The Division of the Circle

Author(s): Camillo Manzitti

Source: *The Mathematical Gazette*, Vol. 4, No. 76 (Dec., 1908), pp. 377-381

Published by: The Mathematical Association

Stable URL: <http://www.jstor.org/stable/3602018>

Accessed: 18-06-2016 09:14 UTC

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THE MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON :
GEORGE BELL & SONS, PORTUGAL STREET, LINCOLN'S INN,
AND BOMBAY.

VOL. IV.

DECEMBER, 1908.

No. 76.

OUR LOCAL BRANCH.

THE usual meeting of the North Wales branch was held at Dr. Bryan's house, Bangor, on Nov. 21. Mr. Madoc Jones, head-master of Beaumaris County School, read an interesting paper on the teaching of algebra in schools, in which he expressed a strong opinion that algebra should not be started too early, the transition from arithmetic being made almost imperceptibly; that the solving of equations should be introduced as soon as possible, and applied to simple problems; and that the principles underlying algebraic methods and problems should be insisted upon, mere mechanical proficiency not being regarded as the chief end. The difficulty of making the "rules of signs" intelligible was also discussed, and graphical illustrations were recommended.

Dr. Bryan urged the early introduction of logarithms, pointing out the mistake of postponing the subject until indices had been mastered, partly because logarithms had been invented without reference to an incommensurable base.

Then followed a discussion about examinations, complaint being made of the difficulty in Wales of satisfying two or sometimes three different bodies of examiners, of the want of sympathy between teachers and examiners, and of the consequent absence of scope for individual work.

The President and Secretary were re-elected for 1909, and it was agreed to hold the next meeting, on Feb. 13, at Bangor.

THE DIVISION OF THE CIRCLE.

THE possibility of solving the problem of the elementary division of a circumference into equal parts was known to the ancients for the cases

$n = 2^p$, $n = 3$, $n = 5$, and their products.

Gauss in his *Disquisitiones Arithmeticae* proved the possibility of solving the problem by quadratics, and therefore geometrically with ruler and the compass, for all the prime numbers of the series

$$n = 2^{2^p} + 1.$$

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The prime numbers n of this series now known are 5, including the values $n=3$, $n=5$, for

$p=0$,	$n=3$;
$p=1$,	$n=5$;
$p=2$,	$n=17$;
$p=3$,	$n=257$;
$p=4$,	$n=65537$.

The case of $n=17$, investigated by Gauss himself with the solution of $x^{17}=1$, was geometrically interpreted by Serret in his *Algèbre Supérieure* by the geometrical representation of the roots of the quadratic equations obtained with the help of ruler and compass. The demonstration as given by Serret was modified by Bachmann and reproduced in his book, *Die Lehre von der Kreisteilung* (1872).

Another geometrical interpretation was given by von Staudt (*Crelle*, 1842) by the help of the ruler and a fixed circle in the plane of the figure; more recently a geometrical construction of the roots solving the problem by means of the compass alone was given by Gérard (*Math. Annalen*, 1897).

The case $n=257$, investigated by Richelot in his work, *De resolutione algebricâ aequationis $x^{257}=1$, sive de divisione circuli per bisectionem anguli septies repetitam in partes 257 inter se aequales commentatio coronata*, received geometrical interpretation at the hands of Affolter and Pascal (*Rendiconto della R. Acc. di Napoli*, 1887).

Case $n=65537$ was investigated by Prof. Hermes of Lingen (*Göttinger Nachrichten*, No. 3, 1894): the papers containing his very laborious researches, which took up ten years of his life, are preserved in the mathematical seminary at Gottingen.

Besides the cases already mentioned, others also exist in which the problem of the division of the circle can be solved by quadratics: these cases belong to the series

$$n=2^p, (2^{p_1}+1), (2^{p_2}+1), \dots (2^{p_s}+1),$$

$p, p_1, p_2, \dots p_s$ being different.

These cases were dealt with by Prof. Enriquez of the University of Bologna in his brochure entitled *Sulle equazioni algebriche risolubili per radicali quadratici e sulla costruibilità dei poligoni regolari*.

In all other cases solution geometrically by means of the rule and compass or algebraically by means of quadratic is impossible.

Thus, e.g. for n a prime number not comprised in the Gaussian series, writing $n-1$ under the form of the product of its prime factors,

$$n-1=2^a 3^b 5^c \dots p^d,$$

it is proved that the problem is reducible to the solution of

α	equations of the 2nd degree.		
β	"	"	3rd "
γ	"	"	5th "
δ	"	"	p th "

It follows that for the geometrical construction of the roots of such equations, instruments more complicated than the rule and compass are necessary, and such instruments should be used for drawing the algebraic curves solving the said equations.

The problems of the 'trisection of an angle,' the 'duplication of the cube' (the Delian problem), and the 'squaring of the circle,' tyrannised, as Prof. Gino Loria has it, all the geometry of the ancients. We may remind the reader that many solutions of that famous problem are known, such as those founded on the conic sections, the 'conchoid of Nicomedes,' the 'Pascal's snail,' besides those given by various instruments for trisection based generally speaking on the metrical properties of the aforesaid curves. Klein, referring to constructions for the polysection of angle or circle, states that the practical execution of all such constructions by means of higher algebraic curves remains unsolved.

Hereafter is shown the solution of the problem of polysection of any angle and the circle, for any whole value of n , by means of algebraic curves that can be graphically obtained.

FIG. 1.

"Given a circle and an angle at its centre, to find a family of curves which shall determine the n -section of the angle and shall divide the circle according to the vertices of the regular inscribed n -gon, n being any whole number."

This is equivalent to proving that these curves must give all the roots of the problem of the n -section of the angle.

Let ABC be any angle at the centre of the circle (B). Let a point M on the circumference divide the arc AC , which measures the given angle, in the ratio $\frac{AC}{MC} = n$.

Let M be conceived as the intersection of BM with DM (D being the image of A in BC), and supposing BM to have rotated through the angle ADM measured by

$$\frac{1}{2}AM = \frac{1}{2} \frac{n-1}{n} \cdot CA.$$

The ratio of the two arcs is the ratio of velocity of rotation of BM and DM , as shown by the arrows in Fig. 1, which will result in their crossing on the circle in the point M cutting off the n -section of the angle; this ratio is a definite function of n expressed by

$$K = \frac{\frac{1}{2}AM}{\frac{AC}{n}} = \frac{n-1}{2}.$$

The problem of n -section of the angle is thus kinematically analogous to the rotation of two straight lines with a given ratio of velocity, being a simple function of n .

The order of the curve-locus of intersections of the two straight lines is also, as will be proved, a definite function of n .

For $n=3$, the case of trisection of the angle, the ratio of velocity will be $\frac{n-1}{2} = 1$.

Noting the sense of rotation of the generating straight lines we see that we have as locus an equilateral hyperbola, since the kinematical definition is the same as the projective definition of two pencils of straight lines inversely congruous and not perspective. However, the kinematical view gives in addition a variety of curves for the trisection of the angle—as for the n -section—essentially differing from the known curves already mentioned.

The equilateral hyperbola not only passes through the poles of rotation B and D , but gives as intersections on the circle points corresponding to the three roots of the cubic equation defining the problem, *i.e.*

$$\frac{ABC}{3}, \quad \frac{ABC+2\pi}{3}, \quad \frac{ABC+4\pi}{3}.$$

This property of giving all the roots of the problem is common to the loci defined by the intersection of the straight lines rotating as explained.

We must, however, distinguish two separate cases corresponding to $n = 2p + 1$, $n = 2p$.

Let n be an odd number.

The ratio $k = \frac{n-1}{2} = p$ is an integer: that is to say the two generating straight lines BM and DM return to their initial mutual position after a semi-rotation of BM ; in the successive semi-rotation the locus is repeated.

The intersections of the curve with the circle, in addition to that corresponding to the point D , are n in number and are distributed in the vertices of the regular inscribed n -gon, one vertex being at M .

FIG. 2.

Let us then consider a point M_1 on the circumference such that the arc $MM_1 = \frac{2\pi}{n}$; it will suffice to show that this point must belong to the locus. In fact, the ratio of the angle MDm_1 measured in the sense of its generation, to the corresponding angle M_1BM is $\frac{n-1}{2}$, as follows from the value of MDm_1 , the supplementary angle to M_1DM measured by $\frac{1}{2}M_1M = \frac{\pi}{n}$, is

$$\frac{n-1}{n}\pi \text{ or } \frac{n-1}{2} \text{ times } \frac{2\pi}{n} = M_1BM. \quad \text{Q.E.D.}$$

The same demonstration can be applied to the other $n-2$ vertices.

CAMILLO MANZITTI.

(To be continued.)

NOTE ON THE GROUPS OF SUBTRACTION AND DIVISION, AND ON THE HYPERBOLIC FUNCTIONS.

THE objects of the present note are to exhibit an interesting property of some groups of subtraction and division, and to present a method of using such groups in the study of some elementary properties of the hyperbolic

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