XXXIV. On the effect of absorption on the resolving power of prism trains, and on methods of mechanically compensating this effect

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more than half the true value, while it made the value of e twice as great as it ought to have been.

If we know the value of e , we can at once deduce the number of molecules in a c.c. of gas at 0° C. and 760 mm. pressure. For if N is this number, then, since e is the same as the charge on the hydrogen ion in the electrolysis of solutions,

$$Ne = 1.22 \times 10^{10}$$

since
$$e = 3.4 \times 10^{-10}, \quad N = 3.6 \times 10^{19}.$$

This number is well within the limits of the various determinations made by the methods of the kinetic theory. The above method for determining N has the advantage of not involving a knowledge of the shape of the molecules, or any assumption as to the nature of the effects produced when two molecules come into collision.

In making the experiments described in this paper, I have had the help of my assistant Mr. Everett.

XXXIV. *On the Effect of Absorption on the Resolving Power of Prism Trains, and on Methods of Mechanically Compensating this Effect.* By F. L. O. WADSWORTH*.

[Plate VIII.]

IN previous investigations † of the resolving-power of prism spectroscopes, it has been generally assumed, 1st, that the illumination is uniform over the wave-front passing

* Communicated by the Author. The general investigation on resolving-power, of which the present paper forms a part, was begun nearly eight years ago, but the work has been much interrupted and delayed by other more imperative and immediate demands upon the writer's time and attention. The publication of various parts of the work has therefore been more scattered and irregular than might otherwise have been the case. Since this paper was written, Prof. Campbell, with whom I fortunately had an opportunity of discussing it, has called my attention to a paper on the same subject that was published about a year ago by Dr. Reese of the Lick Observatory Staff. On examining this paper (*Astrophys. Journ.* xiii. p. 199, April 1901) I find that Dr. Reese has deduced the general equation corresponding to (11) of this paper (for $\delta_1 = 0$), but has confined his investigations of the effects of absorption to the particular case where the value of β is very small. Hence his conclusion (p. 206) "that the resolving power of the Mills Spectrograph is diminished by less than one half of one per cent. by the absorption of the prisms" is incomplete, and for that reason erroneous and misleading as a general conclusion. I have therefore allowed my own paper to stand as it was originally written, acknowledging Dr. Reese's priority of publication as above.

† See Rayleigh, *Phil. Mag.* vol. viii. pp. 261-264; vol. ix. pp. 49-56; vol. xlii. p. 167; *Enc. Brit.* vol. xxiv. pp. 430-439; Schuster, *ibid.* vol. xxii. pp. 373-374; also papers by the writer, *Astrophysical Journal*, vol. i. p. 52; vol. iii. pp. 176 and 321; vol. iv. p. 54; vol. vi. p. 27; vol. xvi. p. 1; *Phil. Mag.* vol. xliii. p. 317.

through the prism-train ; 2nd, that the material of the latter is of the same temperature and refractive index throughout.

The general theoretical results agree so closely with those obtained in practice that it is certain that the error, if any, introduced by either of these assumptions must be small under usual conditions. It is not, however, *a priori* certain that this would be true when the absorption of the glass, or the differential temperature-change in different parts of the prisms, becomes abnormally large, as it may do in certain regions of the spectrum, and under conditions of use of the instrument such as are met with in astrophysical work. The following investigation was undertaken in order to determine the actual magnitude of the effects to be expected from large variations in absorption and optical density over the transmitted wave-front.

The most general expression for the distribution in intensity in the image of a point formed at the focal plane of a telescope is

$$I^2 = \left[\iint \frac{i}{\lambda \rho} \sin \frac{2\pi}{\lambda} (at - \rho) dx dy \right]^2, \quad . \quad . \quad . \quad (1)$$

where i is the amplitude of vibration of any element dx, dy in the wave-front : ρ is the distance of this element from a point p in the focal plane at which the intensity is desired.

In reducing this general expression to the form usually given*, the following assumptions are made† :—

1st. That the amplitude of vibration is constant, $i = \text{const.}$

2nd. That the wave-front passing the diffracting aperture is truly spherical, *i. e.*,

$$x^2 + y^2 + z^2 = f^2.$$

3rd. The light which unites to form the image is strictly monochromatic, *i. e.*, $\lambda = \text{const.}$

Under these assumptions (1) may be reduced at once to the form given by Lord Rayleigh,

$$I^2 = \frac{1}{\lambda^2 f^2} \left[\iint \sin \left(\frac{2\pi \xi}{\lambda f} + \frac{2\pi \eta}{\lambda f} \right) dx dy \right]^2 \\ + \frac{1}{\lambda^2 f^2} \left[\iint \cos \left(\frac{2\pi \xi}{\lambda f} + \frac{2\pi \eta}{\lambda f} \right) dx dy \right]^2, \quad . \quad (2)$$

where ξ, η are the coordinates of any point in the focal-plane image.

* Rayleigh, article "Wave Theory," *Enc. Brit.* vol. xxiv. p. 430.

† See also Popular Astronomy, "Problems Relating to . . . Resolving Power of Telescopes," vol. v. pp. 528–536 ; and Astrophysical Journal, vol. xvi. pp. 266 *et seq.*

If we retain i as a variable function,

$$i = \phi(xy), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

but consider the second and third assumptions as correct, we obtain similarly

$$I^2 = \frac{1}{\lambda^2 f^2} \left[\iint \phi(xy) \sin\left(\frac{2\pi\xi}{\lambda f} + \frac{2\pi\eta}{\lambda f'}\right) dx dy \right]^2 \\ + \frac{1}{\lambda^2 f'^2} \left[\iint \phi(xy) \cos\left(\frac{2\pi\xi}{\lambda f} + \frac{2\pi\eta}{\lambda f'}\right) dx dy \right]^2 \quad . \quad (4)$$

In the case of the prism spectroscope the aperture of the prism-train is generally rectangular, and if all parts of the beam of light falling on the first prism-face are of uniform intensity, the diminution in the amplitude of vibration due to the absorption of the glass of the train will be uniform along any line parallel to the refracting edge of the prisms. Along any line at right angles to this the change will be expressed by the law

$$i = i_0 e^{-\frac{\beta}{2}(l-l_0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where i and i_0 are the amplitudes of vibration of the ray after traversing thicknesses of l and l_0 centimetres of glass, and β is the coefficient of absorption.

If we choose the two lines just defined as the axes of y and x respectively we have for $\phi(xy)$ in (3)

$$\phi(xy) = i_0 e^{-\beta x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

since the length of path, $l-l_0$, through any given point in the prism-train is directly proportional to x , the distance of that point from the central ray.

The function $oe^{-\beta x}$ is independent of y , and, for a rectangular aperture, the limits of integration for y in both terms of (4) are constant. Hence we obtain at once for the distribution in intensity along the axis of ξ in the focal plane

$$I_\xi^2 = \frac{i_0^2 d^2}{\lambda^2 f^2} \left[\int_{-\frac{b}{2}}^{+\frac{b}{2}} e^{-\beta x} \sin \frac{2\pi\xi}{\lambda f} x dx \right]^2 + \frac{i_0^2 d^2}{\lambda^2 f'^2} \left[\int_{-\frac{b}{2}}^{+\frac{b}{2}} e^{-\beta x} \cos \frac{2\pi\xi}{\lambda f} x dx \right]^2 \\ = \frac{i_0^2 d^2}{\lambda^2 f^2} [C^2 + S^2]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For convenience put

$$\frac{2\pi\xi}{\lambda f} = k.$$

Then integrating each of the two terms of (7) by parts we obtain

$$\begin{aligned} C &= k \left(e^{\frac{Bb}{2}} - e^{-\frac{Bb}{2}} \right) \cos \frac{kb}{2} - B \left(e^{\frac{Bb}{2}} + e^{-\frac{Bb}{2}} \right) \sin \frac{kb}{2} \\ S &= k \left(e^{\frac{Bb}{2}} + e^{-\frac{Bb}{2}} \right) \sin \frac{kb}{2} + B \left(e^{\frac{Bb}{2}} - e^{-\frac{Bb}{2}} \right) \cos \frac{kb}{2} \end{aligned} \quad (8)$$

and therefore

$$I_{\xi}^2 = C^2 + S^2 = \frac{e^{Bb} + e^{-Bb} - 2 \cos kb}{k^2 + B^2} \quad (9)$$

Resubstituting for k its value in terms of ξ , and also expressing $\cos kb$ in terms of $\sin \frac{kb}{2}$ we finally obtain

$$I_{\xi}^2 = i_0^2 \frac{b^2 d^2}{\lambda^2 f^2} \frac{e^{Bb} + e^{-Bb} - 2 - 4 \sin^2 \frac{\pi \xi b}{\lambda f}}{(Bb)^2 + 4 \left(\frac{\pi \xi b}{\lambda f} \right)^2} \quad (10)$$

When the origin of coordinates is not at the centre of the horizontal aperture, and the limits of integration, b_1 and $-b_2$, are unequal, we obtain similarly

$$I_{\xi}^2 = i_0^2 \frac{d^2}{\lambda^2 f^2 (k^2 + B^2)} \left[e^{-2Bb_1} + e^{2Bb_2} - 2 \cos k(b_1 + b_2) e^{-B(b_1 + b_2)} \right] \quad (11)$$

When there is no absorption $B=0$ and $i=1$, and both (9) and (10) reduce at once to the usual form for rectangular aperture, *i. e.*,

$$I^2 = \frac{b^2 d^2}{\lambda^2 f^2} \frac{\sin^2 \frac{\pi \xi b}{\lambda f}}{\left(\frac{\pi \xi b}{\lambda f} \right)^2} \quad (12)$$

For very small values of B the distribution in intensity is practically the same as when there is no absorption. For very large values of B , on the other hand, the term e^{Bb} remains the only one of importance in the numerator, and we have

$$I_{\xi}^2 = i_0^2 \frac{b^2 d^2}{\lambda^2 f^2} \frac{e^{Bb}}{(Bb)^2 + 4 \left(\frac{\pi \xi b}{\lambda f} \right)^2} \quad (13)$$

In order better to show the transition between these two extreme cases, the values of the function I_a^2 have been computed for six intermediate values of B as follows:—

$$\begin{aligned} Bb &= 0.5754, \\ Bb &= 1.3862, \\ Bb &= 2.1972, \\ Bb &= 2.7726, \\ Bb &= 4.6052, \\ Bb &= 7.825. \end{aligned}$$

and

In these computations the quantity $i_0^2 \cdot \frac{b^2 d^2}{\lambda^2 f^2}$, whose value is determined simply by the intensity of illumination in the incident wave-front and by the dimensions of the instrument,

TABLE I.

$\pi \frac{a}{a_0}$	$I^2 = \frac{\sin^2 x}{x^2}$	I_a^2 for different values of Bb .					
		$Bb = .5754$	1.3862.	2.1972.	2.7726.	4.6052.	7.825.
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.10 π	.9675	.9681	.9696	.9739	.9766	.9862	.9925
0.20 π	.8751	.8769	.8870	.8997	.9100	.9436	.9745
0.30 π	.7368	.7411	.7598	.7880	.8114	.8791	.9459
0.4 π	.5754	.5792	.6086	.6535	.6900	.7988	.9079
0.5 π	.4053	.4138	.4527	.5132	.5628	.7102	.8620
0.6 π	.2545	.2646	.3105	.3826	.4416	.6207	.8128
0.7 π	.1353	.1462	.1955	.2732	.3313	.5351	.7599
0.8 π	.0547	.0654	.1140	.1916	.2558	.4624	.7079
0.9 π	.0119	.0217	.0663	.1382	.1989	.4003	.6571
1.0 π	.0000	.0084	.0464	.1089	.1629	.3494	.6079
1.1 π	.0080	.0146	.0452	.0968	.1422	.3087	.5621
1.2 π	.0243	.0292	.0528	.0935	.1303	.2752	.5194
1.3 π	.0392	.0428	.0605	.0923	.1222	.2477	.4798
1.4 π	.0468	.0497	.0632	.0886	.1139	.2232	.4432
1.5 π	.0450	.0472	.0587	.0805	.1026	.2008	.4095
1.6 π	.0358	.0381	.0486	.0686	.0889	.1798	.3773
1.7 π	.0229	.0250	.0358	.0554	.0736	.1611	.3495
1.8 π	.0108	.0130	.0239	.0434	.0626	.1441	.3233
1.9 π	.0027	.0049	.0155	.0344	.0529	.1298	.3002
2.0 π	.0000	.0020	.0120	.0297	.0462	.1182	.2789
2.2 π	.0072	...	...	...	.0422	.1011	.2429
2.4 π	.0159	...	...	...	.0402	.0885	.2116
2.6 π	.0136	...	...	...	.0347	.0764	.1871
2.8 π	.0045	...	...	...	.0270	.0651	.1655
3.0 π	.0000	...	...	...	.0210	.0562	.1470
3.2 π	...	...	...	...	...	.049	.1317
3.5 π	...	...	...	...	...	.043	.1169
4.0 π	...	...	...	...	...	...	.0888
5.0 π	...	...	...	...	...	...	.0582
6.0 π	...	...	...	...	...	...	.0413

is so chosen that the intensity I_ξ^2 at the centre of the diffraction image, $\xi=0$, is the same (unity) in each case. The abscissæ are expressed in terms of the angular coordinates $\frac{\alpha}{\alpha_0}$ which are determined by the relations

$$\alpha = \frac{\xi}{f}, \quad \alpha_0 = \frac{\lambda}{b},$$

and therefore

$$\frac{\pi \xi b}{\lambda f} = \pi \frac{\alpha}{\alpha_0} = x. \quad . \quad . \quad . \quad (14)$$

The comparative values of I_α^2 for values of α ranging from 0 to $6\alpha_0$, and for the values of Bb given above, are tabulated in columns 3, 4, 5, 6, 7, and 8 of Table I. (p. 359) and plotted as full-line curves in fig. 1 (Pl. VIII.). For comparison the value of I^2 for $B=0$ is also tabulated in column 2 of this table, and plotted as a dotted curve in fig. 1.

An inspection of this table, or the corresponding curves of fig. 1, shows that the effect of increasing absorption is to change the form of the curve $I^2=f(\alpha)$, 1st, by gradually obliterating the points of maxima and minima on each side of the centre, and, 2nd, by gradually increasing its apparent width *mm*. We have to consider this change in form (a) on the resolving-power of the instrument, (b) on the visual or photographic appearance of the spectral lines.

(a) The visual criterion for resolving-power is that two lines of equal intensity may be considered as definitely "resolved" when the illumination at the centre of their superimposed diffraction images is not more than 0.81 of the illumination at the centres of the images themselves. In the case of an instrument unaffected by absorption (equation 11), this corresponds to a separation of the two lines by an amount

$$\sigma_0 = \alpha_0 = \frac{\lambda}{b}$$

as defined by (14).

In the case of two equal lines separated by an angular interval σ the relative intensities in the physical image of the double source at the points corresponding to the centres of the lines themselves and a point midway between them will be

$$\frac{I_m^2}{I_c^2} = \frac{2I_{\sigma/2}^2}{I_0^2 + I_\sigma^2} \quad . \quad . \quad . \quad . \quad (15)$$

and the lines will be resolved or not according as I_m^2 is less or greater than $0.81 I_c^2$.

Applying this criterion to the case of the images I_α^2 shown in fig. 1, we find for a constant separation $\sigma = \alpha_0$ the following values of $\frac{I_m^2}{I_c^2}$:—

$$\begin{aligned} \text{For } Bb = .5754 : \frac{I_m^2}{I_c^2} &= \frac{.8276}{1.0084} = .82 + \\ Bb = 1.3862 : \frac{I_m^2}{I_c^2} &= \frac{.9054}{1.0464} = .86 + \\ Bb = 2.1972 : \frac{I_m^2}{I_c^2} &= \frac{1.0264}{1.1089} = .92 + \\ Bb = 2.7726 : \frac{I_m^2}{I_c^2} &= \frac{1.1256}{1.1629} = .97 - \\ Bb = 4.6052 : \frac{I_m^2}{I_c^2} &= \frac{1.4204}{1.3494} = 1.05 + \\ Bb = 7.825 : \frac{I_m^2}{I_c^2} &= \frac{1.724}{1.608} = 1.07 + \end{aligned}$$

which shows that the effect of absorption is to decrease the resolving-power of the instrument by an amount depending both on the value of B and of b .

To find the separation necessary for resolution we must select a value of σ such that

$$\frac{2I_{\sigma/2}^2}{I_0^2 + I_\sigma^2} = 0.81. \quad . \quad . \quad . \quad . \quad (16)$$

If we put

$$e^{Bb} + e^{-Bb} - 2 = 4D,$$

$$(B'b)^2 = 4F^2,$$

and

$$\frac{\pi}{\alpha_0} \cdot \frac{\sigma}{2} = x,$$

we obtain from (9) and (15)

$$(F^2 + x^2) \left[\frac{D}{F^2} + \frac{D + \sin^2 2x}{F^2 + (2x)^2} \right] \simeq 2.5(F^2 + \sin^2 x), \quad . \quad (17)$$

from which we can determine x and σ necessary for resolution for any given values of Bb . We thus obtain

For	$Bb = .5754$	$\sigma = 1.007\alpha_0$
	$Bb = 1.3862$	$\sigma = 1.04\alpha_0$
	$Bb = 2.1972$	$\sigma = 1.11\alpha_0$
	$Bb = 2.7726$	$\sigma = 1.18\alpha_0$
	$Bb = 4.6052$	$\sigma = 1.54\alpha_0$
	$Bb = 7.825$	$\sigma = 2.56\alpha_0$

These relative values of Bb and σ are plotted (circles) in fig. 2 (Pl. VIII.). As is there shown, the relation between these quantities can be closely represented by the dotted curve, the equation of which is

$$\sigma = \alpha_0 \{1 + .0253 (Bb)^2\} \quad . \quad . \quad . \quad . \quad (18)$$

an expression which is simpler and more convenient for computation than the more exact formula (16).

An inspection of the curve in fig. 2 shows that for small values of Bb , *i. e.* for small absorptions, the actual resolving-power of the instrument σ is very nearly the same as the theoretical resolving-power α_0 . For large values, however, it is very considerably reduced; nearly one-third for a value of Bb equal to 4.

To obtain a better idea of the physical conditions corresponding to the different values of Bb considered above, it is desirable to express these values in terms of the intensities at the centre and edges of the transmitted wave-front. We have from (5) and (6)

$$\frac{i}{i_0} = e^{-Bx}.$$

Since the intensities at any two points are proportional to the square of the amplitudes of vibration, we have for the centre of the transmitted wave-front $x=0$, and the edge

$x = \frac{b}{2}$, the ratio

$$\frac{i_c^2}{i_e^2} = e^{\mp Bb},$$

and therefore for the different values of Bb already considered

$$\begin{aligned}
\frac{i_{\epsilon}^2}{i_0^2} &= \frac{9}{16} = \left(\frac{3}{4}\right)^2 \text{ for } Bb = .5754 \\
\frac{i_{\epsilon}^2}{i_0^2} &= \frac{1}{4} = \left(\frac{1}{2}\right)^2 \quad , \quad = 1.3862 \\
\frac{i_{\epsilon}^2}{i_0^2} &= \frac{1}{9} = \left(\frac{1}{3}\right)^2 \quad , \quad = 2.1972 \\
\frac{i_{\epsilon}^2}{i_0^2} &= \frac{1}{16} = \left(\frac{1}{4}\right)^2 \quad , \quad = 2.7726 \\
\frac{i_{\epsilon}^2}{i_0^2} &= \frac{1}{100} = \left(\frac{1}{10}\right)^2 \quad , \quad = 4.6052 \\
\frac{i_{\epsilon}^2}{i_0^2} &= \frac{1}{2500} = \left(\frac{1}{50}\right)^2 \quad , \quad = 7.825
\end{aligned}$$

or for the extreme cases the intensity at the edge of the beam traversing the base of the prism-train is fifty-six per cent. and one twenty-fifth of one per cent. respectively, of the intensity at the centre.

To find the relation between the quantities Bb , $\frac{i_{\epsilon}^2}{i_0^2}$, and β , the coefficient of absorption of the glass, we have similarly from (5) and (6)

$$Bx = \frac{\beta}{2} (l - l_0).$$

Also from the geometry of the prism-train

$$\frac{x}{l_1 - l_0} = \frac{\sqrt{1 - n^2 \sin^2 \phi/2}}{2 \sin \phi/2},$$

for a single prism of refracting angle ϕ placed at minimum deviation. For N similar prisms placed *en train* we have similarly

$$\frac{x}{l - l_0} = \frac{x}{N(l_1 - l_0)} = \frac{\sqrt{1 - n^2 \sin^2 \phi/2}}{2N \sin \phi/2},$$

and therefore in general

$$B = \beta N \frac{\sin \phi/2}{\sqrt{1 - n^2 \sin^2 \phi/2}}, \quad . \quad . \quad . \quad (19)$$

from which we can determine the corresponding values of Bb and β for any given spectroscope. Some examples may be of interest.

For the Bruce spectrograph of the Yerkes Observatory,

which is one of the largest and most recently constructed instruments of this class, we have for b , ϕ , N , and n the following values* :—

$$b = 5.1 \text{ cms.}$$

$$\phi = 63^\circ 35' \text{ (mean value).}$$

$$N = 3.$$

$$n = \begin{cases} 1.653 & \text{for } \lambda = 5500 \dagger \\ 1.678 & \text{,, } \lambda = 4300 \\ 1.693 & \text{,, } \lambda = 3900 \dagger. \end{cases}$$

Introducing these values and solving for Bb we find

$$Bb = \beta \times \begin{cases} 16.4 & \text{for } \lambda = 5500 \\ 17.2 & \text{,, } \lambda = 4300 \\ 17.8 & \text{,, } \lambda = 3900 \end{cases}$$

The absorption of the glass of which this prism-train is composed (0.102) has been determined by Vögel and Müller‡. The values of β computed from their results are as follows:—

$$\begin{array}{lll} \text{For wave-length } 5500, & \beta = .021 \\ \text{,,} & \text{,,} & 4300, \quad \beta = .071 \\ \text{,,} & \text{,,} & 3900, \quad \beta = .37 \end{array}$$

For the visual region of the spectrum, therefore, the value of Bb for this spectroscope is about $.34$, and the corresponding value of σ , as given by (18), is less than $1.003 \alpha_0$, *i. e.* the resolving-power is diminished less than one-third of one per cent. by the result of absorption.

In the photographic region of the spectrum, however, the case is quite different. At wave-length $\lambda = 4300$, which is about the middle of the photographic region, the value of Bb for the Bruce spectroscope is $17.2 \times .071 = 1.22$, and the corresponding value of σ is 1.04 , *i. e.* the resolving-power at this point in the spectrum is reduced about four per cent. by the result of absorption. At wave-length $\lambda = 3900$ the value of Bb is 6.59 , the value of σ is $2.10 \alpha_0$, and the resolving-power is therefore diminished by more than fifty per cent.

For prism spectroscopes of larger size or greater resolving-power than the one above considered the effect of absorption may become serious, even in the visible part of the spectrum.

* "The Bruce Spectroscope of the Yerkes Observatory," E. B. Frost, *Astrophysical Journal*, vol. xv. pp. 12-17, Jan. 1902.

† Estimated by extrapolation from the values given by Frost.

‡ *Astrophysical Journal*, vol. v. p. 82 (1897).

particularly if a denser glass than 0.102 is used. Prism spectroscopes have been constructed having an aperture of from 2.2 cms. to 2.5 cms., and prism-trains of from 10 to 13 dense flint prisms*. These prism-trains have been constructed of very dense flint-glass whose coefficients of absorption are probably considerably higher than those for 0.102, which is unusually white and transparent. Assume the coefficients to be only 33 per cent. (1/3) larger. Then, since the ratio $Bb \div \beta$ is roughly only one-half as large for the Bruce instrument as for the Young and Grubb instruments, we have for the latter:

In the visible spectrum

$$Bb \simeq 1.00 \quad \sigma = 1.025 \alpha_0.$$

In the photographic region

$$\lambda = 4300 \quad Bb \simeq 3.7 \quad \sigma = 1.35 \alpha_0$$

$$\lambda = 3900 \quad Bb \simeq 20.0 \quad \sigma = 11.0 \alpha_0.$$

In this last case the resolving-power of the spectroscope is reduced more than 90 per cent. by absorption in the neighbourhood of the H and K lines.

It is evident from these results that it is useless to increase the theoretical resolving-power r of our prism spectroscopes beyond a certain point. For every value of β there will be a certain limiting value of α_0 , and hence of r , for which the practical resolving-power of the instrument as affected by absorption (which we will call R_β) will be a maximum. To find this value we must transform equations (18) and (19) so that all variables are expressed in terms of r and R_β .

From the well-known relation

$$r = \frac{m}{\alpha_0} = \frac{\lambda}{\Delta\lambda}$$

we have obviously

$$\frac{R_\beta}{r} = \frac{\alpha_0}{\sigma}.$$

We have also the relation

$$Nb \frac{\sin \phi/2}{\sqrt{1-n^2 \sin^2 \phi/2}} = \frac{r}{2} \cdot \frac{d\lambda}{dn} \dagger \quad \dots \quad (20)$$

* Young, 'Nature,' vol. iii. p. 110; Grubb, Monthly Notices R. A. S. vol. xxxi. pp. 36-38.

† See Astrophysical Journal, vol. i. p. 55 (1895).

From (18), (19), and (20) we then obtain at once

$$R_{\beta} = \frac{r}{1 + \cdot 0063 \left(\beta r \frac{d\lambda}{dn} \right)^2}, \quad \dots \quad (21)$$

which is similar in general form to the expression for the purity of the spectrum with slits of finite width*.

To find the value of r for which R_{β} is a maximum we have

$$\frac{d}{dr} R_{\beta} = \frac{1 - \cdot 0063 \left(\beta \frac{d\lambda}{dn} r \right)^2}{\left\{ 1 + \cdot 0063 \left(\beta \frac{d\lambda}{dn} r \right)^2 \right\}^2} = 0, \quad \dots \quad (22)$$

or for $R_{\beta} = \text{max.}$

$$r_m = 12 \cdot 59 \frac{1}{\beta} \cdot \frac{dn}{d\lambda}, \quad \dots \quad (23)$$

and for the corresponding maximum value of R_{β}

$$R_{\beta}(\text{max.}) = 6 \cdot 29 \frac{1}{\beta} \cdot \frac{dn}{d\lambda} = \frac{1}{2} r_m, \quad \dots \quad (24)$$

that is, the maximum practical resolving-power that can be attained with a prism spectroscope is one-half what we could attain with the same spectroscope if there were no absorption.

For the flint glass 0.102 which we have been considering, the value of the ratio

$$\frac{dn}{d\lambda}$$

for wave-length $\lambda = 3900$ is about 4250. The value of β for the same region is, as we have seen, about .37. Hence for this glass we have

$$\begin{aligned} R_{\beta}(\text{max.}) &\simeq 72000, \\ \text{for } r_m &\simeq 144000. \end{aligned}$$

The relation between r and R_{β} , as given by (21), is tabulated in Table II. and plotted in Curve 1 of fig. 3 (Pl. VIII.). It will be seen that the increase in R_{β} is very small (less than 7 per cent. of the maximum) between the points $r = 100000$ and $r_m = 144000$; and it would therefore in general be inadvisable to go beyond the first point, if we proposed to work in the neighbourhood of the H and K lines with a prism spectroscope of dense flint glass. The theoretical resolving-power of the Bruce instrument for this region is about 153000, $\left\{ \frac{dn}{d\lambda} (t_1 - t_2) \right\}$, and is therefore

* See Phil. Mag. vol. xliii. pp. 333-339.

TABLE II.

r .	R_g .		
	II.	III.	IV.
25,000	24375		
50,000	44680	48765	49275
75,000	59152		
100,000	67750	90830	95970
125,000	71675		
150,000	72420	122220	137070
200,000	68880	142460	169820
300,000	56760	157140	217830
400,000	46440	152720	239440
500,000	38750	141850	244200
600,000	33060	129420	239160
800,000	25440	107200	217280
1000,000		90100	192700

considerably larger than can be used to best advantage. The Young spectroscope has a theoretical resolving-power r of nearly 300000 in the same region of the spectrum, but as will be seen from the Table or the Curve the effect of absorption cuts down the practical resolving-power to about 57000, actually less than that which an instrument of only one-fourth the size would possess.

It is at once evident from these results that if high-power prism spectroscopes are to be used in the investigation of the photographic region of the spectrum, the use of the extra dense flint glass, so commonly employed in the past, must be avoided, not only on the score of light-efficiency, but, as now appears, on the score of photographic resolving-power and purity as well. The use of lighter flint reduces the theoretical resolving-power, r , of any given prism-train by

decreasing the value of the dispersion-coefficient $\frac{dn}{d\lambda}$; but this may be easily and even advantageously compensated by increasing the refracting angle ϕ of the prisms. The advantages of this latter construction on the score both of light-efficiency and economy of material were pointed out by the writer several years ago*, and it has been adopted in all the large spectroscopes recently constructed at Allegheny†.

* Astrophysical Journal, vol. ii. p. 264 (1895).

† Among others may be mentioned the spectroscope of the Philadelphia Observatory, the spectroscope of the Lowell Observatory, and the spectroheliograph of the Philadelphia Observatory (just completed). Prof. Lord, of the Emerson McMillen Observatory, has also recently replaced the battery of dense flint prisms first constructed for his star spectroscope with one of light flint; and I believe Dr. Gill, of the Cape Observatory, is likewise considering this change.

To show the advantage of the lighter glasses on the score of photographic purity the values of $R\beta$ have also been computed for the dense flint, 0.93, and the light flint, 0.340, of Schott's table. The constants n , β , and $\frac{dn}{d\lambda}$ for wave-length 3900 for these two glasses are as follows:—

$$\begin{array}{lll} 0.93 & n_{3900} = 1.66; & \frac{dn}{d\lambda} = 3500; \quad \beta \simeq 14 \\ 0.340 & n_{3900} = 1.61; & \frac{dn}{d\lambda} = 3100; \quad \beta \simeq 0.8. \end{array}$$

The values of $R\beta$ for these values of $\frac{dn}{d\lambda}$ and β are tabulated in columns III. and IV. of Table II., and plotted as curves II. and III. of fig. 3 (Pl. VIII.). The points of maximum for these curves are found at

$$\begin{array}{ll} r_m \simeq 315000 & \text{for glass } 0.93, \\ r_m \simeq 488000 & \text{,, , } 0.340, \end{array}$$

and the corresponding values of $R\beta$ (max.) are

$$\begin{array}{ll} 157000 & \text{for } 0.93, \\ 244000 & \text{for } 0.340. \end{array}$$

As in the preceding case the form of the curves is such that it is not desirable to go to the extreme maximum values of $R\beta$, but to stop at points corresponding to $r \simeq 200000$ and $\simeq 350000$ respectively.

For the visible portion of the spectrum the values of n , $\frac{dn}{d\lambda}$, and β for the first glass considered, 0.102, are as follows:—

$$\begin{array}{lll} n = 1.653; & \frac{dn}{d\lambda} \simeq 1150, & \beta = 0.21. \\ \therefore \beta \frac{d\lambda}{dn} = 0.00018; \\ r_m \simeq 740000; \\ R\beta \text{ (max.)} \simeq 370000. \end{array}$$

The above results show that while the Bruce spectroscope is quite efficient for the visible part of the spectrum, it is quite otherwise for the photographic region in the neighbourhood of the H and K lines. Such an instrument would not be well suited, therefore, for solar work, particularly of a spectroheliographic nature.

For regions of the spectrum lying below wave-length 3900, materials having a smaller coefficient of absorption than light flint or even crown-glass must be used. In the case of glass, indeed, we soon reach a limit beyond which a further reduction in density will be actually disadvantageous. As will be seen from (21) and (23) we reach an absolute maximum for R_β when the quantity

$$\beta \frac{d\lambda}{dn}$$

becomes an absolute minimum. For the various flint glasses already considered, the absorption-coefficient β decreases more rapidly than the dispersion-coefficient $\frac{dn}{d\lambda}$; hence the quotient of these two coefficients decreases with diminishing index of refraction.

But as we pass from the flints to the crowns the reverse takes place, and the value of β decreases less rapidly than the dispersion. Hence we soon reach a point at which the quotient

$$\beta \frac{d\lambda}{dn}$$

is an absolute minimum for glass. Thus for the crown 0.203 examined by Vogel we have in the neighbourhood of the H line

$$n_{3900} \simeq 1.532, \quad \frac{dn}{d\lambda} \simeq 1650, \quad \text{and} \quad \beta \simeq 0.054.$$

Hence for crown 0.203

$$\beta \frac{d\lambda}{dn} = 0.000032,$$

which is nearly 25 per cent. larger than the corresponding quantity (0.000026) for the light flint 0.340. Hence, in spite of its diminished absorption crown-glass would not be as good a material to use for the prisms as light flint.

We might perhaps find a flint for which the quotient of the two coefficients is somewhat less than in the case of 0.340. It is, however, safe to say that the value of this quotient cannot be much reduced, and that the maximum practical resolving-power which we can attain in spectroscopes with glass prisms will not exceed 250,000 units in the photographic region of the spectrum, nor 400,000 in the visual region.

In order to attain still higher resolving-powers we must use other materials than glass. Of those available for this

purpose quartz, rock-salt, and fluorite are among the best. Unfortunately I have not been able to find any reliable data on the coefficients of absorption of these materials for different wave-lengths. Although they are considerably less than those for glass for both very short and very long wave-lengths, they are sufficiently large to exercise an appreciable effect on the resolving-power in these ultra-visible portions of the spectrum. It is also necessary to remember that certain kinds of glass, as well as other materials of which prisms may be constructed, frequently show strong selective absorption; and certain regions even in the visible part of the spectrum where this occurs may exhibit a lack of apparent definition in consequence. In fact, in all discussions involving questions of resolving-power, the possibility of the effects of large local or general absorptions must be kept in mind.

(*b*) The second effect of absorption which we have to consider is that on the general form and apparent "width" of the individual spectral lines. This effect is in many respects more pronounced and striking than that on the resolving power just considered; for as will be seen by an inspection of fig. 1 (Pl. VIII.), the apparent width is increased in a much greater ratio than the resolving-power is diminished by increased absorption.

As long as there is a well-marked minimum on each side of the centre, we may consider the edges of the spectral image to be defined by the position of these minima; or, in other words, consider that the apparent "width" of the line is $m_0 m_0$ (fig. 1). For the value $\beta=0$ this width for an infinitely narrow slit and monochromatic radiation is of course $2\alpha_0$. The effect of an increasing absorption is to increase all the ordinates of the diffraction pattern relatively to the central one (for $\alpha=0$); but the ones in the neighbourhood of the first minima are increased proportionally more than the others, so that these points of minima are gradually obliterated, and become indistinguishable for values of Bb greater than 1.5. The edges of the line then become less and less sharply defined and clearly marked. Their apparent position depends somewhat on the absolute as well as the relative intensity at different parts in the diffraction pattern; but in general they may be taken to lie near the limits $\pm m$ defined by the relations

$$f(\pm m) = I_{\pm m}^2 \simeq \cdot 04 I_0^2;$$

i. e. near the points at which the intensity of illumination in the image first falls to less than 4 per cent. of the intensity at the geometrical centre.

Assuming this criterion, we have for the relation between

Bb (or β) and w , the apparent "width," the following values (from Table I. and fig. 1):—

For Bb = 000	$w = m_0 m_0 = 2.00 \alpha_0$,
Bb = .575	$w = 2.04 \alpha_0$,
Bb = 1.386	$w = m_1 m_1 = 2.16 \alpha_0$,
Bb = 2.197	$w = 3.66 \alpha_0$,
Bb = 2.77	$w = m_2 m_2 = 4.8 \alpha_0$,
Bb = 4.61	$w = m_3 m_3 \simeq 7.6 \alpha_0$,
Bb = 7.83	$w \simeq 13.5 \alpha_0$.

The apparent width of the line for the last value of Bb is therefore nearly seven times as great as the theoretical width, $2\alpha_0$, for zero absorption. This large increase in w , together with the accompanying reduction in the practical resolving-power, R_β , is quite sufficient to explain the apparent haziness and lack of definition frequently observed in certain portions of a spectrum formed by prisms with strong selective absorption.

The effect of absorption on the form and distribution in intensity in the spectral image, and hence on the resolving-power of the prism-train, may be entirely eliminated by mechanical diaphragming of the prism aperture. By referring to equation (7) we see that the form of the expression for I^2 is the same as would be obtained if the illumination were uniform over the wave-front, and the latter were limited by an aperture whose length (along the x axis) is b , as before, but whose width at any point is the ordinate to the exponential curve

$$y = ae^{-Bx} \dots \dots \dots (25)$$

If, therefore, we limit the aperture of the prism-train by a diaphragm whose width is y^{-1} , the illumination remaining the same as in (7), we obtain for the distribution in intensity in the image of the spectral line

$$\begin{aligned} I_\xi^2 &= \frac{i_0^2 a^2}{\lambda^2 f^2} \left[\int_{-\frac{b}{2}}^{+\frac{b}{2}} e^{-Bx} e^{Bx} \sin \frac{2\pi\xi}{\lambda f} x dx \right]^2 \\ &+ \frac{i_0^2 a^2}{\lambda^2 f^2} \left[\int_{-\frac{b}{2}}^{+\frac{b}{2}} e^{-Bx} e^{Bx} \cos \frac{2\pi\xi}{\lambda f} x dx \right]^2 \\ &= \frac{i_0^2 a^2}{\lambda^2 f^2} \left[\int_{-\frac{b}{2}}^{+\frac{b}{2}} \cos \frac{2\pi\xi}{\lambda f} x dx \right]^2 = \frac{i_0^2 a^2 b^2 \sin^2 \frac{\pi\xi b}{\lambda f}}{\lambda^2 f^2 \left(\frac{\pi\xi b}{\lambda f} \right)^2}, \quad (26) \end{aligned}$$

which is the same as obtained with rectangular aperture of height d , unaffected by absorption, save that the absolute intensities at every point in the pattern are diminished in the ratio

$$\frac{I_{00}^2}{I^2} = \frac{i_0^2 a^2}{d^2} \cdot \cdot \cdot \cdot \cdot \cdot \quad (27)$$

Since the intensity in the transmitted wave-front at the thin edge of the prism is the same as in the incident beam ($l=0$), we have, by making $x = -\frac{b}{2}$ in (6),

$$i_0^2 e^{Bb} = 1, \quad \text{or} \quad i_0^2 = e^{-Bb} \cdot \cdot \cdot \cdot \cdot \quad (28)$$

Also if we choose a in (25) such that the greatest width of the diaphragm is equal to d , we have

$$d = ae^{\frac{Bb}{2}},$$

or
$$a^2 = d^2 e^{-Bb} \cdot \cdot \cdot \cdot \cdot \quad (29)$$

Hence we have from (27), (28), and (29)

$$\frac{I_{00}^2}{I^2} = e^{-Bb} \cdot \cdot \cdot \cdot \cdot \quad (30)$$

For the Bruce spectroscope before considered the value of Bb for wave-length 3900 is, as already computed,

$$Bb \simeq 6.59;$$

$$\therefore \frac{I_{00}^2}{I^2} \simeq .000\,002.$$

Although so much light is lost in diaphragming the aperture as above described, the intensity in the resultant spectral image will compare somewhat more favourably with that secured under the actual conditions of absorption with full aperture, because the "width" of the line is so much reduced when the diaphragm is used. The intensity at the centre of the diffraction-image with full aperture bd (see 10) is

$$I_0^2 = i_0^2 \frac{b^2 d^2}{\lambda^2 f^2} \cdot \frac{e^{Bb} + e^{-Bb} - 2}{(Bb)^2} \cdot \cdot \cdot \cdot \quad (31)$$

The ratio between this intensity and the intensity of the image formed by the same instrument, unaffected by absorption, will be

$$\frac{I_0^2}{I^2} = \frac{e^{-Bb}}{(Bb)^2} \{ e^{Bb} + e^{-Bb} - 2 \} \cdot \cdot \cdot \quad (32)$$

For the same case considered before for which Bb is 6.59, we have

$$\frac{I_0^2}{I^2} \sim .023.$$

Finally, for the ratio between the intensities with and without a diaphragm we have from (30) and (32),

$$\frac{I_{00}^2}{I_0^2} = \frac{e^{-Bb}(Bb)^2}{e^{Bb} + e^{-Bb} - 2} \quad \dots \quad (33)$$

which gives us for the Bruce instrument in the H regions

$$\frac{I_{00}^2}{I_0^2} \sim .0001. \quad \dots \quad (33a)$$

The curve bounding the edges of the diaphragm for any given case is given by the equation [see (25) and (29)],

$$y = \pm e^{+\beta N \frac{\sin \phi/2}{\sqrt{1-n^2 \sin^2 \phi/2}} \left(x - \frac{b}{2}\right)} \quad \dots \quad (34)$$

For the above case

$$B = \beta N \frac{\sin \phi/2}{\sqrt{1-n^2 \sin^2 \phi/2}} = 1.29, \\ b = 5.1.$$

If with these values of b and B we make $d=b$ and compute y for different values of x , we shall find the form of the required diaphragm to be that shown in fig. 4 (Pl. VIII.). The widths at the two ends are 0.008 cm. and 5.1 cms. respectively, with the wide end toward the base of the prism.

When so placed this diaphragm will entirely eliminate the effect of the variable absorption, but only, as indicated by (33) and (33a), at a very large sacrifice of intensity of illumination. This device can therefore be used only when we have abundance of light. In other cases the only means we have of reducing the effect of absorption for any given material is to decrease the resolving-power of the prism-train either by reducing the aperture or the number of prisms, or both.

General Conclusion.

The results of the preceding investigation show that in the case of prismatic spectroscopes we are confined to certain resolving-powers. We are limited in one direction by the physical properties of the glass now at our disposal, in another by the difficulty, we might almost say impracticability, of obtaining blocks of crystalline minerals of larger than a certain size, and in a third by lack of sufficient illumination.

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Our only means of obtaining more powerful instruments seems to lie, therefore, in the further development of the grating-spectroscope. The maximum resolving-powers that have thus far been attained with the latter class of instruments are somewhat less than 400,000 units *, about the maximum which, as we have now shown, is attainable with prismatic spectroscopes in the visible region of the spectrum. As the writer has already shown † instruments having at least three times this resolving-power are not only exceedingly desirable, but absolutely necessary to the successful investigation of certain spectral phenomena. We must therefore aim to ultimately produce gratings at least 40 cms. diameter. A machine designed to rule gratings of this size was begun by the writer seven years ago (1896), but owing to the many interruptions to which reference was made at the beginning of the article, work on it was only recently completed ‡. The construction of a machine of about the same size was begun in 1899 by Professors Michelson and Stratton at the University of Chicago; but work on this has also been much interrupted and delayed by the resignation of Professor Stratton to accept the directorship of the new National Bureau of Standards at Washington. More recently still Jewell, at Johns Hopkins, has been remodelling one of the Rowland dividing-engines to rule 10 in. (25 cm.) gratings.

It is to be hoped, therefore, that through the united efforts of all of those now working on this problem, gratings of a considerably larger size than we now have may soon become a commercial possibility.

Allegheny Observatory, November, 1902.

XXXV. *The Use of Contour Integration in the Problem of Diffraction by a Wedge of any Angle.* By H. S. CARSLAW, Professor of Pure and Applied Mathematics in the University of Sydney, N.S.W. §

IN the Appendix on Diffraction to his Adams Prize Essay || Macdonald gives a discussion of the two-dimensional

* See paper on "Resolving Power of Telescopes and Spectroscopes for Lines of Finite Width," *Phil. Mag.* vol. xliii. p. 320, May 1897.

† *Loc. cit.* The large solar grating-spectroscopes of the new Allegheny Observatory are designed for gratings of 25 cms. and 40 cms. aperture Report of Director, 1900.

‡ See Report of the Director, Allegheny Observatory, 1900, p. 22. The final setting up of the new engine has been still further delayed by the delay in the completion of the new Observatory and Laboratories.

§ Communicated by the Author.

|| 'Electric Waves,' by H. M. Macdonald, Camb. 1902.

Fig 1.

Fig. 2

Fig. 3.

Fig. 4