LXXV. The bakerian lecture.—On the theory of the astronomical refractions

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To cite this article: James Ivory K.H. M.A. F.R.S. L. & E. (1839) LXXV. The bakerian lecture.—On the theory of the astronomical refractions , Philosophical Magazine Series 3, 15:99, 497-507, DOI: [10.1080/14786443908649947](https://doi.org/10.1080/14786443908649947)

To link to this article: <http://dx.doi.org/10.1080/14786443908649947>

Published online: 01 Jun 2009.

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THE
LONDON AND EDINBURGH
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE,
SUPPLEMENT TO VOL. XV. THIRD SERIES.

LXXV. *The Bakerian Lecture.*—*On the Theory of the Astronomical Refractions.* By JAMES IVORY, K.H., M.A., F.R.S. L. & E., *Instit. Reg. Sc. Paris, Corresp. et Reg. Sc. Götting. Corresp.*

[Continued from p. 395.]

10. **T**HE equation (C.) supposes that the atmosphere consists entirely of dry air: we have next to consider what modification must be made when it contains a portion of aqueous vapour.

In the first place, when p' and τ' , the pressure and temperature at the surface of the earth, are given, as they are in the mean atmosphere which produces the refractions, the quantity α , or the refractive power of the air, is not liable to be altered by any possible mixture of aqueous vapour. For if an addition of vapour to dry air diminish the refractive power by making the density less, the greater action of the vapour upon light is found almost exactly to compensate the defect. Laplace first made this observation; which has been confirmed by MM. Biot and Arago, who have established by experiments, that the refractive power of air, whether dry or mixed with vapour, is the same, when the pressure and temperature are the same. It thus appears that, as far as the quantity α , or the refractive power of the air at the earth's surface, is concerned, the astronomical refractions are independent on the hygrometric condition of the atmosphere.

But a mixture of vapour may produce changes in the expression of the refraction, by altering the coefficients or the integrals. Now, if we attend to the formulas that have been found for an atmosphere of moist air, and in the equation (10.) make the same substitution as in the case of dry air, viz.

$$\frac{\sigma}{a} = \frac{s}{a} + \alpha w,$$

we shall obtain

$$s \cdot \frac{(\rho')}{p'} = \frac{s \left(1 - \frac{3}{8} \frac{\phi'}{p'}\right)}{L(1 + \beta \tau')} = x,$$

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$$\frac{p'}{(\rho')} \cdot \frac{1}{a} = \frac{L(1 + \beta \tau')}{a \left(1 - \frac{3}{8} \cdot \frac{\phi'}{p'}\right)} = i,$$

$$\alpha \cdot \frac{a(\rho')}{p'} = \frac{\alpha}{i} = \frac{a \left(1 - \frac{3}{8} \cdot \frac{\phi'}{p'}\right)}{L(1 + \beta \tau')} = \lambda;$$

and further, it will appear that the same relation subsists between x and u in the atmosphere of moist air as between the quantities represented by the same letters in the atmosphere of dry air. The same procedure will therefore lead, in both cases, to the same integrals extending between the same limits. The only difference lies in the values of λ and i , which in the case of moist air acquire, as a multiplier or divisor,

the small factor $\left(1 - \frac{3}{8} \cdot \frac{\phi'}{p'}\right)$ depending on the tension of

the vapour at the earth's surface. If the hygrometer afforded an easy practical method of ascertaining the tension of the vapour, the minute variations of the refractions, arising from moisture in the atmosphere, might be corrected by the method usually employed for compensating the small changes which a difference of temperature causes in the mean constants.

Experience confirms what has been said; for all the astronomers who have attended to aqueous vapour in the atmosphere, agree in admitting that it either has no influence, or but a very small and imperceptible effect, to alter the refractions. On this head it will be sufficient to cite the authority of M. Biot*, who seems carefully to have studied this point, on which he expresses himself very strongly. The very exact coincidence of the theoretical with the observed refractions as far as 88° or $88\frac{1}{2}^\circ$ from the zenith, concurs to prove that the variable quantity of vapour in the air has little influence so long as it retains the gaseous form; but at lower altitudes, when the rays of light become almost parallel to the horizon, it is very probable that particular and local causes may come into play.

11. Nothing is now wanted for completing the solution of the problem, except the reducing of the expression (C.) to a form fit for numerical calculation.

* *Précis Elem. de Physique*, p. 229, tom. ii. edit. 2nd. *Addit. à la Contr. des Temps*, 1839, p. 36.

Investigation of the integral Q_0 .

We have

$$Q_0 = \int_0^m \frac{e \, dx \, c^{-x}}{\Delta} = \int_0^m \frac{e \, dx \, c^{-x}}{\sqrt{(1-e^2)^2 + 4e^2 \cdot \frac{x}{m}}};$$

assume,

$$\Delta = \sqrt{(1-e^2)^2 + 4e^2 \cdot \frac{x}{m}} = 1 - e^2 + 2e^2 z;$$

then,

$$\frac{e \, dx}{\Delta} = e \cdot m \, dz,$$

$$\frac{x}{m} = t = 1 - e^2 (z - z^0).$$

By Lagrange's theorem,

$$\Psi = t - t^2,$$

$$z = t + e^2 \cdot \Psi + \frac{e^4}{1.2} \cdot \frac{d \cdot \Psi^2}{dt} + \frac{e^6}{1.2.3} \cdot \frac{d \, d \cdot \Psi^3}{dt} + \&c.;$$

$$m \, dz = m \, dt \cdot \left\{ 1 + e^2 \cdot \frac{d \cdot \Psi}{dt} + \frac{e^4}{1.2} \cdot \frac{d^2 \Psi^2}{dt^2} + \frac{e^6}{1.2.3} \cdot \frac{d^3 \Psi^3}{dt^3} + \&c. \right\};$$

consequently,

$$\int_0^m \frac{e \, dx \, c^{-x}}{\Delta} = \int_0^1 m \, dt \, c^{-mt} \left\{ e + e^3 \frac{d \Psi}{dt} + \frac{e^5}{1.2} \cdot \frac{d^2 \Psi^2}{dt^2} + \&c. \right\}.$$

Wherefore, if we assume

$$Q_0 = A_1 e + A_3 e^3 + A_5 e^5 + \&c.,$$

we shall have

$$A_{2n+1} = \frac{1}{1.2.3...n} \int_0^1 m \, dt \, c^{-mt} \cdot \frac{d^n \cdot \Psi^n}{dt^n}.$$

In the first place, it may be proper to show that all the coefficients in the series for Q^0 are positive. For this purpose integrate by parts, and the results will be

$$A_{2n+1} = \frac{m}{1.2.3...n} \times \left\{ c^{-mt} \cdot \frac{d^{n-1} \Psi^n}{dt^{n-1}} + \int m \, dt \, c^{-mt} \cdot \frac{d^{n-1} \Psi^n}{dt^{n-1}} \right\}.$$

Now it is evident that

$$\frac{d^{n-1} \Psi^n}{dt^{n-1}}$$

is divisible both by t and $1-t$: it is therefore zero at both the limits of the integral; so that we have simply

$$A_{2n+1} = \frac{m}{1.2.3...n} \cdot \int_0^1 m \, dt \, c^{-mt} \cdot \frac{d^{n-1} \Psi^n}{dt^{n-1}}.$$

Continuing to integrate in like manner, we shall find after n successive operations,

$$A_{2n+1} = \frac{m^n}{1.2.3\dots n} \int_0^1 m dt e^{-mt} \Psi^n,$$

which is obviously a positive quantity.

By expanding, we get

$$\Psi^n = t^n (1-t)^n = t^n - n \cdot t^{n+1} + n \cdot \frac{n-1}{2} \cdot t^{n+2} - \&c.:$$

and, by performing the differential operations,

$$\begin{aligned} \frac{1}{1.2.3\dots n} \cdot \frac{d^n \Psi^n}{d t^n} &= 1 - n \cdot n + 1 \cdot \frac{t}{1} + \left(n \cdot \frac{n-1}{2} \right) \cdot n + 1 \cdot n \\ &\quad + 2 \cdot \frac{t^2}{1.2} - \&c. \end{aligned}$$

Now, because $t = \frac{x}{m}$, if we put,

$$\Psi'(x) = \frac{1}{1.2.3\dots n} \cdot \frac{d^n \Psi^n}{d t^n},$$

we shall have

$$\Psi'(x) = 1 - n \cdot \frac{n+1}{m} \cdot \frac{x}{1} + n \cdot \frac{n-1}{2} \cdot \frac{n+1 \cdot n+2}{m^2} \cdot \frac{x^2}{1.2} - \&c.$$

Another form may be given to this function; for, without any variation in quantity, t and $1-t$ may be interchanged, not only in

$$\Psi^n = t^n (1-t)^n,$$

but in all its differentials, observing that the results equal in quantity will have opposite signs when the number of differentiations is odd, and the same sign when the number is even.

Now if, instead of $t = \frac{x}{m}$, we substitute $1-t = \frac{m-x}{m}$, we shall have

$$\begin{aligned} \Psi'(x) &= \pm \left\{ 1 - n \cdot \frac{n+1}{m} \cdot \frac{m-x}{1} \right. \\ &\quad \left. + n \cdot \frac{n-1}{2} \cdot \frac{n+1 \cdot n+2}{m^2} \cdot \frac{(m-x)^2}{1.2} - \&c. \right\} \end{aligned}$$

The coefficient A_{2n+1} is thus expressed in terms of x :

$$A_{2n+1} = \int_0^m dx c^{-x} \Psi'(x);$$

the indefinite integral is

$$- c^{-x} \cdot \left\{ \Psi'(x) + \frac{d \cdot \Psi'(x)}{dx} + \frac{d d \cdot \Psi'(x)}{d x^2} + \&c. \right\}.$$

This integral, taken between the limits $x = 0$ and $x = m$, is equal to A_{2n+1} : the first form of $\Psi'(x)$ will give the values of all the differentials at the limit $x = 0$; and the second form of the same function will give the like values at the other limit $x = m$: Thus we obtain,

$$A_{2n+1} \left\{ 1 - n \cdot \frac{n+1}{m} + n \cdot \frac{n-1}{2} \cdot \frac{n+1 \cdot n+2}{m^3} - \&c. \right\} \\ \mp c^{-m} \cdot \left\{ 1 + n \cdot \frac{n+1}{m} + n \cdot \frac{n-1}{2} \cdot \frac{n+1 \cdot n+2}{m^3} + \&c. \right\},$$

the upper or lower sign taking place according as n is even or odd.

The numerical coefficients, computed by the formula, are as follows:

$$c^{-m} = c^{-10} = \cdot 0000454$$

$$A_1 = 1 - c^{-m} = 0\cdot 9999546$$

$$A_3 = \frac{4}{5} + \frac{6}{5} c^{-m} = 0\cdot 8000545$$

$$A_5 = \frac{13}{25} - \frac{43}{25} c^{-m} = 0\cdot 5199219$$

$$A_7 = \frac{7}{25} + \frac{73}{25} c^{-m} = 0\cdot 2801326$$

$$A_9 = \frac{16}{125} - \frac{726}{125} c^{-m} = 0\cdot 1277363$$

$$A_{11} = \frac{31}{625} + \frac{8359}{625} c^{-m} = 0\cdot 0502072$$

$$A_{13} = 0\cdot 0172805$$

$$A_{15} = 0\cdot 0052779$$

$$A_{17} = 0\cdot 0014467$$

$$A_{19} = 0\cdot 0003593$$

$$A_{21} = 0\cdot 0000815$$

$$A_{23} = 0\cdot 0000170$$

$$A_{25} = 0\cdot 0000036.$$

The horizontal refraction answers to $\cos \theta = 0$, $e = 1$; and the part of it depending on Q_0 is found by adding all the coefficients, viz.

$$\frac{\alpha(1+\alpha)}{\sqrt{5i}} \times 2.8024736 = 2036''.52.$$

If we take the integral between the limits $x = 0$, $x = \infty$, the result is not sensibly different, viz.

$$\alpha(1+\alpha) \int_0^\infty \frac{dx c^{-x}}{\sqrt{2ix}} = \frac{\alpha(1+\alpha) \sqrt{\pi}}{\sqrt{2i}} = 2036''.52.$$

Investigation of $\lambda \times Q_1$.

For this purpose we must find the value of

$$\begin{aligned} \int_0^m \frac{2 dx c^{-2x}}{\Delta} &= \int_0^m \frac{2 dx c^{-2x}}{\sqrt{(1-e^2)^2 + 4e^2 \cdot \frac{x}{m}}} \\ &= \int_0^m \frac{2 dx c^{-2x}}{\sqrt{(1-e^2)^2 + 4e^2 \cdot \frac{2x}{2m}}}: \end{aligned}$$

this integral has therefore the same form as Q_0 , the quantities $2x$ and $2m$ taking the place of x and m . Wherefore, if we assume

$$e \times \int \frac{2 dx c^{-2x}}{\Delta} = a_1 e + a_3 e^3 + a_5 e^5 + \&c.,$$

the value of a_{2n+1} will be found merely by writing $2m$ for m in the expression of A_{2n+1} ; but as $c^{-2m} = c^{-20}$ is extremely minute, the part multiplied by it may be neglected. Thus,

$$a_{2n+1} = 1 - n \cdot \frac{n+1}{2m} + n \cdot \frac{n-1}{2} \cdot \frac{n+1 \cdot n+2}{(2m)^2} - \&c.$$

The numerical coefficients are as follows:

$$\begin{aligned} a_1 &= 1 \\ a_3 &= 0.9 \\ a_5 &= 0.73 \\ a_7 &= 0.535 \\ a_9 &= 0.3555 \\ a_{11} &= 0.21505 \\ a_{13} &= 0.118945 \\ a_{15} &= 0.0604215 \\ a_{17} &= 0.0283127 \\ a_{19} &= 0.0122898 \\ a_{21} &= 0.0049621 \\ a_{23} &= 0.0018695 \\ a_{25} &= 0.0006623. \end{aligned}$$

These values being found, if we assume

$$\lambda \times Q_1 = B_3 e^3 + B_5 e^5 + B_7 e^7 + \&c.$$

the term $\lambda \times c^{-m} \times e$, which is insensible, being omitted, we shall have

$$\begin{aligned} B_3 &= \lambda (a_3 - A_3) = 0.021866 \\ B_5 &= \lambda (a_5 - A_5) = 0.045961 \\ B_7 &= \lambda (a_7 - A_7) = 0.055760 \\ B_9 &= \lambda (a_9 - A_9) = 0.049829 \\ B_{11} &= \lambda (a_{11} - A_{11}) = 0.036064 \\ B_{13} &= \lambda (a_{13} - A_{13}) = 0.022242 \\ B_{15} &= \lambda (a_{15} - A_{15}) = 0.012064 \\ B_{17} &= \lambda (a_{17} - A_{17}) = 0.005878 \\ B_{19} &= \lambda (a_{19} - A_{19}) = 0.002610 \\ B_{21} &= \lambda (a_{21} - A_{21}) = 0.001067 \\ B_{23} &= \lambda (a_{23} - A_{23}) = 0.000405 \\ B_{25} &= \lambda (a_{25} - A_{25}) = 0.000144 \end{aligned}$$

By making $\cos \theta = 0$, $e = 1$, we shall have, for the approximate value of the part of the horizontal refraction depending on λQ_1 ,

$$\frac{\alpha (1 + \alpha)}{\sqrt{5i}} \times 0.253891 = 184''.50.$$

If the integrals be taken from $x = 0$ to $x = \infty$, the same quantity will be

$$\begin{aligned} \lambda \times \alpha (1 + \alpha) \int \frac{dx (2c^{-2x} - c^{-x})}{\sqrt{2ix}} &= \frac{\alpha (1 + \alpha) \sqrt{\pi}}{\sqrt{2i}} \cdot \lambda (\sqrt{2} - 1) \\ &= 184''.56. \end{aligned}$$

Between the two limits, the exact quantity obtained by integrating from $x = 0$ to $x = m = 10$, must lie; so that the error of the series is of no account.

It may be proper to make an observation here, which applies generally to the kind of integrals peculiar to this investigation. The first term of λQ_1 , viz. $B_1 e = \lambda c^{-m} \cdot e$, which is rejected, varies with the height of the atmosphere. If a small number be taken for m , that is, in low atmospheres, the refractions will vary with the height, and will not agree with the observed quantities; if a considerable number be taken, as eight or ten, or any greater number, that is, if the atmosphere extend forty or fifty miles or more above the earth's surface, the refractions will not be sensibly different from what they would be in an atmosphere of unlimited height. The invariability of the refractions concurs with other phenomena to prove that the air reaches an elevation of fifty miles, more or less.

Investigation of $f \times Q_2$.

We have

$$\frac{Q_2}{e} = \int_0^m \frac{dx}{\Delta} (4c^{-2x} - 3c^{-x} + xc^{-x}).$$

Now the following formula is easily proved by differentiating,

$$\int \frac{dx}{\Delta} xc^{-x} = \frac{1}{2} \int \frac{dx c^{-x}}{\Delta} - \frac{m}{4} \cdot \frac{(1-e^2)^2}{e^2} \cdot \int \frac{dx c^{-x}}{\Delta} + \frac{m}{4} \cdot \frac{1-e^2-c^{-x}\Delta}{e^2},$$

all the integrals vanishing when $x = 0$. By extending the integrals to $x = m = 10$, in which case $\Delta = 1 + e^2$, the result will be

$$\int_0^m \frac{dx}{\Delta} xc^{-x} = \frac{1}{2} \int_0^m \frac{dx c^{-x}}{\Delta} - \frac{5}{2} \cdot \frac{(1-e^2)^2}{e^2} \cdot \int_0^m \frac{dx c^{-x}}{\Delta} + \frac{5}{2} \cdot \frac{1-e^2+c^{-m}(1+e^2)}{e^2};$$

and, by substituting this value, we shall have

$$Q_2 = 2e \int_0^m \frac{2dx c^{-2x}}{\Delta} - \frac{5}{2} \int_0^m \frac{e dx c^{-x}}{\Delta} - \frac{5}{2} \cdot \frac{(1-e^2)^2}{e^2} \cdot \int_0^m \frac{e dx c^{-x}}{\Delta} + \frac{5}{2} \cdot \frac{1-c^{-m}}{e} - \frac{5}{2} (1+c^{-m}) e.$$

The value of Q_2 will now be obtained in a series of the powers of e by putting for the integrals the equivalent series that have already been investigated. When this is done, the three first terms will be as follows:

$$\begin{aligned} & \frac{5}{2} (1-c^{-m} - A_1) \cdot \frac{1}{e} \\ & + \left(2a_1 - \frac{5}{2} A_1 + 5A_1 - \frac{5}{2} A_3 - \frac{5}{2} (1+c^{-m}) \right) \cdot e \\ & + \left(2a_3 - \frac{5}{2} A_3 - \frac{5}{2} A_1 + 5A_3 - \frac{5}{2} A_5 \right) \cdot e^3. \end{aligned}$$

Upon substituting the exact values of A_1 , A_3 , &c., the first of these terms is zero: the other two are as follows:

$$\begin{aligned} & - 8c^{-m} \times e \\ & + \frac{49}{5} c^{-m} \times e^3; \end{aligned}$$

the amount of which is very small even at the horizon; and, when multiplied by $f = \frac{2}{9}$, it becomes insensible. These terms being neglected, we may assume

$$Q_2 = C_5 e^5 + C_7 e^7 + C_9 e^9 + \&c.;$$

and we shall find

$$\begin{aligned} C_5 &= 2 a_5 - \frac{5}{2} A_5 - \frac{5}{2} \Delta^2 A_3 \\ C_7 &= 2 a_7 - \frac{5}{2} A_7 - \frac{5}{2} \Delta^2 A_5 \\ &\vdots \\ C_{2n+1} &= 2 a_{2n+1} - \frac{5}{2} A_{2n+1} - \frac{5}{2} \Delta^2 A_{2n-1}. \end{aligned}$$

The numerical coefficients will now be obtained :

$$\begin{array}{ll} \Delta^2 A_1 = -\cdot 0802325 & \\ \Delta^2 A_3 = +\cdot 0403433 & \dots\dots C_5 = \cdot 059337 \\ \Delta^2 A_5 = \cdot 0873930 & \dots\dots C_7 = \cdot 151186 \\ \Delta^2 A_7 = \cdot 0748672 & \dots\dots C_9 = \cdot 204491 \\ \Delta^2 A_9 = \cdot 0446024 & \dots\dots C_{11} = \cdot 193076 \\ \Delta^2 A_{11} = \cdot 0209241 & \dots\dots C_{13} = \cdot 142381 \\ \Delta^2 A_{13} = \cdot 0081714 & \dots\dots C_{15} = \cdot 087220 \\ \Delta^2 A_{15} = \cdot 0027438 & \dots\dots C_{17} = \cdot 046149 \\ \Delta^2 A_{17} = \cdot 0008096 & \dots\dots C_{19} = \cdot 021658 \\ \Delta^2 A_{19} = \cdot 0002133 & \dots\dots C_{21} = \cdot 009187 \\ \Delta^2 A_{21} = \cdot 0000511 & \dots\dots C_{23} = \cdot 003569 \\ \Delta^2 A_{23} = \cdot 0000105 & \dots\dots C_{25} = \cdot 001290. \end{array}$$

As the value of f is not fixed with the same certainty as that of λ , the coefficients of Q_2 have not been multiplied by f : the intention of which is to make it more easy to determine a variation of the refraction, viz. $\delta f \times Q_2$, answering to δf any variation of f that good observations may require.

The part of the horizontal refraction depending on Q_2 is

$$\frac{2}{9} \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \times 0\cdot 919534 = 148''\cdot 51.$$

If we integrate the original expression of Q_2 from $x = 0$ to $x = \infty$, e being 1, we shall have

$$\begin{aligned} &\frac{2}{9} \times \alpha(1+\alpha) \times \int \frac{dx (4c^{-2x} - 3c^{-x} + xc^{-x})}{\sqrt{2i}x} \\ &= \frac{2}{9} \cdot \frac{\alpha(1+\alpha)}{\sqrt{2i}} \cdot \frac{\sqrt{\pi}}{2} \cdot \left(2\sqrt{2} - \frac{5}{2} \right) = 148''\cdot 63. \end{aligned}$$

It thus appears that the error is less than $0''\cdot 12$; for the exact integral from $x = 0$ to $x = m = 10$ is less than the second number, and greater than the first on account of the terms of the series left out.

The next point that should engage attention is to find the value of $f' \times Q_3$. In the present state of our knowledge

of the phænomena of the atmosphere, it seems impossible to determine f' by experiments. The probability is, that it is much less than f or $\frac{2}{9}$; and as the integral Q_3 is inconsiderable except within a degree or two above the horizon, and even at such low altitudes is not great; it follows that the part of the refraction depending on $f' Q_3$ will only be sensible, if at all, when a star is distant 88° or more from the zenith. At present the probability is, that there is no other way of ascertaining the value of f' but by good observed refractions at great distances from the zenith; which observations are neither numerous nor easily collected. From the uncertainty of the term $f' \times Q_3$, it cannot be estimated in constructing a table of mean refractions, which must therefore be deduced entirely from the other three terms, as in the paper of 1823. In this manner has the table in this paper been computed, by means of the formulas now to be explained. But the term $f' Q_3$ will afterwards be discussed, and its value investigated, in order that it may be taken into account, if this should be found necessary, in the progressive improvement of the theory.

When the term $f' Q_3$ is left out, the expression of the refraction will be

$$\delta \theta = \sin \theta \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \cdot (Q_0 + \lambda Q_1 - f Q_2):$$

and if the equivalent series be substituted for the first two terms, and the series for Q_2 be multiplied by $f = \frac{2}{9}$, the result will

$$\begin{aligned} \delta \theta = \sin \theta \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \times \left\{ e \right. \\ &+ 0.821921 \cdot e^3 \\ &+ 0.552697 \cdot e^5 \\ &+ 0.302296 \cdot e^7 \\ &+ 0.132123 \cdot e^9 \\ &+ 0.043365 \cdot e^{11} \\ &+ 0.007883 \cdot e^{13} \\ &- 0.002040 \cdot e^{15} \\ &- 0.002930 \cdot e^{17} \\ &- 0.001842 \cdot e^{19} \\ &- 0.000893 \cdot e^{21} \\ &- 0.000371 \cdot e^{23} \\ &- 0.000139 \cdot e^{25} \end{aligned}$$

To bring this formula to a form more convenient for calculation, all the coefficients must be reduced to seconds. The negative terms are all very small, never amounting to so much as $6''$, and of no account whatever, except the apparent altitude be equal to 2° or less; it will therefore be proper to

separate these terms from the rest, representing their sum by the symbol $V(\theta)$. These things being attended to, we have, in the first place, this formula for computing e , viz.

$$\log . \tan \phi = \log . \sec . \theta + 19 \cdot 2067840 - 20 : e = \tan \frac{\phi}{2}.$$

Next, reducing the arcs to seconds,

$$\frac{\alpha(1+\alpha)}{\sqrt{5i}} = 726'' \cdot 687 :$$

$$\delta \theta = \sin \theta \times \left\{ \begin{array}{ll} e \times 726'' \cdot 687, & 2 \cdot 8613472^{\log.} \\ + e^3 \times 597 \cdot 280, & 2 \cdot 7761772 \\ + e^5 \times 401 \cdot 638, & 2 \cdot 6038343 \\ + e^7 \times 219 \cdot 674, & 2 \cdot 3417796 \\ + e^9 \times 96 \cdot 012, & 1 \cdot 9823255 \\ + e^{11} \times 31 \cdot 513, & 1 \cdot 4984866 \\ + e^{13} \times 5 \cdot 728, & 0 \cdot 7580287 \\ - V(\theta). \end{array} \right.$$

$$V(\theta) = \sin \theta \times \left\{ \begin{array}{ll} e^{15} \times 1 \cdot 483, & 0 \cdot 1710^{\log.} \\ + e^{17} \times 2 \cdot 129, & 0 \cdot 3282 \\ + e^{19} \times 1 \cdot 337, & 0 \cdot 1266 \\ + e^{21} \times 0 \cdot 649, & -1 \cdot 8122 \\ + e^{23} & 0 \cdot 270, -1 \cdot 4307 \\ + e^{25} & 0 \cdot 102, -1 \cdot 0072 \end{array} \right.$$

When $\theta = 87^\circ$, $V(\theta)$ is zero; and if this function be computed for every succeeding half-degree, the quantity answering to any intermediate value of θ will be found by an easy interpolation. Such is the intention of the following table; by the help of which any refraction from the zenith to the horizon may be computed by a series of the simplest form, and consisting of no more than seven terms.

θ .	$V(\delta \theta)$.
$87\frac{1}{2}$	0 ^{''} 06
88	0 ^{''} 14
$88\frac{1}{2}$	0 ^{''} 38
89	0 ^{''} 86
$89\frac{1}{2}$	2 ^{''} 30
90	5 ^{''} 97

If $e = 1$, the result will be the horizontal refraction, viz.

$$2078'' \cdot 53 - 5'' \cdot 97 = 2072'' \cdot 56,$$

which is almost exactly the same with $2072'' \cdot 46$, the quantity before computed in § 10 by a very different method.

[To be continued.]