$4'5'6'$ and $1'2'3'$ in the planes A, C respectively correspond to the triangles β_1, β_2 ; and the conditions that $4', 5', 6'$ shall be in a line and that $1', 2', 3'$ shall be in a line, are the same condition, namely, that the triangles β_1, β_2 shall be inscribed in the same conic. Analogous properties must apparently belong to Cremona's other transformations, and the investigation of them will form an interesting part of the theory.

It is important, also, to notice the relation of the transformation to Hesse's "Uebertragungsprincip," Crelle, Vol. 66, p. 15, which establishes a correspondence between the points of a plane and the point-pairs on a line. If $Ax^2 + 2Bxy + Cy^2 = 0$ is the equation of a point-pair, the co-ordinates in the plane are taken by Hesse *directly*, but in the present Paper *inversely* proportional to A, B, C .

The President and Professor Hirst made some remarks on the Paper, the latter pointing out its connection with his method of Quadric Inversion.

The following presents were received :—

"Société Philomathique de Paris. Extrait de la séance du 18 Janvier, 1868;" from Prof. Hirst.

"Proceedings of the Literary and Philosophic Society, Oct. 1868. Recherches sur les surfaces réglées tétraédrales Symmetriques, par Jules de la Gournerie;" from Prof. Cayley.

"Crelle," Band 69, 2^{es} Heft.

"On the curves which satisfy given conditions;" by Prof. Cayley, from the Author.

"A Photographic Diagram consisting of 10,206 lines connecting 144 points on a circle, and involving 17,178,876 intersections, to illustrate Mr. Woolhouse's Question in the Lady's and Gentleman's Diary for 1867;" from Mr. Woolhouse.

January 14th, 1869.

Prof. CAYLEY, in the Chair.

Mr. G. H. Darwin, B. A., was admitted into the Society.

Mr. S. Roberts then read a paper

On the Mechanical Description of some species of Circular Curves of the third and fourth degrees.

"At æquatio non est, sed descriptio, quæ curvam Geometricam efficit."—Newton.

In the present state of the theory of quartic and cubic curves, it is very desirable that we should be able to draw them continuously.

There are to be found in the books several isolated examples of such description. To cite a familiar and ancient instance, Nicomedes drew the conchoid, named after him, by means of a simple instrument called the "Trammel." This curve is generated, it will be remembered, by any point on a straight line which constantly passes through a fixed point, while another point upon it is constrained to move rectilinearly. The instrument, therefore, consists of a grooved bar moving over a fixed pin, and at the same time turning on another pin which travels along a straight groove. The method given by Newton, for drawing the Cissoid of Diocles, is similar. In this case, one leg of a right angle always passes through a fixed point, while a point on the other leg moves along a straight line. When the distance of the fixed point from the given line is equal to the length intercepted between the angle and the point constrained to move rectilinearly, the middle point of the intercepted length will generate the Cissoid. In effect, Newton generalizes this by considering the case of a point on the leg not being the middle point of the intercepted length. I may mention also the Limaçon of Pascal, which is generated by making a straight line pass constantly through a given point upon a circle. A point on the line at a constant distance from its other intersection with the circle will describe the Limaçon, that is to say, a Cartesian oval having a finite double point.

And it is to be observed, that in this instance the curve is general, except as to its essential singularities. For the double point is the fixed point, and, being given, is equivalent to three conditions. We have also the two constants of the circle and the arbitrary distance of the generating point. Now two given cusps and a given finite double point determine 11 of the 14 conditions which define a quartic curve, so that there remain only 3 disposable constants.

I have endeavoured to obtain results similar to the foregoing, having regard to mechanical convenience, and employing only points, straight lines, and circles. As might be anticipated, the curves which I am able to describe by these means are circular. The system of equations from which the equation representing the locus is evolved, is usually circular with respect to the uneliminated variables, and since the so called circular points at infinity satisfy all such equations, the result contains a circle, or the straight line at infinity, or the circular points without degeneration.

What follows, therefore, may be considered as an extension of the principle of the compasses to curves of the third and fourth degrees, and notwithstanding practical limitations the curves capable of being thus described are of considerable variety and elegance.

1. I find that the "Trammel" of Nicomedes may be generalised in three particulars without elevating the algebraical degree of the locus.

We may substitute for the fixed point or pole through which the moving line passes a fixed circle touched by the moving line.

We may substitute for the point which moves rectilinearly a circle touched by the moving line and having its centre always on the given line or "rule."

We may attach an arm to the moving tangent of given length and inclination.

Although I did not arrive at these conclusions *per saltum*, it is better here to take at once the most general case.

2. The centre of the fixed circle is taken as origin of rectangular coordinates; x_1, y_1 are the coordinates of the centre of the moving circle, x, y are the current coordinates of the locus, k is the distance of the generating point from the centre x_1, y_1 , and m is the tangent of the angle which the attached arm makes with the moving tangent.

We have therefore the following conditions in which X, Y are general coordinates.

The line $AX + BY + C = 0$ is to touch the circles $X^2 + Y^2 = r^2$, $(X - x_1)^2 + (Y - y_1)^2 = s^2$.

Also we may write $A'(x - x_1) + B'(y - y_1) = 0$,

$$m = \frac{BA' - AB'}{AA' + BB'},$$

$$(x - x_1)^2 + (y - y_1)^2 = k^2,$$

$$ax_1 + by_1 + c = 0.$$

Putting L, M for $x - x_1, y - y_1$, respectively, we must eliminate A, B, C, A', B', L, M , from

$$(1) \quad r^2 (A^2 + B^2) = C^2,$$

$$(2) \quad s^2 (A^2 + B^2) = \{AL + BM - Ax - By - C\}^2,$$

$$(3) \quad L^2 + M^2 = k^2,$$

$$(4) \quad A'L + B'M = 0,$$

$$(5) \quad aL + bM - ax - by - c = 0,$$

$$(6) \quad A(mA' + B') + B(mB' - A') = 0,$$

By (4) and (6), $A(mM - L) - B(mL + M) = 0$;

and therefore, by (1) and (2),

$$s^2 (1 + m^2) (L^2 + M^2) = \{m(L^2 + M^2) - x(mL + M) - y(mM - L) - r\sqrt{(1 + m^2)(L^2 + M^2)}\}^2;$$

that is to say, by (3), $(mx - y)L + (my + x)M = t$,

where $t = mk^2 + (s - r)k\sqrt{(1 + m^2)}$.

$$\text{By this and (5),} \quad L = \frac{P(my + x) - bt}{a(my + x) - b(mx - y)},$$

$$M = -\frac{P(mx - y) - at}{a(my + x) - b(mx - y)},$$

where P is put for $ax + by + c$;

so that finally, by (3), the required locus is given by

$$\{P(my+x)-bt\}^2 + \{P(mx-y)-at\}^2 = k^2 \{a(my+x)-b(mx-y)\}^2 \dots\dots\dots (A),$$

a circular curve of the fourth degree.

The equation may also be written in the form

$$(yQ-bt)^2 + (xQ-at)^2 = k^2 (ay-bx)^2 \dots\dots\dots (B),$$

Q being a general linear function.

We may now suppose that $Q=0$ is a line parallel to the axis of x , and the equation is

$$\{y(y+c)-bt\}^2 + \{x(y+c)-at\}^2 = k^2 (ay-bx)^2 \dots\dots\dots (C).$$

3. From equation (C) we derive, according to the common notation,

$$\frac{du}{dx} = 2(y+c) \{x(y+c)-at\} + 2k^2b(ay-bx),$$

$$\frac{du}{dy} = (4y+2c) \{y(y+c)-bt\} + 2x \{x(y+c)-at\} - 2k^2a(ay-bx).$$

These and the original equation are satisfied by the coordinates of the point where the line $y+c=0$ meets the line at infinity, and also by the coordinates of two points on the line $ay-bx=0$. For

$$y(y+c)-bt=0, \quad x(y+c)-at=0, \quad ay-bx=0 \dots\dots\dots (a)$$

have two common sets of roots.

The equation represents, therefore, a unicursal quartic curve, having a double point at infinity. The double focus and the two double points at a finite distance lie in a straight line. The equation is general for a quartic so conditioned; for, after determining the double point at infinity, we have at our disposal one constant of the "rule" line, three constants of the arbitrary circle, and the quantities k and m , making six in all. But the curve is subjected to eight known conditions, the linear relation of the double focus and the two finite double points being one.

If $b=0$ in (C), or $m=0$ in (A), there are two double points united at infinity, and the line $y+c=0$, or in (A) the line $P=0$, passes through four consecutive points. There is, however, a double point at a finite distance; and if $s-r=0$, the curve is the conchoid of Nicomedes, which consequently is not the most general curve of its kind. When $c=0$, $b=0$, $a=1$, the "rule" line passes through the centre of the given circle, and we have a peculiar special case.

The equation then is $(xy-t)^2 + y^2(y^2-k^2) = 0$.

4. In a practical point of view, it is convenient that the radii of the circles enter only by their sum or difference. We may consequently reduce either to a point without affecting the locus, or, what is more

7

8

10

4

5

1

2

6

3

(a)

(c)

(d)

(e)

important, can make compensation for the sensible breadth of the moving bar.

The few figures which accompany this paper will give, though inadequately, some idea of the nature of the curves drawn in the way proposed. Fig. (1) represents a quartic curve having three real double points, two of them visible, the other at infinity. The common solutions of (a) may, however, be imaginary.

5. The quartic locus can be reduced to a cubic curve and a straight line. This straight line must manifestly be parallel to the given fixed line, since the distance of a point on the locus from that line cannot exceed k , a finite quantity. Referring to equation (A), and for simplicity's sake making $a=0$, $b=1$, we find the following conditions that a line parallel to the axis of x may be contained in the locus. They are

$$\begin{aligned} y^2 + (y^2 - k^2) m^2 &= 0, \\ ty - mk^2 (y + c) &= 0, \\ \{t - my(y + c)\}^2 - (y^2 - k^2)(y + c)^2 &= 0, \\ t &= mk^2 + (s - r)k\sqrt{1 + m^2}. \end{aligned}$$

These equations are uniquely satisfied by $s - r = 0$, $y = \frac{km}{\sqrt{1 + m^2}}$.

Thus, in order that the curve described may be a circular cubic, the necessary and sufficient condition is that the difference of the radii of the circles must be equal to the distance of the fixed line from the centre of the fixed circle. It will be seen that Newton's method of drawing the Cissoid fulfils this condition, which is also evident geometrically. The word "difference" may be taken in its algebraical sense.

Writing in (A) $\tan \theta$ for m , putting c for $s - r$, and still taking $a=0$, $b=1$, we have

$$\begin{aligned} \{(y + c)(y \sin \theta + x \cos \theta) - k(c - k \sin \theta)\}^2 \\ + \{(y + c)^2 - k^2\}(x \sin \theta - y \cos \theta)^2 = 0. \end{aligned}$$

After some easy reductions, we can divide out $y + c + k \sin \theta$, leaving
 $(y + c - k \sin \theta)(x^2 + y^2) - k^2(y - c - k \sin \theta) - 2kc(y \sin \theta + x \cos \theta) = 0$
..... (D),*

a general equation for circular cubics having a double point, that is to say, the "pedal curve" of a parabola.

Suppose, in fact, that we take a line parallel to the real asymptote for the axis of x , the equation of a circular cubic may be written

$$(y + A)(x^2 + y^2) + Bx + Cy + D = 0.$$

* Besides the Cissoid and other known curves, Dr. Booth's "Logocyclic Curve" is included in the class here represented. We have to make $\theta = \frac{\pi}{2}$, $c = -k$.

Let x_1, y_1 be the coordinates of the double point, then we have

$$\begin{aligned}x_1^2 + y_1^2 + 2y_1(y_1 + A) + C &= 0, \\A(x_1^2 + y_1^2) + 2Bx_1 + 2Cy_1 + 3D &= 0, \\2x_1(y_1 + A) + B &= 0;\end{aligned}$$

or putting $c - k \sin \theta$ for A , we have

$$\begin{aligned}B &= -2kc \cos \theta, \\C &= -2kc \sin \theta - k^2, \\D &= k^2(c + k \sin \theta).\end{aligned}$$

These results agree with the equation (D).

Fig. (2) represents a circular cubic curve drawn in the manner above described.

7. An instrument fulfilling the conditions indicated can be easily made.

In Fig. (a), AB represents a rather stiff wire stretched across a board of convenient size. CD is a bar caused to touch a cylinder F by means of an elastic band. At E is a pin fixed in a short grooved bar G, while an arm EP carries a pencil at P, and is fixed to CD by a screw in any required position. It is also well to make CD capable of carrying a pencil.

In Fig. (b), GD represents a bar capable of moving about G, while G moves along the wire AB. The bar CD, which touches the cylinder F, and the arm PD, which carries a pencil at P, are clamped at D.

In a more elaborate instrument, the breaks which the above simple arrangements necessitate might be avoided by raising the moving apparatus above the level of the paper in the manner of Suardi's geometric pen.

8. If, instead of a fixed line, we take another fixed circle, the corresponding locus is usually of the sixth degree. When, however, the new fixed circle passes through the centre of the original fixed circle, and $s - r = 0$, or the radii of the circles are equal, the locus contains a circle, as is geometrically evident. The remaining part of the locus is then a Limaçon. I have already remarked that the well-known mode of drawing the Limaçon is general. We only gain, therefore, in convenience by the extended mode of generation.

It is, perhaps, unnecessary to give here the details of the elimination, which are merely tedious. I have found the equation in its general form to be

$$\begin{aligned}&\{ (my - x + mB)^2 + (mx + y + mA)^2 \} (x^2 + y^2 - k^2) \\&+ 2 \{ m(Bx - Ay) - x^2 - y^2 \} \{ A(my + x) - B(mx - y) + 2x^2 + 2y^2 \} \\&+ \{ A(my + x) - B(mx - y) + 2x^2 + 2y^2 \}^2 = 0;\end{aligned}$$

the equation of the second fixed circle is

$$x^2 + y^2 + Ax + By = 0.$$

It will be remembered that the Limaçon can also be generated as an epitrochoid, or as the "pedal curve" of a circle.

9. The next method which I propose is also adapted to mechanical application. I find that if one point of a line moves in a circle, while another point of the line moves in a straight line, any point rigidly connected with the moving line will describe a curve of the fourth degree.*

Let the coordinates of the points upon the circle and line be x_1, y_1, x_2, y_2 respectively, so that we may take

$$x_1^2 + y_1^2 = r^2,$$

$$ax_2 + by_2 + c = 0.$$

Let l be the length of a perpendicular to the moving line, and let its extremity be considered as the generating point. If p, q are the parts into which the constant intercepted length is divided by the foot of the perpendicular l , and x', y' are the coordinates of the last-mentioned point, it is plain that

$$\frac{y_1 - y'}{y' - y_2} = \frac{x' - x_1}{x_2 - x'} = \frac{p}{q},$$

$$x = x' + \frac{l}{p+q} (y_1 - y_2), \quad y = y' + \frac{l}{p+q} (x_1 - x_2).$$

From which, writing L, M for $x - x_1, y - y_1, L_2, M_2$ for $x_2 - x_1, y_2 - y_1$, we get

$$(1) \quad (p+q) L = pL_2 - lM_2,$$

$$(2) \quad (p+q) M = pM_2 + lL_2.$$

We have also

$$(3) \quad a(L_2 - L + x) + b(M_2 - M + y) + c = 0,$$

$$(4) \quad L^2 + M^2 = l^2 + p^2,$$

$$(5) \quad L^2 + M^2 - 2xL - 2yM + x^2 + y^2 - r^2 = 0.$$

By (1), (2), (3) we get

$$a \{ (p+q) (pL + lM) - (l^2 + p^2) (L - x) \} \\ + b \{ (p+q) (pM - lL) - (l^2 + p^2) (M - y) \} + (l^2 + p^2) c = 0;$$

which, since l, p, q are independent, may be written

$$\alpha L + \beta M + ax + by + c = 0.$$

Also by (4) and (5),

$$2xL + 2yM - (x^2 + y^2 - r^2 + l^2 + p^2) = 0;$$

* If we take two straight lines instead of the circle and line, we get the "Eliptic Trammel." The locus degenerates to an ellipse.

and, eliminating L, M successively,

$$2(\alpha y + \beta x) L + 2yP + \beta\phi = 0,$$

$$2(\alpha y - \beta x) M - 2xP - \alpha\phi = 0,$$

where P is put for $ax + by + c$, and ϕ for $x^2 + y^2 - r^2 + l^2 + p^2$.

Finally, by (4),

$$(2yP + \beta\phi)^2 + (2xP + \alpha\phi)^2 = 4(l^2 + p^2)^2(\alpha y - \beta x)^2 \dots\dots\dots (E).$$

10. If we compare our last result with equation (B) of § 2, there is seen to be a remarkable analogy. And, in point of fact, the last-named equation is a particular case of the present one. Mechanically, however, it is not so. Some of the quantities which were finite are now infinite.

Let α, β be indefinitely small, while

$$\frac{\alpha}{2}(r^2 - l^2 - p^2) = a't, \quad \frac{\beta}{2}(r^2 - l^2 - p^2) = b't,$$

$$(l^2 + p^2)\alpha = a'k, \quad (l^2 + p^2)\beta = b'k,$$

then the present equation becomes identical in form with (B). Thus the two methods of description are connected in an unanticipated way.

The equation (E) represents a quartic curve having two double points on the line $\alpha y - \beta x = 0$; for if we write the equation in the form

$$\psi^2 + \chi^2 - k^2(\alpha y - \beta x)^2 = 0,$$

then $\psi = 0, \chi = 0, \alpha y - \beta x = 0$ have two common solutions.

We have at our disposal the two constants of P, the three constants of the circle α, β , and k , being eight in all. The quartic is therefore subjected to six conditions. It passes through the circular points and has two double points, and these and the double focus are connected by a linear relation. The remaining condition does not appear to be of a critical kind.

Practically we cannot translate the line $\alpha y - \beta x = 0$ to infinity. The curve has no real points at infinity, except in limiting cases which practically cannot occur, though they may be approximated to.

11. The following easy construction enables us to draw a variety of circular quartics with two double points, or three double points, or a triple point:—

Three arms are fixed at D (Fig. *e*) by means of a screw. One arm carries a pencil P, another is attached to a radius AC which moves about A as centre, the third is attached to a short grooved bar G which moves with it along the line EF.

In this Paper, I have not thought it expedient to enter upon a detailed discussion of the equations which have been obtained.* I may remark, however, that we can determine a third double point by making the distance of the centre of the fixed circle from the fixed line equal to the length $p+q$ plus or minus the radius.

Thus, if CGD, AB (Fig. *d*) be the circle and line, CF being the position of the moving bar when it is perpendicular to AB, the extremity P of a perpendicular to CF at E will generate a curve with a double point at P. Moreover, it does not necessarily lie on a straight line through D, the centre of CGD, and another double point. Hence we shall have three double points. We may also arrange for a triple point. In fact, let GH be another position of the moving bar, which meets CF in K. Take $CL=GK$; then, if we raise a perpendicular at the middle point of LK, and another at the same distance from K on KH, they will meet in a point M, which will determine the required length of arm. The point M will be a triple point on the locus, for we have provided that another branch of the curve shall pass through it.

Figs. (3), (4), (5), (6), (7), (8) illustrate the curves drawn by this method. Fig. (6) is analogous to the curve which Plücker traced from its equation, and used for the purpose of showing that the twenty-eight double tangents of a quartic curve may be all real.

12. It remains to notice a mode of description which bears to the last a relation similar to that which the mode of describing the Limaçon bears to the generalized "Trammel" of Nicomedes.

Let a circle C' (Fig. *e*) pass through the centre of the circle C , and let a line AB, equal in length to the radius of C , rest with its extremities on C and C' respectively; I say that any point rigidly connected with AB will describe a quartic curve.

Taking the centre of C for the origin of rectangular coordinates, and the line joining the centres for the axis of x ; writing also l for the perpendicular DP, p for AD, q for BD; x_1y_1 for the coordinates of A, x_2y_2 for the coordinates of B, L, M for $x-x_1$, $y-y_1$ respectively; L' , M' for $x-x_2$, $y-y_2$ respectively; we have, in the same way as in §9, the following equations:—

- (1) ..... $(l+p^2) L' + (pq-l^2) L + rLM = 0$,
- (2) ..... $(l^2+p^2) M' + (pq-l^2) M - rLM = 0$,
- (3) ..... $(L-x)^2 + (M-y)^2 = (p+q)^2 = r^2$,
- (4) ..... $L^2 + M^2 = l^2 + p^2$.

* It may be worth while to notice that the method of M. Chasles for drawing tangents to certain classes of curves is applicable to all the curves here dealt with.

Also, by means of

$$\begin{aligned}x_1^2 + y_1^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 = r^2, \\x_2^2 + y_2^2 - 2ax_2 &= 0,\end{aligned}$$

we have

$$(x_1 - a)x_2 + y_1y_2 = 0,$$

or, what is the same thing,

$$(x - L - a)(L' - x) + (y - M)(M' - y) = 0;$$

and substituting the values of L' , M' given by (1), (2), we get

$$\begin{aligned}&\{(pq - l^2)(x - a) - (l^2 + p^2)x - rly\}L \\&+ \{(pq - l^2)y - (l^2 + p^2)y + rl(x - a)\}M \\&+ (l^2 + p^2)\{x(x - a) + y^2 - pq + l^2\} = 0.\end{aligned}$$

Also, by (3), (4), $2xL + 2yM - x^2 - y^2 + r^2 - l^2 - p^2 = 0$.

By the successive elimination of L , M ,

$$\begin{aligned}&\{rl[x(x - a) + y^2] + a(pq - l^2)y\}2M \\&+ 2(l^2 + p^2)x[x(x - a) + y^2 - pq + l^2] \\&+ \{(pq - p^2 - 2l^2)x - rly - a(pq - l^2)\}(x^2 + y^2 - r^2 + l^2 + p^2) = 0; \\&\{rl[x(x - a) + y^2] + a(pq - l^2)y\}2L \\&- 2(l^2 + p^2)y[x(x - a) + y^2 - pq + l^2] \\&- \{(pq - p^2 - 2l^2)y + rl(x - a)\}\{x^2 + y^2 - r^2 + l^2 + p^2\} = 0.\end{aligned}$$

The substitution of the values of L , M in (4) gives

$$\begin{aligned}&\left\{ \begin{aligned} &[(pq - p^2 - 2l^2)y + rl(x - a)][x^2 + y^2 - r^2 + l^2 + p^2] \\ &+ 2(l^2 + p^2)y[x(x - a) + y^2 - pq + l^2] \end{aligned} \right\}^2 \\&+ \left\{ \begin{aligned} &[(pq - p^2 - 2l^2)x - rly - a(pq - l^2)][x^2 + y^2 - r^2 + l^2 + p^2] \\ &+ 2(l^2 + p^2)x[x(x - a) + y^2 - pq + l^2] \end{aligned} \right\}^2 \\&= 4\{l^2 + p^2\}\{rl[x(x - a) + y^2] + a(pq - l^2)y\}^2.\end{aligned}$$

Put χ for $x^2 + y^2 - l^2 - q^2$, then the equation becomes

$$\begin{aligned}&\{(pry + rlx - arl)\chi - 2(l^2 + p^2)axy + 2r^2l^2y - 2(pq - l^2)rlx + 2(pq - l^2)arl\}^2 \\&+ \left\{ \begin{aligned} &[prx - rly - a(pq - l^2)]\chi - 2(l^2 + p^2)ax^2 + 2r^2l^2x \\ &+ 2(pq - l^2)rlx + 2a(pq - l^2)^2 \end{aligned} \right\}^2 \\&= 4(l^2 + p^2)\{rl[x(x - a) + y^2] + a(pq - l^2)y\}^2.\end{aligned}$$

Now, if we consider successively the terms of this equation not involving χ explicitly, and multiplied by a^2 , a^1 , a^0 , we find that each such term is in fact divisible by χ ; and, by a somewhat tedious reduction, the following result will be obtained, after dividing out χ :

$$\begin{aligned}&\{r^2(x^2 + y^2) - 2arlx - 2arqy + a^2(l^2 + q^2)\}\{x^2 + y^2 - l^2 - q^2 - 4pq + 4l^2\} \\&- 4(ax - rq)\{pr(x^2 + y^2) - arly - a(pq - l^2)x\} \\&+ 4a^2(l^2 + p^2)x^2 - 8arl(pq - l^2)y - 4a^2(pq - l^2)^2 - 4r^2l^2(x^2 + y^2) = 0 \dots (F);\end{aligned}$$

or putting ρ^2 for $x^2 + y^2$,

$$\begin{aligned} r^2 \rho^4 + \{ (a^2 - r^2) (l^2 + q^2) - 2arly - 2ar(r+p)x \} \rho^2 \\ + 4a^2rx(px + ly) + 2arq \{ l^2 + q^2 + 2(pq - l^2) \} x \\ + 2arl(l^2 + q^2 - 2rq)y - a^2 \{ l^2 + q^2 + 2(pq - l^2) \}^2 = 0. \end{aligned}$$

The curve represented is evidently bicircular. If $l=0$, the equation becomes $\{ r^2y^2 + [rx - a(r+p)]^2 \} (x^2 + y^2 - q^2) - 4a^2pry^2 = 0 \dots (G)$;

and the curve has a double point at $y=0$, $x = a \frac{r+p}{r}$. From this it

may at once be inferred that the curve represented by the more general equation has also a finite double point. For the description by the extremity of l is continuous or discontinuous in the same measure as the description by a point taken generally on the moving line. The curve is therefore the "pedal curve" of a conic section.

13. The equation (F) is also general. For we have at our disposal the three constants of the arbitrary circle, the two constants of the circle passing through the centre of the first circle together with p and l , making 7 constants in all.

The simpler equation (G) contains a large class of curves. We may write it in the form,

$$(\rho^2 - \mu^2) \{ y^2 + (x - \nu)^2 \} - \pi^2 y^2 = 0.$$

If $\nu = 0$, the origin is a double point, and we have

$$\rho^4 = \mu^2 x^2 + (\pi^2 + \mu^2) y^2,$$

representing the well known locus of the feet of the perpendiculars on the tangents of a conic section from its centre.

In order to obtain the Lemniscate of Bernoulli, we must make

$$p = -r, \quad a = r\sqrt{2}.$$

We are thus enabled to describe the curve continuously by very simple means.

In order that the equation should represent a Limaçon, we must have $r=p$ or $a=0$. Consequently we can only approximate to the curve. Figs. (9), (10) represent curves described in the above manner.

Here I leave the subject. Although the methods noticed appear at first sight very different in mechanical relation and result, they are nevertheless theoretically connected, as we have seen. Nor is the practical diversity so great as to prevent the combination of the processes in one apparatus. It is better not to interfere with this principle of unity, by discussing other graphical contrivances by means of extended strings, the intersections of conditioned lines or trochoidal motion; nor

could justice be done to the subject without unduly extending my limits.

The author exhibited some simple contrivances by which his figures had been drawn. The President made some remarks on the subject, more particularly bearing upon the *point*-description of such curves, whereas Mr. Roberts' constructions were confined to their *continuous* description.

Prof. Hirst next read a paper "On degenerate forms of Conics," in the course of which he referred to the Chairman's paper, "On the curves which satisfy given conditions."

The following presents were received:—

"Crelle's Journal," 69 Band, iii^e Heft.

"Annali di Matematica," Serie ii^a, tomo ii., fasc. i. and ii., Agosto 1868, Ottobre 1868.

"Memorie del Reale Istituto Lombardo," vol xi. della Serie iii., fasc. iv., v. ed ultimo, 1867.

"Mémoire de Géométrie pure sur les Surfaces du troisième ordre."

"Rendiconti adunanza," del 7 May, 1868, Serie ii., vol. iv. 1867, vol. i. 1868.

"Solenni adunanze," vol. i., fasc. iv.

"Royal Society's Catalogue of Scientific Papers," vol. ii., COA—GRA.

In the Proceedings of the Society, under date of 26th November, 1868,* it will be seen that Professor Sylvester gave an account of "Spirals whose arcs and the arcs of whose pedals are algebraically related"; the subject having grown under his hands, he made a further communication to the Society (March 11th, 1869), on a "Theorem in Logarithmic Waves," and went into details connected with the subject, and its bearing upon Cyclodes and the properties of Numbers. The Theory still continued to occupy the Author's mind (and from the importance of its results he has been led to style it his *capo d'opera*), so that he subsequently communicated to the Society (June 10th, 1869) his matured results in the following paper.†

* No. xvi., p. 118.

† The interesting character and importance of this paper, in addition to some delay having occurred with reference to other communications, and Professor Sylvester's desire to have it put into circulation, will perhaps sufficiently account for its thus appearing out of its proper position.