LIV. On the vibrations of an elastic fluid

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LVI. *On the Vibrations of an Elastic Fluid.* By the Rev. J. CHALLIS, M.A., F.R.S., F.R.A.S., Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge*.

IN accordance with the intention expressed at the close of my communication to the Philosophical Magazine for August last, I proceed to draw some inferences from the equations (B.) and (C.). But, first, it will be proper to point out two inaccuracies contained in that communication. The third term of equation (C.) in p. 100 should be $\frac{b^2 f}{a^2}$ instead of $b^2 f$; and in the same page, in the series for the form of ϕ , the quantities in brackets should be $z - a't + c_1$, $z - a't + c_2$, $z - a't + c_3$, &c., c_1 , c_2 , c_3 , &c. being certain constants. No inferences are in the least degree affected by these alterations.

Let us now, for the purpose of testing equation (C.), suppose that f is a function of $x^2 + y^2$. By this supposition the equation becomes, on putting r^2 for $x^2 + y^2$,

$$0 = \frac{d^2 f}{dr^2} \left(1 - \frac{\phi_0^2}{a^2} \cdot \frac{df^2}{dr^2} \right) + \frac{1}{r} \cdot \frac{df}{dr} + \frac{b^2}{a^2} \cdot f,$$

showing that the supposition is allowable. In any case of vibrations not very large, the second term in the brackets is too small to have any appreciable effect. Omitting this term, and putting $4e$ for $\frac{b^2}{a^2}$, the integral of the resulting equation in a series is

$$f = 1 - er^2 + \frac{e^2 r^4}{1^2 \cdot 2^2} - \frac{e^3 r^6}{1^2 \cdot 2^2 \cdot 3^2} + \&c.$$

I have already stated (Phil. Mag. for April, p. 281) that according to this result f and $\frac{df}{dr}$ cannot vanish together. Hence, since the velocity of the vibrating particle in any direction perpendicular to the axis of z is $\phi \cdot \frac{df}{dr}$, and parallel to the axis of z , $f \cdot \frac{d\phi}{dz}$, it follows that the fluid cannot be constantly quiescent at a certain distance from that axis. This conclusion is inconsistent with the non-divergence of the vibrations, and would be enough to condemn the whole of the previous investigation, if the reasoning by which it was arrived at be good.

It is now proper for me to state the grounds on which that conclusion is to be rejected. I have maintained, and still

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maintain, that to obtain true and consistent results in hydrodynamics another general equation, distinct from those commonly recognised, is absolutely necessary. In the foregoing investigation no reference has been made to that equation, and the reasoning is on that account defective. Before stating the modification which the reasoning must undergo, I will briefly advert to the evidence for the new equation which I have given in a communication to the Philosophical Magazine for May 1845 (p. 425). I have there obtained, by elementary considerations which admit of no dispute, the following equation,

$$\frac{d\rho}{dt} + \frac{d.\rho V}{ds} + \rho V \left(\frac{1}{R} + \frac{1}{R'} \right) = 0, \quad . \quad . \quad . \quad (I.)$$

in which ρ is the density and V the velocity at the time t , at a point where the principal radii of curvature of the surface cutting at right angles the directions of motion are R and R' ; and ds is the increment of a line drawn always in the direction of the motion of the particles through which it passes. This equation being *true* and *general*, is also *necessary* in the solution of every hydrodynamical problem. It is not identical with the recognised equation,

$$\frac{d\rho}{dt} + \frac{d.\rho u}{dx} + \frac{d.\rho v}{dy} + \frac{d.\rho w}{dz} = 0, \quad . \quad . \quad . \quad (II.)$$

because it rests on *two* principles, that of constancy of mass, and the principle that the directions of motion are normals to a surface of continued curvature; whilst equation (II.) rests only on the principle of the constancy of mass. As however equation (II.) is true, there must exist another equation, which, combined with it, conducts to (I.). That additional equation is the one I contend for. I derive it from the single principle, that the directions of motion in a *given* element are at each instant normals to a surface of continued curvature,—a condition necessary for the continuity of the motion,—and I have shown how, by its combination with (II.), the equation (I.) is arrived at (see Camb. Phil. Trans., vol. vii. part 3, p. 385; and Phil. Mag., S. 3, vol. xx. April 1842). Lastly, in a particular instance I have obtained an absurd result by employing legitimately only the commonly received equations (Phil. Mag. for May 1845, p. 429). The evidence for the necessity of the new equation seems therefore to be in every respect complete. I am not, however, at present aware of any other use required to be made of it than that of deducing equation (I.). That deduction being once made, it follows that in every hydrodynamical problem equation (I.) must *supersede* equation (II.).

I proceed now to state the modification which the reasoning in the question under consideration receives by using equation (I.). The equation (A.), cited in my last communication from the *Mécanique Analytique*, was derived from equation (II.), and from the following three equations,

$$\frac{a^2 \cdot d\rho}{\rho dx} + \left(\frac{du}{dt}\right) = 0, \quad \frac{a^2 \cdot d\rho}{\rho dy} + \left(\frac{dv}{dt}\right) = 0, \quad \frac{a^2 \cdot d\rho}{\rho dz} + \left(\frac{dw}{dt}\right),$$

the supposition being at the same time made that $(d \cdot \phi f) = udx + vdy + wdz$, ϕ being a function of z and t only, and f a function of x and y only. The analogous equation obtained by using (I.) in the place of (II.) differs from (A.) in only one, but that a very important particular; viz. *that the variation of the co-ordinates must be from point to point in the direction of motion*, whilst in (A.) x, y, z are independent variables. Introducing that limitation into (A.), and putting ds for the increment of space along the line of motion, the equation becomes

$$\frac{d^2 \cdot f\phi}{dt^2} - \left(a^2 - \left(\frac{d \cdot f\phi}{ds}\right)^2\right) \frac{d^2 \cdot f\phi}{ds^2} + 2 \cdot \frac{d \cdot f\phi}{ds} \cdot \frac{d^2 \cdot f\phi}{ds dt} \Big] - a^2 \cdot \frac{d \cdot f\phi}{ds} \left(\frac{1}{R} + \frac{1}{R'}\right) = 0; \quad (A')$$

and at the same time, by analytical geometry,

$$\begin{aligned} \left(\frac{1}{R} + \frac{1}{R'}\right) \left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} + \frac{f^2}{\phi^2} \cdot \frac{d\phi^2}{dz^2}\right)^{\frac{1}{2}} &= \frac{d^2 f}{dy^2} \cdot \frac{df^2}{dx^2} + \frac{d^2 f}{dx^2} \cdot \frac{df^2}{dy^2} \\ &- 2 \frac{d^2 f}{dx dy} \cdot \frac{df}{dx} \cdot \frac{df}{dy} + \frac{f^2}{\phi^2} \cdot \frac{d\phi^2}{dz^2} \left(\frac{df}{dx^2} + \frac{df}{dy^2}\right) \\ &+ \left(\frac{f}{\phi} \cdot \frac{d^2 \phi}{dz^2} - \frac{2f}{\phi^2} \cdot \frac{d\phi^2}{dz^2}\right) \left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2}\right). \end{aligned}$$

Now since

$$u = \phi \cdot \frac{df}{dx}, \quad v = \phi \cdot \frac{df}{dy}, \quad \text{and} \quad w = f \cdot \frac{d\phi}{dz},$$

it follows that, corresponding to the maximum value of f (which we have supposed to be unity),

$$u = 0, \quad v = 0, \quad \text{and} \quad w = \frac{d\phi}{dz}.$$

The motion is consequently parallel to the axis of z , and for this line of motion the above equations give

$$\begin{aligned} \frac{d^2 \phi}{dt^2} - \left(a^2 - \frac{d\phi^2}{dz^2}\right) \frac{d^2 \phi}{dz^2} + 2 \frac{d\phi}{dz} \cdot \frac{d^2 \phi}{dz dt} - a^2 \cdot \frac{d\phi}{dz} \left(\frac{1}{R} + \frac{1}{R'}\right) &= 0, \\ \left(\frac{1}{R} + \frac{1}{R'}\right) \cdot \frac{d\phi}{\phi dz} &= \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2}. \end{aligned}$$

For the right-hand side of the second equation we have already substituted $-\frac{b^2}{a^2}$, so that the first equation becomes

$$\frac{d^2\phi}{dt^2} - \left(a^2 - \frac{d\phi^2}{dz^2}\right) \frac{d^2\phi}{dz^2} + 2 \frac{d\phi}{dz} \cdot \frac{d^2\phi}{dzdt} + b^2\phi = 0. \quad (B')$$

This is the general equation for determining ϕ . It is identical, as might have been anticipated, with equation (B.).

By integrating (B') to the first order of approximation (see Phil. Mag. for April, p. 279), I found the only value of ϕ expressible in finite terms to be

$$\phi = m \cos q(z - a't + c),$$

the constant a' being such that

$$a'^2 = a^2 + \frac{b^2}{q^2}.$$

Setting out with this first approximation, and including terms which contain the third power of m , I find the following expressions for ϕ and a'^2 :

$$\left. \begin{aligned} \phi &= m \cos q(z - a't + c) \\ &- \frac{m^2 q^3 a'}{3b^2} \sin 2q(z - a't + c) \\ &- \frac{m^3 q^4}{4b^2} \left(\frac{q^2 a^2}{b^2} + \frac{7}{8} \right) \cos 3q(z - a't + c) \end{aligned} \right\} \quad \dots \quad (1.)$$

$$a'^2 = a^2 + \frac{b^2}{q^2} + m^2 q^2 \left(\frac{2q^2 a^2}{3b^2} + \frac{5}{12} \right) \dots \dots \quad (2.)$$

The equation (1.) serves to define the motion parallel to the axis of z , and the equation (2.) determines the rate of propagation (a'), which, it thus appears, is not altogether independent of m . I have before stated, that the different terms of the expression for ϕ account for *harmonics*.

Let us now investigate the equation which defines the motion transverse to the axis of z . For this purpose, on account of the independence of the variables in ϕ and f , suppose $\phi = 0$. Then $u=0$ and $v=0$. Hence the motion at each point of the plane corresponding to $\phi=0$ is parallel to the axis of z . For any point of this plane R and R' each become infinitely great.

But $\left(\frac{1}{R} + \frac{1}{R'}\right) \cdot \frac{1}{\phi}$ may have a finite value. Hence the general expression for $\frac{1}{R} + \frac{1}{R'}$ gives for this particular case,

$$\left(\frac{1}{R} + \frac{1}{R'}\right) \frac{f}{\phi} \cdot \frac{d\phi}{dz} = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} - \frac{2}{f} \cdot \left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} \right);$$

2 B 2

at the same time the equation (A'.) becomes

$$f \cdot \frac{d^2 \phi}{dt^2} - \left(a^2 - f^2 \cdot \frac{d\phi^2}{dz^2} \right) f \cdot \frac{d^2 \phi}{dz^2} + 2f^2 \cdot \frac{d\phi}{dz} \cdot \frac{d^2 \phi}{dz dt} \\ - a^2 \cdot f \cdot \frac{d\phi}{dz} \left(\frac{1}{R} + \frac{1}{R'} \right) = 0.$$

Hence substituting the above value of $\frac{1}{R} + \frac{1}{R'}$, we have

$$f \cdot \frac{d^2 \phi}{\phi dt^2} - \left(a^2 - f^2 \cdot \frac{d\phi^2}{dz^2} \right) f \cdot \frac{d^2 \phi}{\phi dz^2} + 2f^2 \cdot \frac{d\phi}{dz} \cdot \frac{d^2 \phi}{\phi dz dt} \\ = a^2 \cdot \left(\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} \right) - \frac{2a^2}{f} \cdot \left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} \right).$$

In this equation must be substituted for the factors involving ϕ , the several values which they have when $\phi=0$. These may be obtained from equation (1.), which clearly shows that when $\phi=0$,

$$q(z - a't + c) = \frac{\pi}{2}.$$

This process having been gone through, the following is the resulting equation for determining f :—

$$\left. \begin{aligned} \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} - \frac{2}{f} \cdot \left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} \right) + \frac{b^2}{a^2} f + \frac{2ma'q^3}{a^2} \cdot f(f-1) \\ - \frac{m^2}{a^2} \cdot \left\{ \frac{16q^6 a'^2}{3b^2} f(f-1) - q^4 f(f^2-1) \right\} = 0 \end{aligned} \right\} (3.)$$

It is now clear that where $f=0$, we have also $\frac{df}{dx} = 0$ and $\frac{df}{dy} = 0$. The equations are consequently consistent with non-divergence of the vibrations.

The equations (1.), (2.) and (3.), define the motion and propagation of the motion of the vibrating particles. To complete the theory of non-divergent vibrations, an expression for the density remains to be obtained. This I reserve for a future opportunity.

Cambridge Observatory,
Oct. 21, 1848.