AT BOTH ENDS.

§ 46. The laws of depth for the two parts will be given by $h(1-x/a)$ and $h'(1+x/a')$, if we take the origin at the junction of the two slopes, and choose the standard case to be that where the bottom slopes upwards on both sides of the junction.

FIG. 8.

Let

$$w = 2na\sqrt{(1-x/a)/\sqrt{(gh)}}, \quad w' = 2na'\sqrt{(1+x'/a')/\sqrt{(gh)}}. \quad (75);$$

and

$$\begin{aligned} \alpha &= 2a/\sqrt{(gh)}, \quad \beta = 2a\sqrt{(1-p/a)/\sqrt{(gh)}}; \\ \alpha' &= 2a'/\sqrt{(gh)}, \quad \beta' = 2a'\sqrt{(1-p'/a')/\sqrt{(gh)}}. \end{aligned}$$

* Readers unacquainted with the properties of BESSEL Functions will find all that is here required in a few pages of the treatise by GRAY and MATHEWS (1895), ch. ii., and pp. 241-292 containing the tables.

The boundary conditions are, that ξ vanish at the ends, and that $\xi = \xi, \zeta = \zeta'$ at 0. These lead readily to the following:—

$$\left. \begin{aligned} w\xi &= \alpha A \frac{Y_1(n\beta)J_1(w) - J_1(n\beta)Y_1(w)}{Y_1(n\beta)J_1(na) - J_1(n\beta)Y_1(na)} \sin n(t - \tau); \\ \zeta &= \frac{2a^2}{h} A \frac{Y_1(n\beta)J_0(w) - J_1(n\beta)Y_0(w)}{Y_1(n\beta)J_1(na) - J_1(n\beta)Y_1(na)} \sin n(t - \tau); \\ w'\xi' &= \alpha' A \frac{Y_1(n\beta')J_1(w') - J_1(n\beta')Y_1(w')}{Y_1(n\beta')J_1(na') - J_1(n\beta')Y_1(na')} \sin n(t - \tau); \\ \zeta' &= -\frac{2a'^2}{h} A \frac{Y_1(n\beta')J_0(w') - J_1(n\beta')Y_0(w')}{Y_1(n\beta')J_1(na') - J_1(n\beta')Y_1(na')} \sin n(t - \tau) \end{aligned} \right\} \quad (76).$$

And the period equation is

$$\begin{aligned} &\alpha^2 \{Y_1(n\beta)J_0(na) - J_1(n\beta)Y_0(na)\} \{Y_1(n\beta')J_1(na') - J_1(n\beta')Y_1(na')\} \\ &+ \alpha^2 \{Y_1(n\beta')J_0(na') - J_1(n\beta')Y_0(na')\} \{Y_1(n\beta)J_1(na) - J_1(n\beta)Y_1(na)\} = 0 \end{aligned} \quad (77).$$

UNSYMMETRIC LAKE SHELVEING AT BOTH ENDS.

§ 47

FIG. 9.

In this case $\beta = 0, \beta' = 0$.

Therefore, since $\lim_{w \rightarrow 0} J_1(w)/Y_1(w) = 0$, the equations of § 46 reduce to

$$\begin{aligned} w\xi &= \alpha A \frac{J_1(w)}{J_1(na)} \sin n(t - \tau), \\ \zeta &= \frac{2a^2 A}{h} \frac{J_0(w)}{J_1(na)} \sin n(t - \tau); \\ w'\xi' &= \alpha' A \frac{J_1(w')}{J_1(na')} \sin n(t - \tau); \\ \zeta' &= -\frac{2a'^2 A}{h} \frac{J_0(w')}{J_1(na')} \sin n(t - \tau). \end{aligned} \quad (78).$$

Period equation

$$\alpha^2 J_0(na)J_1(na') + \alpha'^2 J_0(na')J_1(na) = 0. \quad (79)$$

The nodes are given by *

$$\begin{aligned} J_0(w) &= 0 \text{ in the part } OA; \\ J_0(w') &= 0 \text{ in the part } OA'; \end{aligned}$$

where for the ν -nodal seiche

$$w = n_\nu a, \quad w' = n_\nu a'.$$

* *Roots of the Bessel Functions.*—In what follows I shall denote the positive roots of the equation $J_0(z) = 0$ by $j_1, j_3, j_5, \dots$; and the positive roots of $J_1(z) = 0$ (excluding the zero root $j_0 = 0$) by $j_2, j_4, j_6, \dots$. So that we have approximately $j_1 = 2.405, j_2 = 3.832, j_3 = 5.520, j_4 = 7.016, j_5 = 8.654, j_6 = 10.173, j_7 = 11.792, j_8 = 13.323, j_9 = 14.931, j_{10} = 16.471$, etc.

For large values of $n, j_n = (2n+1)\pi/4$, approximately: e.g. this formula gives $j_{11} = 18.064$ instead of the correct value 18.071; so that the error after $n = 11$ is less than $1/100$.

If $_{\nu}r_{\nu}$ denote the distance from O of the ν^{th} node in OA counting from A to O, and $_{\nu}r'_{\nu}$ the like for O A', we have

$$n_{\nu}a \sqrt{(1 - _{\nu}r_{\nu}/a)} = j_{2\nu-1}, \quad j_{2\nu-1} < n_{\nu}a.$$

Hence

$$\begin{aligned} 1 - _{\nu}r_{\nu}/a &= j_{2\nu-1}^2/T_{\nu}^2/4\pi^2a^2, \quad j_{2\nu-1} < n_{\nu}a; \\ 1 - _{\nu}r'_{\nu}/a' &= j_{2\nu-1}^2/T_{\nu}^2/4\pi^2a'^2, \quad j_{2\nu-1} < n_{\nu}a'. \end{aligned} \quad (80).$$

There will be ν roots altogether; but the distribution between the two formulæ will depend on circumstances. In some cases the nodes are all on one side of O. These formulæ lead to some remarkable relations which are true accurately for complete rectilinear lakes, and approximately for such as are approximately rectilinear. For example, we have

$$(a - _{\mu}r_1)/(a - _{\nu}r_1) = T_{\mu}^2/T_{\nu}^2. \quad (81).$$

In other words, the distance from the ends of the lake of the first node of any pure seiche in a given complete rectilinear lake is proportional to the square of the period of the seiche.

SYMMETRIC TRUNCATED LAKE.

§ 48. Here $a = a'$, $\beta = \beta'$. The equations can be simplified by the suppression of unnecessary constants; and the period equation breaks up into two. We have then

$$\begin{aligned} w\xi &= A \{ Y_1(n\beta)J_1(w) - J_1(n\beta)Y_1(w) \} \sin n(t - \tau), \\ \zeta &= \frac{2aA}{h} \left\{ Y_1(n\beta)J_0(w) - J_1(n\beta)Y_0(w) \right\} \sin n(t - \tau); \\ w'\xi' &= A \{ Y_1(n\beta)J_1(w') - J_1(n\beta)Y_1(w') \} \sin n(t - \tau), \\ \zeta' &= -\frac{2aA}{h} \left\{ Y_1(n\beta)J_0(w') - J_1(n\beta)Y_0(w') \right\} \sin n(t - \tau). \end{aligned} \quad (82).$$

The period equation is:—for odd nodality,

$$Y_1(n\beta)J_0(na) - J_1(n\beta)Y_0(na) = 0;$$

for even nodality,

$$Y_1(n\beta)J_1(na) - J_1(n\beta)Y_1(na) = 0. \quad (83).$$

SYMMETRIC LAKE SHELVED AT BOTH ENDS.

§ 49. Starting with the formulæ of § 47, we have to put $a = a' = p = p'$; and therefore $\alpha = \alpha'$. Suppressing unnecessary constants, we may now write

$$\begin{aligned} w\xi &= AJ_1(w) \sin n(t - \tau), \\ \zeta &= \frac{2aA}{h} J_0(w) \sin n(t - \tau); \\ w'\xi' &= AJ_1(w') \sin n(t - \tau); \\ \zeta' &= -\frac{2aA}{h} J_0(w') \sin n(t - \tau). \end{aligned} \quad (84).$$

The period equation breaks up into

$$J_0(na) = 0, \quad J_1(na) = 0. \quad (85).$$

Hence we have

$$T_{\nu} = 4\pi a/j_{\nu} \sqrt{(gh)}. \quad (86).$$

* These formulæ are given by LAMB in his *Hydrodynamics* (1895), § 182.

For large values of ν , we have

$$T_\nu = 16a/(2\nu + 1) \sqrt{(gh)}. \quad (87).$$

And, when $1/2\nu$ is negligible, simply

$$\begin{aligned} T_\nu &= 8a/\nu \sqrt{(gh)}, \\ &= 4l/\nu \sqrt{(gh)}. \end{aligned} \quad (88).$$

Hence, as the nodality increases, the periods of the pure seiches tend more and more to follow the harmonic law; and ultimately are the same as the periods in a uniform lake of double the length.

If νx_r have the same meaning as before, and νX_r have a corresponding meaning for the ventral points, we find

$$\begin{aligned} 1 - \nu x_r/\alpha &= j_{2r-1}^2/j_r^2 = T_r^2/T_{2r-1}^2; \\ 1 - \nu X_r/\alpha &= j_{2r}^2/j_r^2 = T_r^2/T_{2r}^2. \end{aligned} \quad (89).$$

Hence, if we compare the different nodes of the same seiche, the distances of the nodes from the end of the lake are inversely proportional to the squares of the periods of the lower seiches of odd nodality.

If we compare nodes of the same order for different seiches, the distances from the end are directly proportional to the squares of the periods of the corresponding seiches.

It is also easy to see from the above formulæ that, when the nodality of the seiche is high, the wave lengths near the ends of the lake increase at first in arithmetic progression.

If we apply the rule of Du Boys to the present case we get

$$\nu T_1 = \frac{1}{2} \sqrt{\left(\frac{l}{2gh}\right) \int_0^{l/2} dx} = \frac{4l}{\sqrt{(gh)}}.$$

Since $T_1 = 2\pi l/j_1 \sqrt{(gh)}$, we have

$$\frac{\nu T_1}{T_1} = \frac{2j_1}{\pi} = \frac{4 \cdot 810}{3 \cdot 142} = 1 \cdot 531;$$

that is, Du Boys' rule gives too great a period, as it does in the case of parabolic or quartic concave lakes: the deviation is even greater than in the case of a symmetric parabolic lake.

As the present case is an interesting one, serving as a standard of comparison for other cases, I add some numerical data.

Table of Ratios of Periods for a Complete Symmetric Rectilinear Lake.

T_2/T_1	T_3/T_1	T_4/T_1	T_5/T_1	T_6/T_1	T_7/T_1	T_8/T_1	T_9/T_1	T_{10}/T_1
·6276	·4357	·3428	·2779	·2365	·2040	·1805	·1609	·1460

Positions of Nodes in one-half of the Lake.

1	0									
2	...	...	...	...	·6057					
3	0	...	...	...	...	...	8102			
4	...	...	·3809	...	...	...	...	·8825		
5	0	...	...	·5930	...	...	...	...	·9228	
6	...	·3763	...	...	...	...	...	...	...	·9441

Positions of Nodes when Depth is uniform.

1	0									
2	...	...	...	...	·5000					
3	0	...	...	...	...	...	·6667			
4	...	...	·2500	...	...	...	...	·7500		
5	0	...	...	·4000	...	...	...	...	·8000	
6	...	·1667	...	...	·5000	...	...	...	...	·8333

LAKE WITH ONE SLOPE.

§ 50.

FIG. 10.

Obviously the seiches are the same as the seiches of even nodality in the symmetric truncated lake of § 48.

Hence we have $\xi v = A \{ Y_1(n\alpha) J_1(w) - J_1(n\alpha) Y_1(w) \} \sin n(t - \tau) ;$

$$\zeta = \frac{2aA}{h} \{ Y_1(n\alpha) J_0(w) - J_1(n\alpha) Y_0(w) \} \sin n(t - \tau) . \quad (90) .$$

The period equation is $Y_1(n\alpha) J_1(n\beta) - J_1(n\alpha) Y_1(n\beta) = 0 . \quad (91) .$

For the nodes and ventral points

$$\begin{aligned} 1 - \nu x_r/a &= j_{2r-1}^2/j_{2r}^2 \\ 1 - \nu X_r/a &= j_{2r}^2/j_{2r+1}^2 = T_r^2/T_{r+1}^2 . \end{aligned}$$

LAKE WITH ONE SLOPE SHELVEING AT ONE END.

§ 51.

FIG. 11.

Here we have simply to take the seiches of even nodality from the case of § 49.

$$\xi w = A J_1(w) \sin n(t - \tau) ;$$

$$\zeta = \frac{2\alpha A}{h} J_0(n) \sin n(t - \tau) \quad (92).$$

The period equation is

$$J_1(na) = 0 ;$$

so that

$$T_v = 4\pi\alpha/\dot{j}_{2v} \sqrt{\langle y/h \rangle} \quad (93).$$

The nodes are, of course, the nodes of the seiches of even nodality in the right-hand half of the complete symmetric rectilinear lake discussed in § 49.

The following data are useful for reference in lake calculations :—

Ratios of Periods in a Semicomplete Rectilinear Lake.

T_2/T_1	T_3/T_1	T_4/T_1	T_5/T_1	T_6/T_1	T_7/T_1	T_8/T_1	T_9/T_1	T_{10}/T_1
·5462	·3767	·2883	·2327	·1954	·1684	·1479	·1319	·1190

Positions of the Nodes.

1	...	...	6057			
2	...	3809	...	...	8825	
3	3763	...	...	7056	...	9441

Positions of Nodes for Uniform Depth.

1	...	...	5000			
2	...	2500	...	...	7500	
3	1667	...	5000	...	...	8333

SEICHES IN QUARTIC LAKES.

§ 52. In a paper recently published in the Society's *Proceedings* (vol. xxv. p. 638, May 11, 1905) I gave the solution of the seiche problem for lakes whose normal curve is a quartic of the form $\sigma = h(1 \mp v^2/a^2)^2$. For convenience of reference I recapitulate the results here, supposing as usual, for simplicity, that the lake has uniform breadth and rectangular cross section, so that the expression for the depth is $h \times (a^2 \mp x^2)^2$.

CONCAVE TRUNCATED QUARTIC LAKE.

The origin is at O over the deepest point (see fig. 12). The length P Q is l , and

FIG. 12.

P and Q correspond to $x = p$, $x = q$. The depths at P, Q, and O are r , s , d respectively ; so that

$$p = a\sqrt{\left(1 - \sqrt{\frac{r}{d}}\right)}, \quad q = \mp a\sqrt{\left(1 - \sqrt{\frac{s}{d}}\right)};$$

$$l = a\left\{\sqrt{\left(1 - \sqrt{\frac{r}{d}}\right)} \pm \sqrt{\left(1 - \sqrt{\frac{s}{d}}\right)}\right\} = a\gamma, \text{ say} \quad (94)$$

The upper signs correspond to the case figured, where P and Q are on opposite sides of O.

Then

$$\xi h(a^2 - x^2)^2 = u = A(a^2 - x^2)^{\frac{1}{2}} \sin \left\{ \frac{\nu\pi}{k} \left(\log \frac{a+x}{a-x} - \log \frac{a+\rho}{a-\rho} \right) \right\} \sin n_\nu(t - \tau),$$

$$\zeta = -\frac{\partial u}{\partial x}; \quad (95);$$

and

$$T_\nu = 2\pi l / \gamma \sqrt{\{gd(4\nu^2\pi^2/k^2 + 1)\}}; \quad (96);$$

where

$$\gamma = \sqrt{\left(1 - \sqrt{\frac{r}{d}}\right) \pm \sqrt{\left(1 - \sqrt{\frac{s}{d}}\right)}};$$

$$k = \log \left\{ \frac{1 + \sqrt{\left(1 - \sqrt{\frac{r}{d}}\right)}}{1 - \sqrt{\left(1 - \sqrt{\frac{r}{d}}\right)}} \middle/ \frac{1 \mp \sqrt{\left(1 - \sqrt{\frac{s}{d}}\right)}}{1 \pm \sqrt{\left(1 - \sqrt{\frac{s}{d}}\right)}} \right\} \quad (97).$$

When the end barriers P and Q approach more and more nearly to the infinitely shallow theoretical ends A and A', the periods of all the seiches become more and more nearly equal to each other and to $\pi l / \sqrt{gd}$, which I have called *the period of the anomalous seiche*.

CONVEX QUARTIC LAKE.

§ 53. The symbols being defined as in § 52, we now have

$$p = a\sqrt{\left(\sqrt{\frac{r}{d}} - 1\right)}, \quad q = \mp a\sqrt{\left(\sqrt{\frac{s}{d}} - 1\right)};$$

$$l = a \left\{ \sqrt{\left(\sqrt{\frac{r}{d}} - 1\right)} \pm \sqrt{\left(\sqrt{\frac{s}{d}} - 1\right)} \right\} = a\gamma, \text{ say.} \quad (98).$$

$$\xi h(a^2 + x^2)^2 = u = (a^2 + x^2)^{\frac{1}{2}} \sin \left\{ \frac{2\nu\pi}{k} \left(\tan^{-1} \frac{x}{a} - \tan^{-1} \frac{p}{a} \right) \right\} \sin n_\nu(t - \tau),$$

$$\zeta = -\frac{\partial u}{\partial x}; \quad (99);$$

$$\text{and} \quad T_\nu = 2\pi l / \gamma \sqrt{\{gd(4\nu^2\pi^2/k^2 - 1)\}}; \quad (100);$$

where

$$\gamma = \sqrt{\left(\sqrt{\frac{r}{d}} - 1\right) \pm \sqrt{\left(\sqrt{\frac{s}{d}} - 1\right)}};$$

$$k = 2 \tan^{-1} \sqrt{\left(\sqrt{\frac{r}{d}} - 1\right)} - 2 \tan^{-1} \sqrt{\left(\sqrt{\frac{s}{d}} - 1\right)}. \quad (101).$$

§ 54. In constructing a theoretical curve to represent the normal curve of any lake deduced from bathymetric data we can, of course, combine pieces of parabolas, straight lines or quartics at will; and the variety of formulæ above given is probably sufficient for most practical purposes, although the labour of the calculation, even in simple cases, is not small. To this part of the subject I shall have occasion to return in subsequent communications to the Society.

PART III.

A SKETCH OF THE BIBLIOGRAPHY OF SEICHES.

The following list of books and memoirs dealing with Seiches does not claim to be exhaustive; and in many cases I have not been able to verify the references directly. It must, therefore, be looked on merely as an endeavour to help others in a matter where I myself have felt the need of help.

What may be called the ancient history of seiche observations is fully dealt with by FOREL; and it is to his bibliography, extending from 1873 to 1891, that I owe most of the information for that period. I must also acknowledge obligations to papers by HALBFASS, mentioned below, and to Messrs CHUMLEY and MACLAGAN WEDDERBURN, both connected with the Scottish Lake Survey, for many of the later references.

The following abbreviations are used:—

<i>A.G.</i> , <i>Archives des Sciences Physiques et Naturelles</i> , Genève.	<i>B.V.</i> , <i>Bulletin de la Société Vaudoise des Sciences Naturelles</i> .
<i>A.H.</i> , <i>Actes de la Société Helvétique des Sciences Naturelles</i> .	<i>C.R.</i> , <i>Comptes Rendus de l'Académie des Sciences</i> , Paris.
<i>A.Hy.</i> , <i>Annalen der Hydrographie</i> .	<i>P.M.</i> , <i>Petermann's Mittheilungen</i> .
<i>A.W.</i> , <i>Sitzungsberichte der K.K. Akademie der Wissenschaft</i> , Wien.	<i>Z.G.</i> , <i>Zeitschrift für Gewässerkunde</i> .
<i>B.A.</i> , <i>British Association Reports</i> .	<i>Z.I.</i> , <i>Zeitschrift für Instrumentenkunde</i> .

The Roman numeral indicates the volume, the Arabic the page.

1755. Disturbances of the Levels of Lakes in Scotland and elsewhere caused by the Earthquake of Lisbon, *Scots Magazine* for 1755.

These notices are interesting, because there is, as yet, little evidence connecting seiches with seismic disturbances; in fact, none at all in the case of ordinary seiches.

1776. LAPLACE. "Sur les Ondes. Suite des Recherches sur plusieurs points du Système du Monde," § xxxvii., *Hist. de l'Ac. Roy. d. Sc. Paris*, Année 1776.

The modern mathematics of wave motion may be said to date from LAPLACE's researches on the tides. In the memoir quoted he considers waves in a canal of uniform depth to be caused by the immersion of a given object, and arrives at the expression $\sqrt{g \tanh mh/m}$ for the velocity of wave propagation. But, as he does not consider oscillatory waves, the connection of m with the wave length is not made clear.

1781. LAGRANGE. "Sur la Théorie du Mouvement des Fluides," *Mém. Ac. Berl.*

In §§ 48 and 49 he gives for the first time the theory of long waves, with the formula $\sqrt{gh}$ for the velocity of propagation in a canal of uniform depth, h .

1804. YOUNG. *Lectures on Natural Philosophy*, xxiii. Also Works (ed. PEACOCK), ii. pp. 141, 262.

1815. CAUCHY. "Sur la Théorie des Ondes." *Ouvres*, 1^e sér., i. 175.

1815. POISSON. "Sur la Théorie des Ondes," *Mém. d. l'Inst.*, i. (1816), etc.

In the works of CAUCHY and POISSON the mathematics of wave motion has already taken its modern form. Both have LAPLACE's formula for velocity of propagation, but both are pre-occupied with the difficult problem of LAPLACE, and do not consider oscillatory surface waves, either progressive or stationary.

1825. WEBER, W. E. and E. H. *Die Wellentheorie auf Experimente gegründet.* Leipzig.

This treatise is a classic in the experimental part of our subject. Here for the first time direct observations of the motion of the fluid particles in wave motions are described. A great variety of different cases of wave motion are discussed; and in particular, standing waves in a canal, of which seiches are a particular example.

1828. MERIAN, J. R. *Ueber die Bewegung tropfbarer Flüssigkeiten in Gefassen.* Basel.

This memoir was really the first after LAGRANGE that dealt fully and effectively with the problem of stationary waves in a canal of uniform depth (h) and length (l). In MERIAN's paper the formula $T = \sqrt{\{\pi l/g \tanh(\pi h/l)\}}$ appears for the first time in connection with stationary waves. Its relation to the formula of LAPLACE, CAUCHY, and POISSON for the velocity of propagation is obvious. Unfortunately the memoir attracted no attention when it was published, and was forgotten until it was reproduced by the author's grandnephew, VON DER MÜHL, *Math. Ann.*, xxviii. 575, 1885. Meanwhile MERIAN's results had been rediscovered by other mathematicians.

1837. RUSSELL, JOHN SCOTT. His beautiful experiments on canal waves, described in the *B.A. Reports* from 1837 onwards, were the starting-point of a long series of English researches by GREEN, KELLAND, AIRY, STOKES, KELVIN, RAYLEIGH, and others.

1837. GREEN, G. "On the Motion of Waves in a variable Canal of small Depth and Width," *Trans. Camb. Phil. Soc.*, 1838. Also *ib.*, 1839.

1839. KELLAND, P. "On the Theory of Waves," *Trans. Roy. Soc. Edin.*, xiv., 1839. Also *ib.*, xv., 1841.

1843. MILNE, D. "On a Remarkable Oscillation of the Sea," July 1843. *Trans. Roy. Soc. Edin.*, xv. 609.

1845. AIRY, G. B. "Tides and Waves," *Ency. Metrop.*

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AIRY's Article and STOKES' Report are classics on the analytical side of our subject.

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