On the Passage of an Oscillator Wave-train through a Plate of Conducting Dielectric

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XXXIII. *On the Passage of an Oscillator Wave-train through a Plate of Conducting Dielectric.* By GEORGE UDNY YULE, Demonstrator in Applied Mathematics, University College, London.

- I. Introduction.
 - II. The Phenomena at the First Surface of the Plate.
 - III. The Phenomena at the Second Surface of the Plate.
 - IV. The Intensity of the Transmitted Ray.
 - V. The Intensity of the Reflected Ray.
 - VI. Discussion of an Experimental Case and a Correction.
 - VII. The Numerical Value of some of the previous Results.
- Synopsis of Symbols.

I. *Introduction.*

THE problem treated in these pages was suggested by the following experimental results, for a fuller account of which I may refer to a previous paper †.

A series of wave-trains generated by an oscillator O were allowed to propagate themselves along a double-wire circuit LL' about 100 metres in length. At the middle of the circuit the wires were run through a jar containing a certain depth AB of distilled water, alcohol, or a very dilute electrolyte.

Fig. 1.

The intensity of the ray transmitted by this slightly absorbent layer was determined by an electrometer E placed at a quarter wave-length (2.25 metres) from the closed end of the circuit. This intensity varied periodically with the thickness of the

* Read January 11, 1895.

† Phil. Mag. vol. xxxvi. p. 531, Dec. 1893. Prelim. Paper in the Proc. Roy. Soc. vol. liv. p. 96. A sketch of an approximate theory was given in the Phil. Mag. paper, but several assumptions there made were unjustifiable.

layer, as in the analogous case of a ray of light and a "thin plate."

The layer AB was what Boltzmann has termed a "conducting dielectric," *i. e.* both its conducting and dielectric properties were of importance. The theory of a non-conducting plate is much simpler: a problem equivalent to it has recently been treated by Dr. E. H. Barton *, to whose work I shall have several occasions to refer. In the more complex problem I have assumed, we have simply a plane-fronted damped wave-train travelling through an insulating medium and falling at normal incidence on an infinite slab of conducting dielectric. The magnetic permeability of both the plate and the surrounding medium are taken as unity.

The theoretical transmission-curve obtained on these assumptions does not agree well with the experimental one, the divergence being the same in sign and order of magnitude as that noticed by Dr. Barton in his case. We have in fact idealized the experimental facts too far in endeavouring to simplify the analysis: the electrolyte was not at all infinite in extent, and at least one important correction is obviously necessary (section VI.). The experiments can only be regarded as a very rough illustration of the problem, and as giving the *raison d'être* of this paper.

The general results obtained seem to be of considerable interest. The intensities of the reflected rays, the phase-changes, and so on, for damped wave-trains reflected from such a plate differ from those for steady rays in some cases very considerably.

For convenience and brevity, the surrounding medium is hereinafter generally referred to as "the dielectric," and the conducting slab as "the plate" or "electrolyte."

II. *The Phenomena at the First Surface of the Plate.*

In this section we will deal with the phenomena occurring at the first surface of the "electrolyte" only.

The direction of propagation of the ray is chosen as the

* Preliminary Paper, Proc. Roy. Soc. vol. liv. p. 85; more fully in Thesis for the D.Sc. London, 1894, or Wied. *Ann.* vol. liii. p. 513, 1894. Final paper, Proc. Roy. Soc., read April 1893 (not yet published).

positive direction of the axis of z and the origin is taken indefinitely close to the interface.

The suffix 1 always refers to the dielectric and the suffix 2 to the plate.

Fig. 2.

Let X_1 be the electric force in the dielectric and X_2 that in the plate. Under the limitations we have set ourselves—plane waves and non-conducting dielectric—Maxwell's equations assume the simple forms

$$\beta_1 \frac{d^2 X_1}{dt^2} = \frac{d^2 X_1}{dz^2} \quad . \quad . \quad . \quad (1)$$

$$\left(\alpha_2 + \beta_2 \frac{d}{dt} \right) \frac{dX_2}{dt} = \frac{d^2 X_2}{dz^2}, \quad . \quad . \quad . \quad (2)$$

where we have abbreviated Maxwell's notation by putting

$$\beta_1 = \mu_1 \kappa_1, \quad \beta_2 = \mu_2 \kappa_2, \quad \alpha_2 = 4\pi \mu_2 C_2, \quad . \quad . \quad (3)$$

μ_2 being the magnetic permeability of the plate, κ_2 its dielectric constant, C_2 its conductivity; $\mu_1 \kappa_1$ are the same constants referring to the dielectric, C_1 being zero. As already stated, we will only take the case of both media being non-magnetic, or

$$\mu_2 = \mu_1 = 1.$$

As a solution of equations (1) and (2) we select

$$\left. \begin{aligned} X_1 &= A_1 e^{p(t - \sqrt{\beta_1} z)} + A_1' e^{p(t - \sqrt{\beta_1} z)} \\ X_2 &= B_2 e^{p(t - qz)} \end{aligned} \right\} \quad . \quad . \quad (4)$$

where t must be not less than zero. p is either wholly imaginary or complex according as the incident ray is steady or damped: q is as yet undetermined; A_1 is the amplitude of the incident wave-train, A_1' of the reflected wave-train, and B_2 of the transmitted wave-train.

Inserting the value of X_2 given in (4) in equation (2), and carrying out the differentiation, we get

$$q = \sqrt{\beta_2} \left(1 + \frac{\alpha_2}{\beta_2 p} \right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

so q is also partly imaginary.

To determine $\frac{A_1'}{A_1}$ and $\frac{B_2}{A_1}$ we require another pair of equations. These will be given us by the interface conditions: the conditions namely, that there must be no discontinuous jump in the values of either the tangential electric force or the tangential magnetic force, at any time, in passing from the dielectric to the electrolyte. That is, we must have

$$\left. \begin{aligned} X_1 &= X_2, \\ \frac{dX_1}{dz} &= \frac{dX_2}{dz}, \end{aligned} \right\} . \quad . \quad . \quad . \quad . \quad (6)$$

where

$$z=0, \quad t=\text{anything}.$$

Inserting the values of X_1 and X_2 in (6) we get, putting $z=0$,

$$\left. \begin{aligned} A_1 + A_1' &= B_2, \\ p \sqrt{\beta_1} (-A_1 + A_1') &= -pq B_2; \end{aligned} \right\} . \quad . \quad . \quad (7)$$

and solving these two equations for $\frac{A_1'}{A_1}$, $\frac{B_2}{A_1}$,

$$\frac{A_1'}{A_1} = - \frac{\frac{q}{\sqrt{\beta_1}} - 1}{\frac{q}{\sqrt{\beta_1}} + 1},$$

$$\frac{B_2}{A_1} = \frac{2 \sqrt{\beta_1}}{\sqrt{\beta_1} + q}.$$

If we write for brevity

$$\left. \begin{aligned} \lambda &= \frac{\sqrt{\beta_1}}{q}, \\ \frac{A_1'}{A_1} &= -\frac{1-\lambda}{1+\lambda}, \\ \frac{B_2}{A_1} &= \frac{2\lambda}{1+\lambda}, \end{aligned} \right\} \dots \dots \dots (8)$$

λ contains q , which is partly imaginary; so we must separate the real and imaginary quantities and rationalize. Let p_1 be the real part of p , and p_2 the imaginary part, so that

$$p = -p_1 + p_2 i,$$

and let us write

$$\left. \begin{aligned} p_1 &= \eta \cos \chi, \\ p_2 &= \eta \sin \chi. \end{aligned} \right\} \dots \dots \dots (9)$$

In this notation then

$$\left. \begin{aligned} p &= -\eta e^{-i\chi}, \\ \text{period} &= \frac{2\pi}{p_2}, \\ \text{logarithmic decrement} &= \frac{2\pi p_1}{p_2}; \end{aligned} \right\}$$

and using the value of q in (5),

$$\lambda = \sqrt{\frac{\beta_1}{\beta_2}} \frac{1}{\left(1 - \frac{\alpha_2}{\eta \beta_2} e^{i\chi}\right)^{\frac{1}{2}}}.$$

The denominator of this expression is

$$\left(1 - \frac{\alpha_2}{\eta \beta_2} \cos \chi - i \frac{\alpha_2}{\eta \beta_2} \sin \chi\right)^{\frac{1}{2}},$$

so if we put

$$\left. \begin{aligned} \rho \cos \theta &= 1 - \frac{\alpha_2}{\eta \beta_2} \cos \chi \\ \rho \sin \theta &= \frac{\alpha_2}{\eta \beta_2} \sin \chi \end{aligned} \right\}, \dots \dots (10)$$

or

$$\begin{aligned} \rho &= \left[1 - \frac{\alpha_2}{p_2 \beta_2} \sin 2\chi + \left(\frac{\alpha_2}{p_2 \beta_2} \sin \chi\right)^2\right]^{\frac{1}{2}} \\ \tan \theta &= \frac{\frac{\alpha_2}{p_2 \beta_2} \sin^2 \chi}{1 - \frac{1}{2} \frac{\alpha_2}{p_2 \beta_2} \sin 2\chi}, \end{aligned}$$

the denominator becomes $\rho e^{-\frac{i\theta}{2}}$. Writing

$$r = \sqrt{\frac{\beta_1}{\beta_2}} \frac{1}{\sqrt{\rho}}, \quad \dots \dots \dots (11)$$

as a further abbreviation we have

$$\lambda = r e^{\frac{i\theta}{2}}.$$

We can now proceed to rationalize the values of $\frac{A_1'}{A_1}$ and $\frac{B_2}{A_1}$ given in (8). Inserting our value of λ ,

$$\frac{A_1'}{A_1} = - \frac{1 - r e^{\frac{i\theta}{2}}}{1 + r e^{\frac{i\theta}{2}}} = - \frac{1 - r \cos \frac{\theta}{2} - i r \sin \frac{\theta}{2}}{1 + r \cos \frac{\theta}{2} + i r \sin \frac{\theta}{2}}.$$

Multiplying numerator and denominator by

$$1 + r \cos \frac{\theta}{2} - i r \sin \frac{\theta}{2},$$

this becomes

$$- \frac{1 - r^2 - 2 i r \sin \frac{\theta}{2}}{1 + r^2 + 2 r \cos \frac{\theta}{2}},$$

and multiplying again by $1 - r^2 + 2 i r \sin \frac{\theta}{2}$,

$$\frac{A_1'}{A_1} = - \frac{1 + r^4 - 2 r^2 \cos \theta}{(1 + r^2 + 2 r \cos \frac{\theta}{2})(1 - r^2 + 2 i r \sin \frac{\theta}{2})}.$$

If then

$$\tan \psi = \frac{2 r \sin \frac{\theta}{2}}{1 - r^2}, \quad \dots \dots \dots (12)$$

we have finally

$$\frac{A_1'}{A_1} = - \frac{(1 + r^4 - 2 r^2 \cos \theta)^{\frac{1}{2}}}{1 + r^2 + 2 r \cos \frac{\theta}{2}} e^{-i\psi};$$

that is, the ray reflected from the first surface of the plate undergoes a phase-change of $(-\psi)$, the reduction factor or ratio of the *amplitude* of the reflected to that of the incident ray being

$$b = - \frac{(1 + r^4 - 2 r^2 \cos \theta)^{\frac{1}{2}}}{1 + r^2 + 2 r \cos \frac{\theta}{2}} \dots \dots \dots (13)$$

b is a function of the period and damping of the wave-train as well as of the properties of the reflector. If the wave-train be not damped, we have

$$\left. \begin{aligned} \chi &= 90^\circ, \\ \rho &= \sqrt{1 + \left(\frac{\alpha_2}{p_2 \beta_2}\right)^2}, \\ \cos \theta &= \frac{1}{\rho}, \end{aligned} \right\} \quad . \quad . \quad . \quad (14)$$

so the expression for b remains unaltered in form, b being still a function of the period.

But if the conductivity of the electrolyte or α_2 be zero, the angle θ becomes likewise zero, ρ is unity, and we get the ordinary value for b , namely,

$$b = \frac{1 - \sqrt{\frac{\beta_1}{\beta_2}}}{1 + \sqrt{\frac{\beta_1}{\beta_2}}}, \quad . \quad . \quad . \quad . \quad (15)$$

the period and damping of the incident ray having vanished from the expression.

The phase-change ψ is zero for either a perfect insulator or a perfect conductor; θ becoming indefinitely small in the first case, and r in the second.

The refracted wave is dealt with in exactly the same manner. Taking the value of $\frac{B_2}{A_1}$ from (8), and substituting in it the value of λ from (11) we have

$$\begin{aligned} \frac{B_2}{A_1} &= \frac{2\lambda}{\lambda + 1} = \frac{2re^{\frac{i\theta}{2}}}{1 + re^{\frac{i\theta}{2}}} \\ &= \frac{2re^{\frac{i\theta}{2}} \left(1 + r \cos \frac{\theta}{2} - ri \sin \frac{\theta}{2}\right)}{1 + r^2 + 2r \cos \frac{\theta}{2}}. \end{aligned}$$

If we write

$$\tan \psi' = \frac{r \sin \frac{\theta}{2}}{1 + r \cos \frac{\theta}{2}}, \quad . \quad . \quad . \quad (16)$$

then
$$\frac{B_2}{A_1} = \frac{2r}{\sqrt{1+r^2+2r \cos \frac{\theta}{2}}} e^{i(\theta/2-\psi')}.$$

Thus for the transmitted ray at the dielectric-electrolyte surface, the change of phase is $\frac{\theta}{2} - \psi'$, and the ratio of its amplitude to that of the incident ray or c is

$$c = \frac{2r}{\sqrt{1+r^2+2r \cos \frac{\theta}{2}}}. \quad \dots \quad (17)$$

c , like b , is a function of the period and damping of the incident ray; like b it remains a function of the period even for an undamped ray; and like b it becomes a function of the dielectric constants only, if the "electrolyte" be a perfect insulator, the expression in (17) becoming

$$c = \frac{2\sqrt{\frac{\beta_1}{\beta_2}}}{1 + \sqrt{\frac{\beta_1}{\beta_2}}}. \quad \dots \quad (17a)$$

We have now sufficient data to determine the speed of propagation and wave-length in the electrolyte, before going on to deal with the reflexions and refractions at the second surface. Referring back to equation (4) for X_2 , and substituting the values we have determined for B_2 , p , and q , we have

$$X_2 = cA_1 e^{i(\theta/2-\psi')} e^{(-p_1+i p_2)t} e^{\eta \sqrt{\beta_2 \rho} z} \{ \cos(\chi + \theta/2) - i \sin(\chi + \theta/2) \}^z,$$

or retaining only the real terms

$$X_2 = cA_1 e^{-p_1 t + \eta \sqrt{\beta_2 \rho} \cos\left(\chi + \frac{\theta}{2}\right)z} \cos \left\{ p_2 t - \eta \sqrt{\beta_2 \rho} \sin\left(\chi + \frac{\theta}{2}\right)z + \frac{\theta}{2} - \psi' \right\}. \quad \dots \quad (18)$$

Hence the speed of the wave in the electrolyte is

$$V_2 = \frac{p_2}{\eta \sqrt{\beta_2 \rho} \sin\left(\chi + \frac{\theta}{2}\right)} = \frac{\sin \chi}{\sqrt{\beta_2 \rho} \sin\left(\chi + \frac{\theta}{2}\right)}, \quad \dots \quad (19)$$

and the wave-length

$$\lambda_2 = \frac{2\pi}{\eta \sqrt{\beta_2 \rho} \sin\left(\chi + \frac{\theta}{2}\right)} \quad \dots \quad (20)$$

But for the incident wave-train

$$\lambda_1 = \frac{2\pi}{p_2 \sqrt{\beta_1}},$$

therefore

$$\begin{aligned} \frac{\lambda_2}{\lambda_1} &= \sqrt{\frac{\beta_1}{\beta_2}} \frac{p_2}{\eta \sqrt{\rho} \sin\left(\chi + \frac{\theta}{2}\right)} \\ &= \sqrt{\frac{\beta_1}{\beta_2}} \frac{\sin \chi}{\sqrt{\rho} \sin\left(\chi + \frac{\theta}{2}\right)} \quad \dots \quad (21) \end{aligned}$$

ρ and θ being both functions of α_2 and of the period, and χ being a function of the rate of damping of the wave-train, $\frac{\lambda_2}{\lambda_1}$ is (like b and c and the phase-changes) a function of period, damping, and conductivity, except when the latter is zero : when we have simply

$$\frac{\lambda_2}{\lambda_1} = \sqrt{\frac{\beta_1}{\beta_2}} \quad \dots \quad (21a)$$

III. *The Phenomena at the Second Surface of the Plate.*

We now proceed to the inverse case where the wave is passing from the electrolyte into the dielectric.

If X_2 represent as before the electric force in the electrolyte, and X_3 represent the electric force in the second dielectric, we may now write

$$X_2 = B_1 e^{p(t-qz)} + B_1' e^{p(t+qz)}, \quad \dots \quad (22)$$

$$X_3 = A_3 e^{p(t-\sqrt{\beta_1}z)}, \quad \dots \quad (23)$$

the wave in the electrolyte consisting of a direct and a reflected train, and that in the dielectric of a direct train only.

Applying the interface conditions as in the last section, we get for $z=0$

$$B_1 + B_1' = A_3, \quad \dots \quad (24)$$

$$\frac{dX_2}{dz} = \frac{dX_3}{dz} \quad \dots \quad (25)$$

Carrying out the differentiation of (25) and putting $z=0$,

$$-p \sqrt{\beta_1} A_3 = pq(-B_1 + B_1'),$$

or

$$B_1 - B_1' = A_3 \frac{\sqrt{\beta_1}}{q} = A_3 \lambda. \quad (26)$$

Solving for $\frac{B_1'}{B_1}$ and $\frac{A_3}{B_1}$ from (24) and (26) we get

$$\left. \begin{aligned} \frac{A_3}{B_1} &= \frac{2}{1+\lambda} \\ \frac{B_1'}{B_1} &= \frac{1-\lambda}{1+\lambda} \end{aligned} \right\} \quad (27)$$

Using our previous notation of

$$\lambda = re^{\frac{i\theta}{2}}$$

and rationalizing, we obtain finally for the transmitted ray

$$\frac{A_3}{B_1} = \frac{2}{\sqrt{1+r^2+2r \cos \frac{\theta}{2}}} e^{-i\psi'},$$

so that the phase-change for the wave emerging from the electrolyte is $-\psi'$ (equation 16), and the reduction factor

$$f = \frac{2}{\sqrt{1+r^2+2r \cos \frac{\theta}{2}}} \quad (28)$$

The expression for $\frac{B_1'}{B_1}$ in (27) is the same as that for $\frac{A_1'}{A_1}$ in (8) except for the reversal of sign, so the phase-change at the second surface of the electrolyte is the same ($-\psi$) in magnitude and sign as at the first surface, and the reduction factor

$$e = -b. \quad (29)$$

If the electrolyte be replaced by an insulator, f takes the ordinary form

$$f = \frac{2}{1 + \sqrt{\frac{\beta_1}{\beta_2}}}, \quad (30)$$

and the relation

$$cf = 1 - b^2$$

holds. In the general case there is no such relation.

IV. *The Intensity of the Transmitted Ray.*

We have now all the data necessary for treating the general case of a wave-train passing through a plate of finite thickness. We will first take the incident wave-train and follow its history in detail to obtain the expressions for the rays emergent after 0, 2, 4, . . . &c. internal reflexions. Let the incident wave-train be given by

$$A e^{p(t - \sqrt{\beta_1} z)}$$

from the time $t=0$ onwards, p being a complex quantity as defined by (9). Passing through the first surface and reaching the second the wave becomes (d being the thickness of the electrolyte)

$$A c e^{p(t - qd)} e^{i(\theta/2 - \psi')},$$

and emerges in the form

$$A c f e^{-pqd} e^{i(\theta/2 - 2\psi')} e^{p(t - \sqrt{\beta_1} z)}. \quad . \quad . \quad . \quad (I.)$$

If we take the instant at which the incident wave first strikes the plate as the origin of time, the wave-train (I.) begins to emerge at time

$$t_2 = \frac{d \cdot \eta \cdot \sqrt{\beta_2 \rho} \cdot \sin\left(\chi + \frac{\theta}{2}\right)}{p_2}, \quad . \quad . \quad . \quad (31)$$

by equation (19).

The wave reflected from the second surface at the time that (I.) emerges is

$$-A \cdot bc e^{-pqd} e^{p(t + qz)} e^{i(\theta/2 - \psi' - \psi)},$$

which becomes on reflexion at the first surface

$$A \cdot b^2 c e^{-2pqd} e^{p(t - qz)} e^{i(\theta/2 - \psi' - \psi)},$$

and emerges from the second surface at time $3t_2$ as

$$Ab^2 c f e^{-3pqd} e^{p(t - \sqrt{\beta_1} z)} e^{i(\theta - 2\psi' - 2\psi)}. \quad . \quad . \quad . \quad (II.)$$

This is the second emergent ray, or the ray emerging after two internal reflexions. The third emergent ray will be

$$Ab^4 c f e^{-5pqd} e^{i(\theta/2 - 2\psi' - 4\psi)} e^{p(t - \sqrt{\beta_1} z)}, \quad . \quad . \quad . \quad (III.)$$

emerging at time $5t_2$; and so on for the others. The $(n+1)$ th emergent ray, or the ray emerging after $2n$ internal reflexions, is

$$Ab^{2n}cf e^{-(2n+1)p_2d} e^{i(\theta/2-2\psi'-2n\psi)} e^{p(t-\sqrt{\beta_1}z)},$$

which emerges at time $(2n+1)t_2$. Remembering that

$$p = -p_1 + p_2 i = \eta(-\cos \chi + i \sin \chi),$$

$$q = \frac{\sqrt{\beta_1}}{\lambda} = \sqrt{\beta_2 \rho} \left(\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \right),$$

this becomes

$$Acf b^{2n} e^{(2n+1)\kappa d} e^{-p_1 t} e^{i[p_2 t + \alpha - (2n+1)(p_2 t_2 + \psi)]}, \quad \dots \quad (32)$$

where we have written for brevity

$$\left. \begin{aligned} \eta \sqrt{\beta_2 \rho} \cos \left(\frac{\theta}{2} + \chi \right) &= \kappa, \\ \frac{\theta}{2} - 2\psi' + \psi &= \alpha. \end{aligned} \right\} \dots \dots \dots (33)$$

Finally, if we take the initial form of the wave-train to be given by the sine terms, we may write this ray emerging after $2n$ internal reflexions, y_n , as

$$y_n = A \cdot cf. b^{2n} e^{(2n+1)\kappa d} e^{-p_1 t} \sin [p_2 t + \alpha - \overline{2n+1}(p_2 t_2 + \psi)]. \quad (34)$$

However small ψ may be it should evidently be retained, as it may become of importance owing to the multiple reflexions.

The wave-train y_n emerges at time $\overline{2n+1}t_2$. From $\overline{2n+1}t_2$ to $\overline{2n+3}t_2$, y_0 to y_n inclusive are the only wave-trains in existence. The total energy of any wave-train varies as the time-integral of the square of the "displacement"

$$(y), \text{ or } (y_0 + y_1 + y_2 + \dots + y_n).$$

We will call this time-integral the *intensity* of the wave-train (or set of wave-trains forming the ray), though we are not using intensity in its ordinary sense.

To get the intensity of the ray transmitted through the plate, we shall then have to sum up a series of integrals partly of squares ($y_0^2, y_1^2, \dots, y_n^2$) and partly of products ($2y_m y_n$ &c.) :

that is to say, if we call the intensity of the transmitted ray I_t ,

$$I_t = \sum_{n=0}^{n=\infty} \int_{\frac{2n+1}{2}t_2}^{\infty} y_n^2 dt + 2 \sum_{m=0}^{m=n-1} \sum_{n=1}^{n=\infty} \int_{\frac{2n+1}{2}t_2}^{\infty} y_m y_n dt. \quad (35)$$

In the experimental case the deflexions of the electrometer at E, fig. 1, are proportional to I_t .

We have now to get out the integrals in (35) and sum them. We will first multiply out the product $y_m y_n$ from the expression for y_n given in (34). This can then be integrated, and the integral of y_n^2 obtained by putting $m=n$.

Writing, for brevity,

$$p_2 t_2 + \psi = \delta, \quad . \quad . \quad . \quad . \quad (36)$$

$$\begin{aligned} y_m y_n &= A^2 c^2 f^2 b^{2(n+m)} e^{-2p_1 t} e^{2(m+n+1)\kappa d} \\ &\quad \times \sin(p_2 t + \alpha - 2n + 1\delta) \sin(p_2 t + \alpha - 2m + 1\delta) \\ &= \frac{1}{2} A^2 c^2 f^2 b^{2(n+m)} e^{-2p_1 t} e^{2(m+n+1)\kappa d} \\ &\quad \times \{ \cos[2(m-n)\delta] - \cos 2[p_2 t + \alpha - (m+n+1)\delta] \}. \end{aligned} \quad (37)$$

Integrating,

$$\begin{aligned} \int_{\frac{2n+1}{2}t_2}^{\infty} y_m y_n \cdot dt &= \frac{1}{2} A^2 c^2 f^2 b^{2(m+n)} e^{2(m+n+1)\kappa d} \\ &\times \left[\frac{-e^{-2p_1 t} \cos 2(m-n)\delta}{2p_1} + \frac{e^{-2p_1 t}}{2\eta} \cos \{ 2(p_2 t + \alpha) + \chi - 2(m+n+1)\delta \} \right]. \end{aligned}$$

The expression in the square brackets vanishes at the upper limit; so we have, substituting $\eta \cos \chi$ for p_1 in the denominator of the first fraction,

$$\begin{aligned} \int_{\frac{2n+1}{2}t_2}^{\infty} y_m y_n \cdot dt &= \frac{1}{4\eta} A^2 c^2 f^2 b^{2(m+n)} e^{2(m+n+1)\kappa d} \\ &\times e^{-2(2n+1)p_1 t_2} \left\{ \frac{\cos 2(m-n)\delta}{\cos \chi} - \cos [2(n-m)p_2 t_2 + 2\alpha + \chi - 2(m+n+1)\psi] \right\}. \end{aligned} \quad (38)$$

If we put $m=n$, we get

$$\begin{aligned} \int_{\frac{2n+1}{2}t_2}^{\infty} y_n^2 dt &= \frac{1}{4\eta} A^2 c^2 f^2 b^{4n} e^{(4n+2)\kappa d} e^{-(4n+2)p_1 t_2} \\ &\quad \times \{ \sec \chi - \cos [2\alpha + \chi - (4n+2)\psi] \}. \end{aligned} \quad (39)$$

Having now got the integrals, we must next perform the summations indicated in (35). Let us take the y_n^2 integrals first: we have to sum the expression on the right-hand side of (39) from $n=0$ to $n=\infty$. The two terms in the bracket are best taken separately: the first forms a simple geometric series whose sum is

$$\frac{1}{4\eta} A^2 \cdot c^2 f^2 \cdot \frac{e^{2(\kappa d - p_1 t_2)} \sec \chi}{1 - b^4 e^{4(\kappa d - p_1 t_2)}} \cdot \dots \quad (40)$$

The second term requires different treatment, as the cosine is a function of n . We may take the cosine to be the real part of

$$e^{i(2\alpha + \chi - \frac{1}{2}n + 2\psi)},$$

and thus obtain again a geometrical series, the sum of which is

$$\frac{1}{4\eta} A^2 \cdot c^2 f^2 \cdot \frac{e^{2(\kappa d - p_1 t_2)} \cdot e^{i(2\alpha + \chi - 2\psi)}}{1 - b^4 e^{4(\kappa d - p_1 t_2)} e^{-4i\psi}}.$$

This is rationalized by multiplying numerator and denominator by $1 - b^4 e^{4(\kappa d - p_1 t_2)} e^{4i\psi}$; and to further abbreviate the expression we will write

$$\left. \begin{aligned} b e^{\kappa d - 2p_1 t_2} &= \xi, \\ b e^{\kappa d} &= \omega, \end{aligned} \right\} \cdot \dots \quad (41)$$

thus obtaining for the sum of the second terms of (39),

$$\frac{1}{4\eta} A^2 c^2 f^2 \cdot \frac{\omega \xi}{b^2} \cdot e^{i(2\alpha + \chi - 2\psi)} \frac{1 - \omega^2 \xi^2 e^{4i\psi}}{1 - 2\omega^2 \xi^2 \cos 4\psi + \omega^4 \xi^4}.$$

Remembering that

$$\alpha = \frac{\theta}{2} - 2\psi' + \psi,$$

and writing

$$\theta - 4\psi' + \chi = \phi, \quad \dots \quad (42)$$

this becomes, retaining the real terms only,

$$\frac{1}{4\eta} A^2 c^2 f^2 \cdot \frac{\omega \xi}{b^2} \cdot \frac{\cos \phi - \omega^2 \xi^2 \cos (\phi + 4\psi)}{1 - 2\omega^2 \xi^2 \cos 4\psi + \omega^4 \xi^4} \cdot \dots \quad (43)$$

Adding (40) and (42) together, we get for I_1 , the portion of the transmitted intensity due to the square terms,

$$I_1 = A^2 \frac{c^2 f^2}{4\eta b^2} \left\{ \frac{\sec \chi}{1 - \omega^2 \xi^2} - \frac{\cos \phi - \omega^2 \xi^2 \cos (\phi + 4\psi)}{1 - 2\omega^2 \xi^2 \cos 4\psi + \omega^4 \xi^4} \right\}. \quad (44)$$

We have now to sum the product terms. We require

$$I_2 = 2 \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \int_{\frac{2n+1}{2}t_2}^{\infty} y_m y_n dt$$

$$= I_2' - I_2'',$$

say; I_2' being the sum of the terms multiplied by $\sec \chi$, and I_2'' the sum of the second terms. The integrated expression is given in (38). Taking the $\sec \chi$ terms first,

$$I_2' = \frac{A^2 \cdot c^2 f^2}{2\eta b^2} \sum_{n=1}^{\infty} \omega^{2n+2} e^{-(4n+2)p_1 t_2} \sec \chi \sum_{m=0}^{n-1} \omega^{2m} \cos 2(m-n)\delta;$$

$\cos 2(m-n)\delta$ may be treated as the real part of $e^{2i(m-n)\delta}$, or

$$\omega^{2m} \cos 2(m-n)\delta = e^{-2in\delta} (\omega^2 e^{2i\delta})^m,$$

which is a geometric series whose sum from 0 to $n-1$ is

$$e^{-2in\delta} \frac{1 - (\omega^2 e^{2i\delta})^n}{1 - \omega^2 e^{2i\delta}}.$$

Multiplying numerator and denominator by $1 - \omega^2 e^{-2i\delta}$ to rationalize, and retaining only the real quantities in the numerator, this becomes

$$\frac{\cos 2n\delta - \omega^{2n} - \omega^2 \cos (2n+2)\delta + \omega^{2n+2} \cos 2\delta}{1 - 2\omega^2 \cos 2\delta + \omega^4}.$$

Therefore

$$I_2' = \frac{A^2 \cdot c^2 f^2}{2\eta b^2} \frac{\sec \chi}{1 - 2\omega^2 \cos 2\delta + \omega^4} \sum_{n=1}^{\infty} (\cos 2n\delta - \omega^{2n} - \omega^2 \cos (2n+2)\delta + \omega^{2n+2} \cos 2\delta) \omega^{2n+2} e^{-(4n+2)p_1 t_2}.$$

The first and third terms in the bracket may be summed together, as also the second and fourth. Carrying out the summation as before, we get

$$I_2' = \frac{A^2 c^2 f^2}{2\eta b^2} \frac{\sec \chi \cdot \omega \xi}{1 - 2\omega^2 \cos 2\delta + \omega^4} \left\{ \frac{\xi^2 (\cos 2\delta - \xi^2) - \omega^2 \xi^2 (\cos 4\delta - \xi^2 \cos 2\delta)}{1 - 2\xi^2 \cos 2\delta + \xi^4} - (1 - \omega^2 \cos 2\delta) \frac{\omega^2 \xi^2}{1 - \omega^2 \xi^2} \right\}$$

Bringing the two fractions to a common denominator, $(1 - 2\omega^2 \cos 2\delta + \omega^4)$ divides out, and

$$I_2' = \frac{A^2 c^2 f^2}{2\eta b^2} \frac{\sec \chi \cdot \omega \xi}{1 - \omega^2 \xi^2} \frac{\xi^2 (\cos 2\delta - \xi^2)}{1 - 2\xi^2 \cos 2\delta + \xi^4} \cdot \cdot \cdot \quad (45)$$

Returning now to the second half of I_2 , the term not containing $\sec \chi$, we have

$$I_2'' = \frac{A^2 \cdot c^2 f^2}{2\eta b^2} \omega \cdot \xi \cdot \sum_{n=1}^{n=\infty} \sum_{m=0}^{m=n-1} \xi^{2n} \omega^{2m} \cos\{2(n-m)\delta - 4n\psi + \phi\},$$

where we have rewritten the expressions (38) in terms of the abbreviated symbols defined by equations (36), (41), and (42). Carrying out the summation first with regard to m ,

$$I_2'' = \frac{A^2 \cdot c^2 f^2}{2\eta b^2} \frac{\omega \xi}{1 - 2\omega^2 \cos 2\delta + \omega^4} \\ \times \sum_{n=1}^{n=\infty} \xi^{2n} [\cos(2n\delta - 4n\psi + \phi) - \omega^{2n} \cos(4n\psi - \phi) \\ - \omega^2 \cos(2n + 2\delta - 4n\psi + \phi) + \omega^{2n+2} \cos(4n\psi - \phi - 2\delta)].$$

The first and third terms in the square bracket may be summed together, and also the second and fourth. Bringing the two fractions thus obtained to a common denominator, $(1 - 2\omega^2 \cos 2\delta + \omega^4)$ divides out, as in the case of I_2' , and we obtain as our final value of I_2''

$$I_2'' = \frac{A^2 \cdot c^2 f^2}{2\eta \cdot b^2} \frac{\omega \xi}{1 - 2\omega^2 \xi^2 \cos 4\psi + \omega^4 \xi^4} \\ \times \frac{\left\{ \begin{array}{l} \xi^2 \cos(2\delta + \phi - 4\psi) - \xi^4 \cos \phi \\ - \omega^2 \xi^4 \cos(2\delta + \phi) + \omega^2 \xi^6 \cos(\phi + 4\psi) \end{array} \right\}}{1 - 2\xi^2 \cos(2\delta - 4\psi) + \xi^4}. \quad (46)$$

Combining (44) (45) and (46) we have the total value of I_t , the intensity of the transmitted ray,

$$I_t = I_1 + I_2 = I_1 + I_2' - I_2'' \\ = \frac{A^2 \cdot c^2 f^2}{4\eta b^2} \omega \xi \left\{ \frac{\sec \chi (1 - \xi^4)}{(1 - 2\xi^2 \cos 2\delta + \xi^4)(1 - \omega^2 \xi^2)} \right. \\ \left. - \frac{\left\{ \begin{array}{l} (1 - \xi^4)[\cos \phi - \omega^2 \xi^2 \cos(\phi + 4\psi)] \\ - 2\xi^2 \sin(2\delta - 4\psi)[\sin \phi - \omega^2 \xi^2 \sin(\phi + 4\psi)] \end{array} \right\}}{(1 - 2\omega^2 \xi^2 \cos 4\psi + \omega^4 \xi^4)(1 - 2\xi^2 \cos(2\delta - 4\psi) + \xi^4)} \right\}. \quad (47)$$

But we want to compare this with the intensity of the ray transmitted when there is no intervening layer of electrolyte.

Calling the intensity of this ray I_0 , we have for its value

$$I_0 = A^2 \int_0^\infty e^{-2p_1 t} \sin^2 p_2 t \cdot dt = \frac{A^2}{4\eta} (\sec \chi - \cos \chi). \quad (48)$$

Dividing (47) by (48),

$$\frac{I_t}{I_0} = \frac{c^2 f^2 \cdot \omega \xi}{b^2 (\sec \chi - \cos \chi)} \left\{ \frac{(1 - \xi^4) \sec \chi}{(1 - 2\xi^2 \cos 2\delta + \xi^4)(1 - \omega^2 \xi^2)} - \frac{\left\{ \begin{aligned} &(1 - \xi^4) [\cos \phi - \omega^2 \xi^2 \cos (\phi + 4\psi)] \\ &- 2\xi^2 \sin (2\delta - 4\psi) [\sin \phi - \omega^2 \xi^2 \sin (\phi + 4\psi)] \end{aligned} \right\}}{(1 - 2\omega^2 \xi^2 \cos 4\psi + \omega^4 \xi^4)(1 - 2\xi^2 \cos (2\delta - 4\psi) + \xi^4)} \right\}. \quad (49)$$

This formula is one of considerable generality, showing the variation in the intensity of the ray transmitted through a thin absorbent plate when the incident ray is logarithmically damped. Two special cases are of interest: (1) when the plate is not conducting; (2) when the incident ray is not damped. Taking the first case,

$$\left. \begin{aligned} \text{if } \alpha_2 = 0, \quad \psi = \psi' = 0, \quad \delta = p_2 t_2, \\ \theta = 0, \quad \rho = 1, \quad \omega \xi = b^2, \\ \xi = be^{-p_1 t_2}, \quad c^2 f^2 = (1 - b^2)^2, \end{aligned} \right\} \quad (50)$$

and the expression for $\frac{I_t}{I_0}$ simplifies to

$$\frac{I_t}{I_0} = \frac{1 - b^2}{1 + b^2} \left[\frac{1 - \xi^4 + 2\xi^2 \sin 2\delta \cot \chi}{1 - 2\xi^2 \cos 2\delta + \xi^4} \right]. \quad (51)$$

This case was recently worked out by Dr. Barton, and our equation (51) may be reduced to his. Dividing out by the denominator in the square bracket, we get

$$\frac{I_t}{I_0} = \frac{1 - b^2}{1 + b^2} \left[1 - \frac{2\xi^2 (\xi^2 - \cos 2\delta - \cot \chi \sin 2\delta)}{1 - 2\xi^2 \cos 2\delta + \xi^4} \right].$$

Dr. Barton's expression is*

$$\frac{I_t}{I_0} = \frac{1 - b^2}{1 + b^2} \left[1 + \frac{2b^2 e^{-\alpha t_2} (\alpha/\beta) \sin \beta t_2 + \cos \beta t_2 - b^2 e^{-\alpha t_2}}{1 - 2b^2 e^{-\alpha t_2} \cos \beta t_2 + b^4 e^{-2\alpha t_2}} \right],$$

which is equivalent to ours; his t_2 being equal to twice our t_2 , and his α/β equivalent to our p_1/p_2 or $\cot \chi$.

* Proc. Roy. Soc. vol. liv. p. 92; 'Thesis,' p. 13.

As a second special case, suppose we are dealing with a steady simple harmonic ray instead of a damped wave-train. χ is then ninety degrees, and the other symbols take the special values

$$\left. \begin{aligned} \chi &= 90^\circ \quad \therefore \sec \chi = \infty, \\ p_1 &= 0, \quad \therefore \omega = \xi = be^{\kappa d}, \\ \rho &= \sqrt{1 + \left(\frac{\alpha^2}{p_2 \beta_2}\right)^2} \end{aligned} \right\} \quad \dots \dots (52)$$

The long second term in (49) vanishes altogether now, being divided by $(\sec \chi - \cos \chi)$, and

$$\frac{I_t}{I_0} = \frac{c^2 f^2}{b^2} \frac{\xi^2}{1 - 2\xi^2 \cos 2\delta + \xi^4} \quad \dots \dots (53)$$

If in this case we again put $\alpha_2 = 0$, we have

$$\omega = \xi = b,$$

$$\delta = p_2 t_2;$$

and so if the plate be a non-conductor,

$$\frac{I_t}{I_0} = \frac{c^2 f^2}{1 - 2b^2 \cos 2p_2 t_2 + b^4} \quad \dots \dots (54)$$

which is the ordinary expression for the ray transmitted through a thin plate.

The general expression (49) is too complex to allow of an offhand statement of its form. One case was, however, worked out in full arithmetically. The values of the chief constants chosen were

$$\frac{\beta_2}{\beta_1} = 70,$$

sp. resistance of electrolyte = 5100 ohm-centim.,

log. decrement of wave-train = 0.4,

wave-length = λ_1 = 9 metres.

These gave the following values for some of our symbols:—

$$\chi = 86^\circ 21' 26'',$$

$$\theta = 8^\circ 44' 57'',$$

$$\kappa d = -\frac{d}{\lambda_1} (\cdot 673402),$$

$$p_1 t_2 = 3 \cdot 3676 \frac{d}{\lambda_1},$$

$$p_2 t_2 = 3017^{\circ} \cdot 825 \frac{d}{\lambda_1},$$

$$b = \cdot 78705, \quad cf = \cdot 38167,$$

$$\log \xi = \bar{1} \cdot 8960008 - \frac{d}{\lambda_1} (3 \cdot 21751421),$$

$$\log \omega \xi = \bar{1} \cdot 9920016 - \frac{d}{\lambda_1} (3 \cdot 5099681),$$

$$\psi = 1^{\circ} 3' 26'', \quad \psi' = 28' 17'',$$

$$\rho = 1 \cdot 0018.$$

The phase-changes at both surfaces are in this case so small that one would not expect any great shifting of the maxima in the transmission-curve, (1) (fig. 3): the maxima appear to occur rather early, but one can hardly say more. I find (49) too complex to get anything out of it by differentiation.

V. *The Intensity of the Reflected Ray.*

We will now proceed to calculate the intensity of the reflected ray. As before, let the incident wave be

$$A e^{p(t - \sqrt{\beta_1} z)}.$$

A portion of this incident wave is at once reflected, namely

$$Y = A \cdot b \cdot e^{p(t + \sqrt{\beta_1} z)} e^{-i\psi}. \quad . \quad . \quad . \quad . \quad (55)$$

A second portion enters the electrolyte and reaches the second surface in the form

$$A \cdot c \cdot e^{p(t - qd)} e^{i(\theta/2 - \psi')},$$

is there reflected, reaches the first surface again and emerges at time $2t_2$,

$$y_1 = -A \cdot bcf \cdot e^{-2pqd} e^{p(t + \sqrt{\beta_1} z)} e^{i(\theta/2 - 2\psi' - \psi)}.$$

The next wave emerges after three internal reflexions at time $4t_2$,

$$y_2 = -A \cdot b^3 cf \cdot e^{-4pqd} e^{p(t + \sqrt{\beta_1} z)} e^{i(\theta/2 - 2\psi' - 3\psi)},$$

Fig. 3.

Curves showing the variation of $\frac{I_r}{I_0}$ and $\frac{I_r}{I_0}$ with the thickness of the plate.

Ratio of dielectric constants = $\frac{\beta}{\beta_0} = 70$. Sp. resistance of plate 5100 ohm-centim.

and so on ; the expression for the wave emergent at time $2nt_2$ being

$$y_n = -A \cdot b^{2n-1} c f \cdot e^{-2npqd} e^{p(t+\sqrt{\beta_1}t)} e^{i(\theta/2-2\psi)} e^{i(2n-1)\psi}. \quad (56)$$

Now

$$pq = \eta \sqrt{\beta_2 \rho} \left[i \sin \left(\frac{\theta}{2} + \chi \right) - \cos \left(\frac{\theta}{2} + \chi \right) \right].$$

Inserting this value in (56), and using our previous abbreviations,

$$\kappa = \eta \sqrt{\beta_2 \rho} \cos \left(\frac{\theta}{2} + \chi \right), \quad \alpha = \left(\frac{\theta}{2} - 2\psi' + \psi \right), \quad (33)$$

$$\delta = p_2 t_2 + \psi, \quad (36)$$

and retaining only the sine terms, we have

$$y_n = -A c f b^{2n-1} e^{2n\kappa d} e^{-p_1 t} \sin (p_2 t + \alpha - 2n\delta). \quad (57)$$

It is to be noted that the first reflected wave, Y, does not fall into this series so the general expression for the reflected intensity will be

$$I_r = \sum_{n=1}^{\infty} \int_{2nt_2}^{\infty} y_n^2 dt + 2 \sum_{m=1}^{n=\infty} \sum_{n=2}^{\infty} \int_{2nt_2}^{\infty} y_m y_n dt \\ + \int_0^{\infty} Y^2 dt + 2 \sum_{n=1}^{\infty} \int_{2nt_2}^{\infty} Y y_n dt. \quad (58)$$

Multiplying out $y_m y_n$ we get

$$y_m y_n = \frac{1}{2} A^2 c^2 f^2 b^{2(m+n-1)} e^{2(m+n)\kappa d} e^{-2p_1 t} \\ \times \{ \cos 2(m-n)\delta - \cos [2(p_2 t + \alpha) - 2(m+n)\delta] \}; \quad (59)$$

and integrating from $t=2nt_2$ to $t=\infty$,

$$\int_{2nt_2}^{\infty} y_m y_n dt = \frac{1}{4\eta} A^2 c^2 f^2 b^{2(m+n-1)} e^{2(m+n)\kappa d} \\ \times e^{-4np_1 t_2} \left\{ \frac{\cos 2(m-n)\delta}{\cos \chi} - \cos [2(n-m)p_2 t_2 + 2\alpha + \chi] - 2(n+m)\psi \right\} \quad (60)$$

Putting $m=n$,

$$\int_{2nt_2}^{\infty} y_n^2 dt = \frac{1}{4\eta} A^2 c^2 f^2 b^{4n-2} e^{4n\kappa d} e^{-4np_1 t_2} \\ \times \{ \sec \chi - \cos (2\alpha + \chi - 4n\psi) \}. \quad (61)$$

The summation of (61) from $n=1$ to $n=\infty$ is performed

in exactly the same way as that of the analogous expression for I_t in equations (39) to (44). The result is

$$\sum_{n=0}^{\infty} \int_{2nt_2}^{\infty} y_n^2 dt = \frac{A^2 c^2 f^2}{4\eta b^2} \omega^2 \xi^2 \times \left\{ \frac{\sec \chi}{1 - \omega^2 \xi^2} - \frac{\cos(\phi - 2\psi) - \omega^2 \xi^2 \cos(\phi + 2\psi)}{1 - 2\omega^2 \xi^2 \cos 4\psi + \omega^4 \xi^4} \right\}. \quad (62)$$

The summation of the product integrals in (60) is also analogous to that of the product integrals in I_t given in equations (44) to (46). Taking the integral in two parts as before, we have for the sum of the terms containing $\sec \chi$:—

$$\frac{A^2 c^2 f^2}{2\eta b^2} \omega^2 \xi^2 \sec \chi \frac{\xi^2 (\cos 2\delta - \xi^2)}{(1 - \omega^2 \xi^2)(1 - 2\xi^2 \cos 2\delta + \xi^4)}; \quad (63)$$

and for the sum of the second terms, those not containing $\sec \chi$:—

$$\left. \begin{aligned} & \frac{A^2 c^2 f^2}{2\eta b^2} \frac{\omega^2 \xi^2}{1 - 2\omega^2 \xi^2 \cos 4\psi + \omega^4 \xi^4} \\ & \times \frac{\xi^2 \cos(2\delta - 6\psi + \phi) - \xi^4 \cos(\phi - 2\psi) - \omega^2 \xi^4 \cos(2\delta - 2\psi + \phi)}{1 - 2\xi^2 \cos 2\delta + \xi^4} \end{aligned} \right\} [+ \omega^2 \xi^6 \cos(2\psi + \phi)] \quad (64)$$

Adding together (62) (63) and (64), bringing the second half of (62) and (64) to a common denominator and simplifying, we get the expression for that portion of I_r which is a function of the uniform series of y 's only. Calling it I'_r , and dividing out by I_0 the "intensity" of the incident ray, we get :—

$$\frac{I'_r}{I_0} = \frac{c^2 f^2 \cdot \omega^2 \xi^2}{b^2 (\sec \chi - \cos \chi)} \left\{ \frac{(1 - \xi^4) \sec \chi}{(1 - 2\xi^2 \cos 2\delta + \xi^4)(1 - \omega^2 \xi^2)} - \frac{(1 - \xi^4) \{ \cos(\phi - 2\psi) - \omega^2 \xi^2 \cos(\phi + 2\psi) \}}{(1 - 2\omega^2 \xi^2 \cos 4\psi + \omega^4 \xi^4)} - \frac{2\xi^2 \sin(2\delta - 4\psi) \{ \sin(\phi - 2\psi) - \omega^2 \xi^2 \sin(\phi + 2\psi) \}}{(1 - 2\xi^2 \cos(2\delta - 4\psi) + \xi^4)} \right\}, \quad (65)$$

an expression very similar to that for I_t/I_0 given in equation (49). Reverting now to the terms containing Y in (58), let

us evaluate

$$2 \sum_{n=1}^{\infty} \int_{2nt_2}^{\infty} Y \cdot y_n dt.$$

Rationalizing (55) we have

$$Y = Ab \cdot e^{-p_1 t} \sin(p_2 t - \psi). \quad . \quad . \quad . \quad (66)$$

The expression for y_n is given in (56) : multiplying up

$$Y y_n = -\frac{1}{2} A^2 c f \omega_n^2 e^{-2p_1 t} [\cos(2n\delta - \alpha - \psi) - \cos(2p_2 t + \alpha - \psi - 2n\delta)].$$

Integrating from $2nt_2$ to infinity, and writing for brevity

$$\Delta = \alpha - \psi + \chi, \quad . \quad . \quad . \quad . \quad (67)$$

we get

$$\int_{2nt_2}^{\infty} Y y_n dt = -A^2 \cdot \frac{cf}{4\eta} \cdot \xi^{2n} \\ \times [\sec \chi \cdot \cos(2n\delta - \alpha - \psi) - \cos\{2n(p_2 t_2 - \psi) + \Delta\}].$$

Summing this from $n=1$ to $n=\infty$, and multiplying by two,

$$2 \sum_{n=1}^{\infty} \int_{2nt_2}^{\infty} Y \cdot y_n dt = -A^2 \frac{cf}{2\eta} \cdot \xi^2 \\ \times \left[\frac{\sec \chi \{ \cos(2\delta - \alpha - \psi) - \xi^2 \cos(\psi + \alpha) \}}{1 - 2\xi^2 \cos 2\delta + \xi^4} \right. \\ \left. - \frac{\cos(\Delta + 2\delta - 4\psi) - \xi^2 \cos \Delta}{1 - 2\xi^2 \cos(2\delta - 4\psi) + \xi^4} \right]. \quad (68)$$

Calling this I_r'' , and dividing through by I_0 we get

$$\frac{I_r''}{I_0} = \frac{-2cf \cdot \xi^2}{\sec \chi - \cos \chi} \left\{ \frac{\sec \chi \{ \cos(\psi + \alpha - 2\delta) - \xi^2 \cos(\psi + \alpha) \}}{1 - 2\xi^2 \cos 2\delta + \xi^4} \right. \\ \left. - \frac{\cos(\Delta + 2\delta - 4\psi) - \xi^2 \cos \Delta}{1 - 2\xi^2 \cos(2\delta - 4\psi) + \xi^4} \right\}. \quad (69)$$

The remaining portion of I_r is

$$I_r''' = \int_0^{\infty} Y^2 dt = \int_0^{\infty} A^2 b^2 e^{-2p_1 t} \sin^2(p_2 t - \psi) dt \\ = \frac{A^2 b^2}{4\eta} \{ \sec \chi - \cos(\chi - 2\psi) \}.$$

Dividing out by I_0 ,

$$\frac{I_r'''}{I_0} = \frac{b^2 [\sec \chi - \cos (\chi - 2\psi)]}{\sec \chi - \cos \chi} \dots \dots \dots (70)$$

This last expression is independent of the thickness of the slab of electrolyte.

The total expression for $\frac{I_r}{I_0}$ is (65) + (69) + (70), or rewriting these equations,

$$\begin{aligned} \frac{I_r}{I_0} = & \frac{c^2 f^2 \cdot \omega^2 \xi^2}{b^2 (\sec \chi - \cos \chi)} \left\{ \frac{(1 - \xi^4) \sec \chi}{(1 - 2\xi^2 \cos 2\delta + \xi^4)(1 - \omega^2 \xi^2)} \right. \\ & - \left. \frac{\left\{ \begin{aligned} & (1 - \xi^4) [\cos (\phi - 2\psi) - \omega^2 \xi^2 \cos (\phi + 2\psi)] \\ & - 2\xi^2 \sin (2\delta - 4\psi) [\sin (\phi - 2\psi) - \omega^2 \xi^2 \sin (\phi + 2\psi)] \end{aligned} \right\}}{(1 - 2\omega^2 \xi^2 \cos 4\psi + \omega^4 \xi^4)(1 - 2\xi^2 \cos (2\delta - 4\psi) + \xi^4)} \right\} \\ & - 2 \frac{cf \cdot \xi^2}{\sec \chi - \cos \chi} \left\{ \frac{\sec \chi [\cos (\psi + \alpha - 2\delta) - \xi^2 \cos (\psi + \alpha)]}{1 - 2\xi^2 \cos 2\delta + \xi^4} \right. \\ & \quad \left. - \frac{\cos (\Delta + 2\delta - 4\psi) - \xi^2 \cos \Delta}{1 - 2\xi^2 \cos (2\delta - 4\psi) + \xi^4} \right\} \\ & + \frac{b^2 [\sec \chi - \cos (\chi - 2\psi)]}{\sec \chi - \cos \chi} \dots \dots \dots (71) \end{aligned}$$

As a matter of arithmetic this expression is fairly quickly calculable if $\frac{I_t}{I_0}$ has been already worked out; a large number of the constants, &c., are the same in the two cases. The curve (2) (fig. 3) has been calculated from the same data as were given on p. 376; the two curves $\frac{I_t}{I_0}$ and $\frac{I_r}{I_0}$ may be thus compared. In the former the maxima were slightly shifted back, in the latter the minima are slightly shifted forward. The proportion of energy absorbed by the plate is

$$1 - \left(\frac{I_t + I_r}{I_0} \right).$$

The curve thus deduced from $\frac{I_t}{I_0}$ and $\frac{I_r}{I_0}$ is shown in fig. 4. As $\frac{I_t}{I_0}$ asymptotes to zero, and $\frac{I_r}{I_0}$ asymptotes to $\frac{I_r'''}{I_0}$ (70), $\left\{ 1 - \frac{I_t + I_r}{I_0} \right\}$ asymptotes towards the value $1 - I_r'''/I_0$; that is

Fig. 4.

Curve showing the variation in the energy absorbed by the plate $= 1 - \frac{I_t + I_r}{I_0}$.

to say, when the layer of electrolyte gets very thick all is absorbed but the ray reflected from the first surface.

The expression for $\frac{I_r}{I_0}$ given in (71) may be checked by putting α_2 zero. Taking the three parts (65) (69) (70) separately, the first becomes at once

$$b^2 \frac{I}{I_0},$$

where $\frac{I_t}{I_0}$ is the expression given in (51) or the following equation. The second part reduces to

$$(1-b^2) \frac{2\xi^4 - 2\xi^2 \cos 2\delta - 2\xi^2 \sin 2\delta \cdot \cot \chi}{1 - 2\xi^2 \cos 2\delta + \xi^4} = 1 - b^2 - (1+b^2) \frac{I_t}{I_0}.$$

The third part gives simply b^2 . Adding up we get

$$\begin{aligned} \left(\frac{I_r}{I_0}\right)_{\alpha_2=0} &= b^2 \frac{I_t}{I_0} + 1 - b^2 - (1+b^2) \frac{I_t}{I_0} + b^2 \\ &= 1 - \left(\frac{I_t}{I_0}\right)_{\alpha_2=0}. \end{aligned}$$

The expression for $\frac{I_r}{I_0}$ also becomes greatly simplified if we take the incident ray to be non-damped. In this case

$$\sec \chi = \infty, \quad \omega = \xi = b e^{kd}$$

as before, and

$$\begin{aligned} \frac{I_r}{I_0} &= \frac{c^2 f^2}{b^2} \frac{\xi^4}{1 - 2\xi^2 \cos 2\delta + \xi^4} \\ &\quad - 2cf \cdot \xi^2 \cdot \frac{\cos(\psi + \alpha - 2\delta) - \xi^2 \cos(\psi + \alpha)}{1 - 2\xi^2 \cos 2\delta + \xi^4} \\ &\quad + b^2. \quad \dots \dots \dots (71a) \end{aligned}$$

If we now put $\alpha_2 = 0$, we have $\omega = \xi = b$, and $\delta = p_2 t_2$, and so on, as given on p. 375, and the expression becomes

$$\frac{I_r}{I_0} = \frac{c^2 f^2 b^2}{1 - 2b^2 \cos 2p_2 t_2 + b^4}$$

$$\begin{aligned}
& - 2cfb^2 \frac{\cos 2p_2t_2 - b^2}{1 - 2b^2 \cos 2p_2t_2 + b^4} \\
& + b^2 \\
& = \frac{2b^2(1 - \cos 2p_2t_2)}{1 - 2b^2 \cos 2p_2t_2 + b^4}, \quad \dots \quad (71b)
\end{aligned}$$

which is the ordinary expression for 'Newton's Rings by Reflexion.'

VI. *Discussion of an Experimental Case and a Correction.*

This section is independent of the rest of the paper, being devoted to the discussion of a correction necessary in the experimental case mentioned at the beginning.

It has been stated already in the introduction that the theory, sections IV. & V., does not give numerical results agreeing closely with the experimental ones. For all the values of β_2/β_1 I tried, the calculated points lay continually below the experimental determinations. Dr. Barton found exactly the same peculiarity in his analogous experiments, and suggested that it might be due to the coursing of the wave-trains backwards and forwards along the wire circuit between the oscillator and the closed end. Working from this idea he arrived at a correction formula*, which, however, is not immediately applicable to our case. The following analysis is equivalent to Dr. Barton's though differing in method.

Let ABC (fig. 5) represent our circuit, B being the electrolyte, A the oscillator, C the closed end.

Fig. 5.

Let r and d be the reflexion and transmission coefficients at the electrolyte, with regard to energy, so that

$$I_t = tI_0, \quad I_r = rI_0.$$

Let e and x be similar coefficients for the two halves of the

* 'Thesis,' p. 16, section V.

wire circuit—reduction coefficients covering loss of energy due either to damping by the imperfectly conducting wires or to imperfect reflexion at the ends of the circuit.

If then

T_n is the intensity of the n^{th} train emerging from B towards C,

R_n is the intensity of the n^{th} train reflected from B towards A,

we have

$$\left. \begin{aligned} T_{n+1} &= ed R_n + r\chi T_n \\ R_{n+1} &= d\chi T_n + re R_n \end{aligned} \right\} \dots \dots \dots (1)$$

The effect on the electrometer is proportional to

$$\sum_{n=0}^{n=\infty} (1 + \chi) T_n,$$

so we have to sum a finite-difference series.

$$\text{Let} \quad \left. \begin{aligned} T_n &= Ab^n \\ R_n &= Bb^n \end{aligned} \right\} \dots \dots \dots (2)$$

Rewriting (1) with these values of T_n and R_n , we have

$$\left. \begin{aligned} Ab &= ed B + r\chi A, \\ Bb &= d\chi A + re B, \end{aligned} \right\}$$

whence, eliminating A and B,

$$b^2 - br(e + \chi) + e\chi(r^2 - d^2) = 0. \dots \dots (3)$$

This is an equation to determine b . Suppose b_1 and b_2 to be the roots; $A_1 A_2, B_1 B_2$ being the corresponding values of A and B. We have then

$$b_1 + b_2 = r(e + \chi), \quad b_1 b_2 = (r^2 - d^2) e\chi. \dots \dots (4)$$

$$\left. \begin{aligned} T_n &= A_1 b_1^n + A_2 b_2^n \\ R_n &= \frac{d\chi}{b_1 - re} A_1 b_1^n + \frac{d\chi}{b_2 - re} A_2 b_2^n \end{aligned} \right\} \dots \dots \dots (5)$$

From this last equation,

$$\left. \begin{aligned} T_0 &= dI_0 = A_1 + A_2, \\ R_0 &= rI_0 = d\chi \left(\frac{A_1}{b_1 - re} + \frac{A_2}{b_2 - re} \right), \end{aligned} \right\} \dots \dots \dots (6)$$

The total effect on the electrometer is consequently proportional to E where

$$E = (1 + \chi) \sum_0^{\infty} T_n = (1 + \chi) \left\{ \frac{A_1}{1 - b_1} + \frac{A_2}{1 - b_2} \right\} \\ = (1 + \chi) \left\{ \frac{A_1 + A_2 - (b_2 A_1 + b_1 A_2)}{1 - (b_1 + b_2) + b_1 b_2} \right\}.$$

And from (6),

$$r I_0 = d \chi \left\{ \frac{b_2 A_1 + b_1 A_2 - r e (A_1 + A_2)}{b_1 b_2 - r e (b_1 + b_2) + r^2 e^2} \right\}.$$

But by the relations (4),

$$1 - (b_1 + b_2) + b_1 b_2 = 1 - r(e + \chi) + e \chi (r^2 - d^2),$$

$$b_1 b_2 - r e (b_1 + b_2) + r^2 e^2 = -e \chi d^2,$$

whence

$$-e d r I_0 = b_2 A_1 + b_1 A_2 - r e d I_0,$$

or

$$b_2 A_1 + b_1 A_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus, finally,

$$E = (1 + \chi) \frac{d I_0}{1 - r(e + \chi) + e \chi (r^2 - d^2)},$$

$$E_0 = (1 + \chi) \frac{I_0}{1 - e \chi},$$

and

$$\frac{E}{E_0} = \frac{(1 - e \chi) d}{1 - r(e + \chi) + e \chi (r^2 - d^2)}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$\frac{E}{E_0}$ is an ordinate of the experimental curve. d and r are ordinates of our calculated transmission and reflexion curves.

If then we determine $\frac{E}{E_0}$, d , and r for the different thicknesses of the electrolyte, we shall have by substituting their values in (8) a pair of simultaneous equations for the determination of e and χ .

This process only led to impossibilities for all values of β_1/β_2 and the log. dec. of the wave-train that were tried. Either $e \chi$ became greater than unity or negative, or $e + \chi$ became negative.

It may be noted that this correction did not work very successfully in Dr. Barton's case either. His method of

applying it was, however, different: having formed an experimental estimate of the rate of decrease of the total energy of a wave-train by its passage along the imperfectly conducting wires, he assumed that there was perfect reflexion at both ends of the circuit, and thence calculated the correction to be applied to the experimental curve. The corrected curve obtained in this way lay below the calculated curve. Taking two points at random on his curves, I tried to calculate $e\chi$ and $e + \chi$ by my method. The results were again impossible.

It must be remembered that in both these cases more than one of the constants of the theoretical curve were simple guesses, γ and β_2/β_1 in my case were both only approximately known. In trying to make things square by the correction of (8), we may have been attempting the impossible and rightly got irrational results. I need not dwell at greater length on this point: possible causes of error are numerous. The non-agreement of experimental results with the equations does not of course prove the wrongness of the latter, but only the wrongness of their application to the case.

I am compelled to admit that, considering the positions of the maxima in my experimental curves are dependent on the phase-changes and on the correction we have been discussing—possibly also on other uneliminated disturbances—no great weight can be attached to the accuracy of the values of the dielectric constants for water, alcohol, &c., deduced on the assumption that such corrections were negligible*.

VII. *The Numerical Value of some of the previous Results.*

When the expressions in sections II. and III. had been obtained, the question at once arose, how far the values of b, c, f, ψ, ψ' , and so on might be practically affected by possible variations in the rate of damping of the incident ray. In working with ordinary oscillators are changes likely to become important or to remain quite negligible? In working with light are differences likely to be appreciable between flashes and steady rays?

As regards oscillator wave-trains, the damping should

* Phil. Mag. vol. xxxvi. p. 539; Wied. Ann. vol. 1. p. 748.

certainly be taken into account. The second question, however, must, I think, be answered in the negative, for the simple reason that one cannot get a quick enough flash*. Even an electric spark lasts for thousands or millions of the vibrations of violet light: the rays from it would be practically steady, not damped.

The set of curves in fig. 6 is drawn for a steady ray, wave-length one metre. It illustrates the variation in b (the ratio of the transmitted to the incident amplitude at the surface of an infinitely thick slab), when the conductivity and dielectric constant of the slab vary. Curve (1) is drawn for

Fig. 6.

Curves showing the variation in " b " with the dielectric constant and conductivity of the plate, for an undamped wave-train. Wave-length 100 cms.

- Dielectric constant of plate.
(1) For a substance of zero conductivity	} $\beta_1 = 1.$
(2) " " conductivity $\cdot 001 \times 10^{-9}$	
(3) " " " $\cdot 010 \times 10^{-9}$	

a non-conducting slab, curve (2) for a slab of conductivity $\cdot 001 \times 10^{-9}$ C.G.S. units, and curve (3) for a slab of conductivity $\cdot 01 \times 10^{-9}$ or ten times the last. These conductivities are all extremely low, that of a 5 p. c. copper sulphate or zinc sulphate solution being roughly $\cdot 2 \times 10^{-9}$. The curves show very well how rapidly the conductivity of the reflecting plate

* Unless light be itself an aggregate of damped wave-trains.

grows in importance relatively to the dielectric constant even for great wave-lengths.

But if we take the case of a charge vibrating on an isolated perfectly conducting sphere, the amplitude falls to $e^{-\frac{\pi}{\sqrt{\epsilon}}}$ or about $\frac{1}{3.8}$ of its original value in the time occupied by a complete vibration*: this gives us $\chi = 54^\circ 44' 8''$. Let us use this as an example of a damped wave-train to compare with the steady ray, retaining the same wave-length—1 metre—corresponding to a sphere about a foot in diameter.

The curve (1) for b corresponding to a non-conducting plate would be the same in both cases. The curves (3) corresponding to a plate of conductivity $\cdot 01 \times 10^{-9}$ are shown together in fig. 7. b is at first greatest for the damped

Fig. 7.

wave-train, but as the dielectric constant is increased this ratio is reversed. Physically speaking, this has very little meaning: any actual method would measure the energy of the reflected ray, and it has been shown † that the energy is a function of the phase-change, the phase-change being itself a function of the rate of damping of the wave-train. Taking the phase-changes, ψ , first, I calculated them for the steady ray, for the damped ray with which we are dealing, and the conductivity $\cdot 01 \times 10^{-9}$. The results are given together in

* J. J. Thomson, 'Recent Researches,' p. 370.

† Equation (70).

fig. 8. For the lower values of the dielectric constant, ψ for the damped wave-train is about half as large again as for the steady ray.

Utilizing these values of ψ for calculating the intensity of the reflected ray, we get the pair of curves in fig. 9. The

Fig. 8.

result is striking. The energy of the reflected waves for the steady ray is from 15–20 p. c. of the incident energy greater than for the damped ray.

Fig. 9.

Curves showing the difference between the intensities of the portions of damped and undamped wave-trains reflected from an infinitely thick plate of conductivity 0.1×10^{-9} .

Experimental evidence for these results would be very interesting; but, so far as I am aware, there is none whatever. In cases where χ was taken between 80° and 90° , the alterations would be much smaller and more difficult to detect. The curves I have given only illustrate, however, a very small section of cases, even larger variations may be possible.

I am glad to have this opportunity of expressing my indebtedness to Professor Karl Pearson, to whom my best thanks are due for much advice and assistance.

TABLE of the principal Symbols used.

The suffix 1 refers to the dielectric; the suffix 2 to the plate. In our case, where all media are assumed to be non-magnetic,

β = dielectric constant,

α = $4\pi \times$ conductivity.

b = ratio of the reflected to the incident amplitude at the first surface of the plate.

e = the corresponding ratio at the second surface.

c = ratio of the refracted to the incident amplitude at the first surface.

f = the corresponding ratio at the second surface.

d = thickness of slab.

λ_1 = wave-length in dielectric.

λ_2 = " " electrolyte.

$-\psi$ = the change of phase on reflexion at the first surface.

$-\psi'$ = " " refraction through the second surface.

$p = -p_1 + ip_2$ is defined by (4).

η, χ are defined by (9).

χ gives the rate of damping of the wave-train, being 90° for a steady ray.

ρ, θ are defined by (10).

r is " " (11).

κ, α are " " (33).

δ is " " (36).

ω, ξ are " " (41).

ϕ is " " (42).

Δ " " (67).

$$I_t/I_0 = \frac{\text{intensity of transmitted ray}}{\text{intensity of incident ray}}.$$

$$I_r/I_0 = \frac{\text{intensity of reflected ray}}{\text{intensity of incident ray}}.$$

DISCUSSION.

A letter from Dr. E. H. BARTON was read emphasizing the necessity of taking into account the damping in the oscillator-train, and at the same time pointing out why, in his opinion, the corrections applied by Mr. Yule were inadequate and failed to yield intelligible results.

Prof. RÜCKER congratulated Mr. Yule on his work, and on the importance of the results he had obtained.

XXXIV. *Energy Movements in the Medium separating Electrified or Gravitating Particles.* By H. N. ALLEN, *University of Nebraska, Lincoln, Neb.**

1. FARADAY and Maxwell have shown that it is possible to look on the potential energy of electric separation as residing in the surrounding dielectric, and that each of the cells, bounded by the walls of a tube of force and two neighbouring equipotential surfaces, can be looked upon as containing a certain definite amount of energy.

This energy-distribution is not in general permanent, and can only be regarded as a step towards some simpler arrangement. Thus a positive and a negative electrified body suspended in space, and acted on only by the electrical forces between them, are never in equilibrium until they are in actual contact. The energy-distribution in the dielectric changes constantly as they approach. Poynting has shown † how energy is transferred from one point to another in an electromagnetic field; and we are quite accustomed to think of energy as flowing from dynamo to motor through the æther, or from primary to secondary in an alternating-current transformer.

In the following pages an attempt is made to deduce a few of the consequences of Maxwell's suggestion with regard to

* Read March 8, 1895.

† Phil. Trans. 1884, Pt. II. p. 343.