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THE MATHEMATICAL GAZETTE.

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ON VON STAUDT'S GEOMETRIE DER LAGE.

It is far too much the custom now to rely on the analogy of algebra to justify the introduction of imaginaries into geometry. Analogy, however, is no justification unless we first prove the exact correspondence of the fields of investigation. In analytical geometry the identification of the two fields is permissible, and is easily explained; but in pure geometry any reference to algebra, expressed or implied, is irrelevant and misleading. The elements of pure geometry have no dependence on calculation.

There is not precisely the same difficulty about the elements at infinity. They are easier of comprehension, geometrically; we feel that at any rate we know *where* they are—where we should encounter them, if we could only get there—but we can form no conception of position for imaginaries. The only attempt at a strictly geometrical theory is to be found in von Staudt's *Geometrie der Lage* (1847), and *Beiträge zur Geometrie der Lage* (1856); and later writers are mostly contented to leave it there, with a passing reference to it as supplying ample justification for the use of imaginary elements. This is unfortunate, for while the theory itself presents no special difficulties except in one or two points of the argument, yet von Staudt's presentation is at times obscure, at times tedious; and as he lacks the conciliatory style that does so much to encourage a student, very few have the patience to persevere to the end. In these pages I propose to give a brief account of von Staudt's system of geometry, premising however that owing to the small amount of explanation that he vouchsafes, I may at times be reading my own interpretation into his text.

The proper introduction of infinitely distant and imaginary elements into the domain of geometry depends, naturally, on the way in which the material is defined, the whole development of the system being but a series of deductions from the definitions.

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In a strictly formal geometry all the material may be regarded as given, absolutely, with the laws of combination; the difference between real and imaginary elements, or between elements at a finite distance and elements at infinity, will make itself felt only when we attempt to compare the given material with the visible universe. The difficulty is then to determine what part of the intellectual domain can be represented by visible realities. But von Staudt's procedure is altogether different. His elements are in the first instance abstractions from the visible universe, but he enlarges the field thus given by adjoining to it other purely intellectual elements, given by formal definition. The crux here is to prove that these formal elements are of the same nature as the original elements.

Thus von Staudt ascends from a visible, actual domain to a more extensive intellectual domain. His work falls naturally into three principal divisions:—

(i.) *intuitive geometry*, dealing with elements regarded as visible entities; this includes the usual propositions of projective geometry, and other investigations such as are now classed as topological (*Geometrie*, pp. 1-23, 30-178; *Beiträge*, pp. 1-76);

(ii.) *constructive geometry*, in which he creates his own enlarged universe by formal definition (*G.* 23-30; *B.* 76-126);

(iii.) *formal geometry*, in which he considers all figures resulting from the combinations of the observed and defined elements (*B.* 131-256).

The pages here named, with those containing certain metric supplements (*G.* 203-216; *B.* 126-129, 256-283), though only a part of the 600 pages of von Staudt's two works, contain all that is necessary for the present purpose. The division by pages is not very exact; (i.) and (ii.), as also (ii.) and (iii.), overlap to a great extent.

On the whole, I do not consider the intuitive part of von Staudt's geometry synthetic; when it is so, it is rather by an afterthought. The material and its laws of combination are derived from observation, and are afterwards extended and restated in an intellectual synthetic form. His view in the first instance is this. We have before us the visible universe; this, in so far as it is visible, we will treat geometrically. But we cannot handle any part of it, and so we cannot measure anything; all we can do is to change our point of view. The objects we see (or might see, for this is what he means by *denkbar*) are bounded portions of space; from these we arrive at the idea of surfaces, lines, and finally points, each as a boundary; regarded as a boundary, each has two sides, one towards the contained figure, one towards the rest of the field (space, surface, or line) from which it is cut. The objects of our researches are therefore solids, surfaces, lines, and points, and all visible relations of these

are matters for investigation. Each of these elements can move in the preceding one, so generating it; and checking its motion at any stage, it becomes again a boundary. When, for example, a point moves in a line ABC , seeing that as regards the line the point has only two sides, it can move only in the two opposed senses, ABC , CBA . Two points thus determine a bounded portion of the line; if the line be not closed, one point can separate two, but if the line be closed, it takes two points to separate two given points A , B , for we can pass from A to B along the line in either sense.

Among lines and surfaces are particularly to be distinguished straight lines and planes. A straight line is a line which is fixed in space by two of its points; hence two straight lines cannot intersect in more than one point. A surface which can be generated by the motion of a straight line is said to be ruled. The straight line being of all lines the one that is most simply determined, is regarded as one of the fundamental elements of the geometry; when looked upon as a particular line described by the motion of a point it is called a straight line, but when looked upon as an element, it is called a ray (*Strahl*). Starting from any point on a ray, we distinguish two opposed, or complementary, half-rays; all half-rays proceeding from any point in space form a sheaf of half-rays (*Halbstrahlenbündel*). A part of a sheaf is a solid angle or angular space, and the surface by which the part of the sheaf is separated from the rest is a simple conical surface. Thus if a surface is such that it can be filled with half-rays of a sheaf, it is a conical surface, and it bounds an angular space. Such a surface can be described by a half-ray moving about the vertex as a pivot, and in any of its positions this half-ray serves as one boundary of a portion of the conical surface; any line joining the vertex to one other point of a conical surface lies entirely on the surface. The complementary half-rays describe the vertical angular space, with the vertical conical surface to bound it. If these vertical surfaces are two distinct surfaces, so that a ray starting from any position and moving on one conical surface can never reverse itself, the surface is of even order; but if the vertical surfaces are one and the same, so that a half-ray can move on the surface till it coincides with its complement, thus reversing itself, the surface is of odd order. The two parts of a conical surface of even order form the complete conical surface; and if these two parts are connected only at the vertex, so that they have not a single half-ray in common, the surface divides the sheaf into two parts, of which one is a complete cone; this lies on the inside of the conical surface. The remainder of the sheaf, outside the conical surface, is such that conical surfaces of odd order can exist in it; this is the definition of "outside the surface." The complete conical surface divides the

sheaf of half-rays into three parts, of which two are simple cones. All this is of use in considering the circuits of plane curves.

A plane is a conical surface of odd order in which any point may be regarded as the vertex. Hence any line that has two points in common with the plane lies entirely in it, for one of the two points may be regarded as the vertex. From this all the ordinary propositions about planes and lines follow, as for instance that a plane and a line can cut only in one point.

It is to be noticed that von Staudt lays particular stress on the divisibility of every kind of field that presents itself; and that every element presents itself as a field, inasmuch as a system of elements of a different kind can be united with (pass through, or lie in) it. Thus in a plane, a point determines a field whose elements are straight lines, namely, the flat pencil; and in space, a line determines a field with, *e.g.* planes for elements; this is an axial pencil. The bounded portions of these two fields are called angles, plane angle and dihedral angle, determined respectively by two half-rays and two half-planes. If the half-rays or half-planes complementary to the boundary ones are included in the angle considered, it is said to be *erhaben*; if they are not included, the angle is *hohl*. Thus two half-rays divide a flat pencil into two angles, one *hohl* and one *erhaben*. If, however, the two boundary half-rays or half-planes are complementary, the angle is *flach* or straight. Thus angles less than, equal to, or greater than, two right angles are respectively *hohl*, *flach*, or *erhaben*.

A straight line, regarded as a field of points (range), has some characteristics in common with the flat pencil and the axial pencil. For joining any external point (or line) to all the points of the range, we obtain a flat pencil (or axial pencil). In each field two elements bound a portion; and an element describing the field can move only in the two opposed senses. These three fields are the primitive forms of the first grade. The primitive forms of the second grade are the plane, regarded as a field of points and lines, thus containing ranges and flat pencils, and the sheaf, that is, the point regarded as a field of planes and lines, containing therefore axial and flat pencils. The primitive form of the third grade is common space, which contains all the others.

Our only concern is with the appearance of figures; figures that appear the same are for our purposes identical. If the various points of two figures are impressed on the eye by the same rays, then these figures are in appearance, *dem Scheine nach*, identical. The set of rays by which either figure is projected to the eye is therefore called its *Schein*; figures that yield the same *Schein* are identical in appearance. Of course if we change the point of view, the figures are no longer identical *dem Scheine nach*. The

word, as von Staudt uses it, seems untranslatable; mathematically projector (used, for example, in Holgate's translation of Reye's *Geometrie der Lage*), answers well enough, but it misses the significant reference to the appearance, to the visual foundation of von Staudt's geometry. Yet *Schein* cannot be translated by any term that implies any attribute of the figure itself, nor even by aspect, for it is rather the medium, the vehicle, by which the aspect is made known to the eye.

So far as this geometry concerns itself, figures that yield the same *Schein* have the same properties; on this is based the theory of projection. Cutting, for example, the *Schein* of a plane figure by a different plane, we obtain a different figure of (*dem Scheine nach*) identical properties; this projective identity of two forms is shown to depend on harmonic equivalence.

Distinct in conception from the elements by which any given field can be generated, though identical with them in fact, are certain figures in the field, which are themselves aggregates of elements. These figures are

(1) in space, a surface;

(2) in a plane, or on any surface, a curve;

(3) on any line, a *Wurf* (cast or throw). This is the simplest set of elements about which anything can be predicated. Now as any three points on a line can be projected into any other three, a set of three points has no characteristic properties; but a set of four points has properties that are projectively indestructible, and this set is a *Wurf*. Just as curve means neither length nor area, nor even shape, though the figure has shape and can be regarded as determining a length or an area, so *Wurf* means simply the set of four points in a determinate order. The same term is applied to a set of four elements in any other primitive form of the first grade.

It is shown that forms can be projectively equivalent without being in such a position that their projective relation can be immediately discovered by the eye; it may however be indirectly tested by comparison with a third form. When two projective forms of the same nature are thus placed, the pairs of corresponding elements determine a set of new elements; for example, two ranges give rise to a set of lines, two flat pencils in the same plane yield a set of points. In this way we obtain, in a plane, a locus of the second order and an envelope of the second class; in a sheaf, a conical surface as a locus of rays or an envelope of planes; these are regarded simply as figures. But two ranges whose bases are not coplanar, or two axial pencils whose bases are not concurrent, give rise to a system of straight lines, a regulus (*Regelschaar*), which is ranked as one of the primitive forms rather than as a figure, though it is taken also as indicating a figure, namely, the ruled surface on which all the rays lie;

every plane section of this surface is either a proper conic or a pair of straight lines.

I suppose the reason for thus distinguishing the regulus is that while the primitive forms of the first grade originally given allow for every possible position of two points or two planes, since these are necessarily elements in a range or axial pencil, they do not allow for every possible relative position of two straight lines. If the lines intersect, they are components of a flat pencil; but if they do not intersect—which is, after all, the most general case—they cannot be placed in any of the primitive forms first mentioned. The regulus fills up this gap; and the primitive forms of the first grade may now be enumerated as follows:

Range of points, Axial pencil, Flat pencil, and Regulus.

Pure geometry deals with the relations of points, lines, and planes; a very small part deals with figures composed of a finite number of these; when the figures are to be continuous, the elements must be indefinite in number, and then they are handled by means of these primitive forms of the first grade.

It was no part of von Staudt's design to say in what other systems of geometry some of the lines of investigation had been started, or when and how the results were originally discovered. His business was to apply his method in all suitable regions. Thus in addition to the usual projective geometry, we find Plücker's conception of the dual generation of curves (and surfaces), with the production of true singularities by stationary elements¹—a conception that has no dependence on any analytical statement of geometrical facts. We find also (pp. 81-110) an investigation in the theory of circuits and regions, which, together with the contemporary memoirs of Listing and Möbius,² may be regarded as the foundation of the modern treatment of topology.

Among the abstractions from observation on which von Staudt bases the intuitive part of his geometry are to be noticed (1) the idea of direction, (2) the axiom on which his treatment of parallels is founded. This axiom is that a straight line cannot lie entirely within a hollow angle (*hohl*). From this it follows that the *Schein* of a line with reference to any external point is a straight angle. It cannot be *erhaben*, for as all the half-rays of the *Schein* are obtained by joining the points of the line singly to the vertex, two such half-rays cannot be complementary, as the straight line thus formed would then meet the given line in two points, one for each half-ray. By the axiom, it cannot be *hohl*; hence it is

¹ Plücker, *System der analytischen Geometrie*, p. 241 (1835); *Theorie der algebraischen Curven*, p. 200 (1839); von Staudt, *Geometrie*, pp. 110-118 (1847).

² Listing, *Vorstudien zur Topologie*, 1847; Möbius, *Grundformen der planen Linien dritter Ordnung*, 1849, *Selbstauszüge*, 1848.

flach or straight. Now the boundary half-rays are not a part of the angle, therefore they do not meet the given line; and as these half-rays, bounding the straight angle, form one straight line through the vertex, this is a straight line that does not meet the given line. But all other lines through the vertex do meet it, for each consists of two half-rays, of which one is an element of the *Schein* of the given line. Hence every flat pencil that lies in a plane with a given straight line contains precisely one ray that does not meet the line.

Two lines in a plane are said to be parallel if they do not meet; and what we have just proved is that through one point one line can be drawn parallel to a given line.

The direction, *Richtung*, of a line is referred to without a word of comment, it being taken for granted that parallel lines have the same direction. Any figure formed of lines (plane or sheaf) is said to contain a specified direction if it contains one line in that direction. Planes are said to be parallel if they have no point in common; it is a matter of observation that parallel planes have something in common, and this is called their aspect, *Stellung*. Two directions determine the aspect; one point and the aspect determine a plane; two aspects determine a direction. It is noticed that in the various statements direction corresponds to point, aspect to line; two lines in a plane have either a common point or a common direction; a plane contains either a point of a line or the direction of the line; two planes have in common a line, or else their aspect, and so on. A mode of referring to *Richtung* and *Stellung* that shall emphasize this likeness is desirable.

Consider the intersection of two lines in a plane; keep one line fixed, and let the other revolve about a point not on the first line. So long as the two lines have different directions, they have a determinate point of intersection; as the second line approaches the first in direction, this common point recedes indefinitely, and finally loses itself. If we now speak of the intersection of the two lines as a point at infinity, an ideal point, we express this phenomenon in symbolical language, and we replace the term *Richtung* by a phrase that exhibits the analogy between direction and point. But this amounts to extending the domain of our investigation by adjoining to it all ideal points, namely, one on every set of parallel lines; we enquire therefore whether this change of base will necessitate any alteration in the geometry we have already constructed.

The ideal point on any line is to be found in either direction along that line; hence putting on the ideal point has changed the straight line from an open line to a closed one. This statement sums up the changes caused.

If we accept these ideal points, all that lie in any one plane

must be spoken of as lying in a straight line, for there is to be one such point on any arbitrary line; we shall have therefore in each plane one ideal line, and as this is determined by two ideal points, it is fitted to replace the idea of the aspect of a plane. By adjoining the ideal line we change the plane from an open surface to a closed one. All these ideal points and lines must be regarded as lying in one ideal plane, in order that there may be one ideal point on every line, one ideal line in every plane.

These considerations lead to the conclusion that the system of geometry will be most harmoniously developed if we throw aside the restriction of our domain to the visible universe. The region of our researches is now to include the visible universe, and as much more as can be formulated and is found convenient. For the present we adjoin simply one plane; the domain is the visible universe increased by one ideal plane.

To show that this ideal plane with its elements is of the same nature as any other plane with its elements, von Staudt examines and compares the constructions and proofs for the different possible combinations of actual and ideal elements. This involves some tedious detail; but it appears unavoidable with this method of handling the subject. The conclusion to which this minute investigation leads is that the geometry of this enlarged domain is the same as that of the restricted domain—the visible universe—without the exceptions; the laws of combination of the elements now hold universally.

We can picture this ideal plane on any arbitrary plane, P_1 , of the visible universe (for example, by representing its points by the points in which its *Schein* from any vertex is cut by P_1); the plane P_1 must then be represented on some other plane P_2 , P_2 on P_3 , and so on; there will always be one plane left over, one plane not pictured on the planes of the universe; the domain is “the universe+one plane.” Von Staudt does not state this explicitly, but his attitude towards imaginary elements suggests that something of this nature may have been in his mind.

C. A. SCOTT.

(To be continued.)

REVIEWS.

Leçons nouvelles sur les applications géométriques du calcul différentiel. Par W. DE TANNENBERG, Professeur à la Faculté des Sciences de l'Université de Bordeaux. Paris, Hermann, 1899; 8vo, 192 pp.

Opinions seem to differ as to how far the geometrical applications of the Calculus ought to be collected from the four corners of mathematics and presented to the student *en bloc*. Many people think it is a mistake to let them occupy so much space as is generally granted to them at present in the text-books. “If I were writing a book on the Differential