TURBINES.

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In his introduction to Mr. Woodbridge's thesis on "Turbines," in the recent November and December numbers of this journal, Prof. Wood seems to imply that the former's solution, as based "upon the resolution of the pressures of elementary volumes and their moments," is the first employment of such a method.

Without the slightest desire to detract from the credit due to Mr. Woodbridge's achievements, it is perhaps permissible, in justice to the present writer, to state that this method of solution was developed by him several years ago and has been used in his classes at this institution ever since; with such variations in the details from Mr. Woodbridge's, however, as to render its publication in this journal of some interest to students of applied mechanics. It would have been published as a whole long before, had not the writer supposed that most teachers of applied mechanics would probably be led to discover it for themselves, in the effort to smooth the pathway of their pupils. Some portions have already appeared in periodicals, as will be mentioned presently, and these will not be repeated here, except as it may be desirable to present clearer and more direct proofs than those previously given.

The theory of the friction-less turbine consists mainly in the establishing of the first six equations of the article by the writer in this journal for May, 1884, (pp. 333 and 334), with the proviso that at whatever *uniform* angular velocity, ω , the wheel be run, the vane-tangent at entrance of wheel-channel is to coincide in direction with the relative velocity of the water issuing from the guides, and also that all channels are filled.

The question of the proper speed for maximum efficiency will not be entered upon here; Weisbach's treatment of that point being simple and clear, and exact enough for practical purposes.

Of these six equations, (3), (4) and (5), are too simple to need explanation here; (2) is Bernoulli's theorem for steady flow in a *fixed* pipe or channel, proved in all text-books on hydraulics, and dealing with the absolute velocity at entrance; (6) may be called

Bernoulli's theorem for a *rotating* pipe (in horizontal plane) and deals with relative velocities (it is also equation (6) in Mr. Woodbridge's thesis); while (1), the equation for *power* (work per unit of time), is founded on the principle of work and energy, and is assumed, without special proof, by Weisbach and Bresse.

Now it is a simple matter, by algebraic combination of some of these six equations, including (1), to deduce Rankine's expression for the power, as being equal to the difference in the *moment of the momentum* (of the mass of water flowing per unit of time) at entrance and exit, multiplied by the angular velocity. Hence, the establishment of Rankine's equation (call it equation (*R*)); it is equation (8) in Mr. Woodbridge's article) by special proof is equivalent to the establishing of equation (1), and in our list of equations to be established, we may discard (1) and replace it by (*R*), if a student objects to (1) as obscure in principle. In such a case, therefore, it is important to prove (*R*) by as direct and convincing a method as possible, and such a method (by elementary volumes) is the one used by Mr. Woodbridge and the writer, not only for equation (*R*), but for (6).

Rankine's proof of equation (*R*) is very obscure to the beginner, and Prof. Unwin's, in the *Encyclopædia Britannica*, though attempting to be more specific, seems extremely vague.

In the March number, 1880, of Van Nostrand's *Eng. Magazine*, may be found an article by the writer, in which the action of jets on vanes, and also the theory of the "turbine of free deviation" are investigated by the method of elementary volumes and their pressures. In the treatment of the former, attention is called to Rankine's neglect of a portion of the stream, as Mr. Woodbridge does in his recent paper; in treating the latter, the pressure of each element of the stream against the vane, at a given instant, is found, its moment about the wheel-axis formed, these moments summed, and this sum multiplied by the angular velocity ω to obtain the power exerted; the result of which (in equation (10) of that article) $= \omega \times$ change of "moment of the momentum" of the flow per unit of time; *i. e.*, this equation shows the truth of equation (*R*) for that particular case.

The form of proof of equation (*R*), prepared for the writer's classes, for any kind of turbine, will now be presented, consisting, as it does, of a consideration of the individual actions of all the

elementary vertical prisms into which we may divide the volume of water, which, at a given instant, occupies the wheel-channels.

Though following the same method (of elementary volumes) as that of Mr. Woodbridge, the present solution is, perhaps, less complicated, in that it does not require us to obtain a detailed expression for the pressure between each prism and the vane, while sacrificing nothing in rigidity of proof, nor in satisfactory minuteness of detail. It is convenient to divide it into three propositions:

PROP. I.—In any (outward- or inward-flow) turbine, if the pressures of the water against the vanes (or partitions between the channels) have reached a state of permanency, the wheel having a uniform angular velocity $= \omega$; then the work done by the water on the wheel per unit of time (or power) is

$$L = \omega \Sigma (P a) \quad (a)$$

in which $\Sigma (P a)$ denotes the sum of the moments, about the wheel axis, of all the elementary pressures of the water against the vanes at any definite instant. (These pressures are, of course, normal to the surface of the vanes at their respective points of application. This proposition calls for no proof here, and we accordingly pass on to

PROP. II.—Let M be the mass of a small particle or "material point," *Fig. 1*, which, in describing a plane curve, A, B, C , under the action of one or more forces. Let AB be any element of the path, described in a time, dt , while P is the resultant of all the forces acting on the particle at this point of the path. Call the velocity at A , w ; that at B , w' ; the acceleration due to P , call p .

Of course, $p = P \div M$. By the principle of the composition of motions, w' is the diagonal formed on w'' at B , parallel and equal to the w at A , as one side, and $p \, dt$ the increment of velocity due to the action of P during dt , as the other side of a parallelogram.

From any point, O , in the plane drop perpendiculars upon w'' , $p \, dt$, and w' ; also, upon the w at A . Denote the lengths of these perpendiculars by the symbols shown in the figure. Then the same geometrical relation will hold between w'' ($= w$), w' , and $p \, dt$, and their perpendiculars, as if ST were a parallelogram of forces. That is,

$$w' \, k' = w'' (k + \overline{DN}) - (p \, dt) a$$

(The negative sign is due to the circumstances of this particular figure).

But as the distance AB is considered shorter and shorter, $\overline{DN}$, ultimately vanishes as compared with the finite k ; hence we may write, since

$$\begin{aligned} w'' &= w \text{ and } p = P \div M, \\ M(w \, k - w' \, k') &= (P a) \, dt \end{aligned} \tag{b}$$

or, more strictly, with full calculus notation,

$$M \, d(w \, k) = P a \, dt \tag{c}$$

That is, by multiplying the mass of the moving particles by the change which the product of its velocity by the perpendicular let fall upon the latter from any point, experiences in an element of time, dt , we obtain the product of $d(w \, k)$ by the moment of the resultant force about the same point. (As is well known, this principle can be used to prove Kepler's law that the radius vector of a planet sweeps over equal areas in equal times.)

PROP. III.—*The sum of the moments about the wheel axis of the pressures exerted by the water in the wheel passages against the vanes at a definite instant, is equal to the loss (or gain) of "moment of momentum," between entrance and exit of the mass of water flowing per unit of time ("moment of momentum" being a name given by Rankine to the product of the mass of a particle $\times$ its absolute velocity, $\times$ perpendicular distance of this absolute velocity from the arbitrary centre of moments).*

Conceive the water which at any instant lies in the turbine passages to be subdivided into a great number of vertical rings,

concentric with the wheel, of *equal volume* and of such small thickness that at the end of the time dt , each ring fills the exact space occupied by its neighbor at the beginning of dt . (See *Fig. 2*.) The dotted line MN shows the absolute path, and initial and final absolute velocities, w_1 and w_n , of a particle, as the ring to which it belongs passes completely through the wheel. In the small time

dt , in which any one ring passes into its consecutive position, the portion A (of the ring) included between any two neighboring partitions (see *Fig. 3*.) passes into a position A' in the next ring-space, and in this new position, on account of the "state of permanency" of the flow, has an absolute velocity, w' , equal to that, w'' , which the portion B had at the beginning of the dt , while the

length of the perpendicular k' , dropped on w' from the wheel axis, is the same in value as for B at the beginning of the dt , since the positions A' and B are in the same ring. In other words, the "moment" of the absolute velocity for A' (i. e., the $w' k'$ of *Fig. 1*), about the wheel axis, at the end of dt , is the same in value as that for B , at the beginning of dt .

Now consider by itself (*i. e.*, as a "free body,") the prism 1, at the entrance of any one of the wheel channels, *Fig. 4*. P'_1 and P''_1 are the pressures of the partitions against it; let P_1 represent their resultant; (it is, of course, the equal and opposite of the resultant pressure of this prism against the wheel at this instant). Since the pressures of the neighboring prisms against 1 have lines of action containing O the wheel axis, those pressures have no moments about O , and the moment ($P_1 a_1$) of P_1 about O is therefore equal to the moment about O of the resultant of P'_1, P''_1 , and the pressures on LK and RS . During the time dt , prism 1 moves to position 1' in the next ring-space, w_1 changes to w_2 , k_1 to k_2 . Hence, from Prop. II, equation (b) with dM = mass of elementary prism, we have

$$dM (w_1 k_1 - w_2 k_2) = P_1 a_1 dt.$$

Similarly for the other prisms in this channel, between RS and n , as they, simultaneously with 1, in time dt , move into their consecutive positions, we may write (remembering that all the dM 's are equal)

$$dM (w_2 k_2 - w_3 k_3) = P_2 a_2 dt$$

$$dM (w_3 k_3 - w_4 k_4) = P_3 a_3 dt$$

and so on, up to

$$dM (w_{n-1} k_{n-1} - w_n k_n) = P_n a_n dt.$$

Adding these equations, member to member, we obtain

$$\Sigma (P a) \text{ for one channel} = \frac{d M}{d t} (w_1 k_1 - w_n k_n).$$

Hence, if the wheel has m channels

$$\Sigma (P a) \text{ for whole wheel} = \frac{m d M}{d t} (w_1 k_1 - w_n k_n).$$

Now $m d M$ is the mass of water which leaves the wheel in time $d t$; hence, denoting by Q the volume of water passing per unit of time, and by γ the weight of a unit of volume, we have

$$m d M = \frac{Q \gamma}{g} d t$$

whence

$$\Sigma (P a) = \frac{Q \gamma}{g} [w_1 k_1 - w_n k_n] \quad (d)$$

= the total moment with which the water acts on the wheel, as some would express it.

Finally, therefore, substituting from equation (a), we have the power of the wheel

$$L = \omega \Sigma (P a) = \omega \frac{Q \gamma}{g} [w_1 k_1 - w_n k_n] \quad (e)$$

which is equation (B), whose truth it was required to establish by a consideration of elementary volumes and pressures.

If, following Rankine's form of expression, we resolve the absolute velocities w_1 and w_n , of entrance and exit, of the water particles into components radial and tangential to the two circumferences of entrance, and exit, we may also write (from similar triangles)

$$L = \omega \frac{Q r}{g} \left[V_1 r_1 - V_n r_n \right] \quad \text{or } \begin{matrix} (f) \\ (R) \end{matrix}$$

Fig. 6.

in which V_1 and V_n are the initial and final "velocities of whirl" as Rankine calls them, and r_1 and r_n the corresponding wheel radii. (This is Mr. Woodbridge's equation (8)).

We now pass to the proof of the equation referred to as (6) [both in Mr. Woodbridge's article and in that of the writer in JOURNAL FRANKLIN INSTITUTE, May, 1884,] showing the relation between the relative velocities and the pressures at entrance and at exit (Bernoulli's theorem for a uniformly rotating horizontal pipe or channel, as the equation might be called). The closely related

problem of the frictionless motion of a small solid or particle in a pipe of any form (of single curvature) rotating with a uniform angular velocity in a horizontal plane about a fixed axis (vertical, of course,) may be treated at the same time with but little extra detail, and is accordingly presented, being solved by a method of the writer's, not heretofore published in any periodical, but according to his experience, of a very direct nature in satisfying the student mind. In *Fig. 6*, OB' is the absolute path described by the particle in a time dt ; *i. e.*, in time dt the pipe has rotated through an angle ωdt , carrying O to O' , and the particle itself, having at this instant a relative velocity $= c$ along the pipe, has described a distance $= c dt$, $= \overline{OB}$, $= \overline{O'B'}$, along the pipe. Let ρ = the radius of curvature of the guide at this point, while r the radial distance of O and O' from the axis C is the radius of the circular arc OO' . The absolute velocity w , of the particle, is the diagonal formed on ωr and c , and by the principle of the composition of motions the absolute motion OB' may be considered to result from the simultaneous compounding (by parallelograms) of the following component motions, *viz.*:

(*To be continued.*)

A NEW METHOD TO DETERMINE THE MOON'S MASS
CORRECTLY, FOLLOWED BY A DISCUSSION UPON
THE THEORY OF TIDES FULLY IDENTIFYING THE PHENOMENON UPON DYNAMICAL PRINCIPLES.

BY L. D'AURIA.

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Let M = mass of the earth;

μ = mass of the moon;

D = distance between their respective centres;

ω = angular velocity around their common centre of gravity;

x = distance between this centre of gravity and that of the earth;

ρ = radius of the earth at a latitude whose *sine* $= \frac{1}{\sqrt{3}}$;