XL. On some properties derivable from the development of a binomial; with a simplified proof of a remarkable theorem of Abel

J.R. Young

To cite this article: J.R. Young (1848) XL. On some properties derivable from the development of a binomial; with a simplified proof of a remarkable theorem of Abel, Philosophical Magazine Series 3, 33:222, 268-275, DOI: 10.1080/14786444808646096

To link to this article: http://dx.doi.org/10.1080/14786444808646096
THE following properties of the binomial development, although for the most part new to me, yet lie so near the surface, that I can scarcely suppose that they have hitherto been overlooked. It is probable, however, that there may at least be something novel in the method of investigation I have employed; and this, added to the circumstance that the proof of Abel's remarkable generalization of the binomial theorem, and of the development of the function $\phi (x + \alpha)$, is a little facilitated by this method, may perhaps justify the publication of the paper in these pages.

If we apply successive differentiation to the formula

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2} x^2 - \frac{n(n-1)(n-2)}{2 \times 3} x^3 + \&c.,$$

regarding $n$ as a positive integer, and always multiply by $x$ previously to each differentiation after the first; then, upon making $x$ equal to 1, we shall obtain these results, viz.

$$0 = - n + \frac{n(n-1)}{2} 2 - \frac{n(n-1)(n-2)}{2 \times 3} 3 + \&c. \ldots (1.)$$

$$0 = - n + \frac{n(n-1)}{2} 2^2 - \frac{n(n-1)(n-2)}{2 \times 3} 3^2 + \&c. \ldots (2.)$$

$$0 = - n + \frac{n(n-1)}{2} 2^3 - \frac{n(n-1)(n-2)}{2 \times 3} 3^3 + \&c. \ldots (3.)$$

$$\vdots$$

$$0 = - n + \frac{n(n-1)}{2} 2^{n-1} - \frac{n(n-1)(n-2)}{2 \times 3} 3^{n-1} + \&c. \ldots (4.)$$

as also

$$(-1)^n 1 \cdot 2 \cdot 3 \cdot 4 \ldots n = - n + \frac{n(n-1)}{2} 2^n - \frac{n(n-1)(n-2)}{2 \times 3} 3^n + \&c. \quad (5.)$$

In this last formula $n$ may be any positive integer without

* Communicated by the Author.

† Certain expressions analogous to these are well-known, viz.

$$0 = 1 - n \cdot 2^n + \frac{n(n-1)}{2} 3^n - \&c.,$$

where, as above, $m$ is less than $n$; but whether the forms in the text have also been given or not I am unable to say. By differentiating oftener than once, before multiplying by $x$, different zero-expressions may be obtained, as also by multiplying by powers of $x$, instead of by $x$ simply.
exception; but from the others the case of $n=1$ is excluded; it is comprehended in the well-known form

$$0 = 1 - n + \frac{n(n-1)}{2} - \frac{n(n-1)(n-2)}{2\cdot3} + \&c. \ldots (6.)$$

supplied at once by the original development.

As in each of the above forms the final coefficient is equal to unit, it follows that, by transposing the final term of (4.), we shall have the expression

$$(-n)^{n-1} = -n + \frac{n(n-1)}{2} 2^{n-1} - \frac{n(n-1)(n-2)}{2\cdot3} 3^{n-1} + \&c. \ldots (7.)$$

to $n-1$ terms; and similarly for the forms which precede (4.).

Now all these formulae may be converted into others of greater generality by aid of the following theorem, viz.

If it be true that

$$0 = -n + \frac{n(n-1)}{2} 2^m - \frac{n(n-1)(n-2)}{2\cdot3} 3^m + \&c. \ldots (8.)$$

throughout a certain range of consecutive positive integer values of $m$, then it is equally true that, throughout the same range, the following more general form will have place:

$$0 = x^m - u(x+\beta)^m + \frac{n(n-1)}{2} (x+2\beta)^m - \frac{n(n-1)(n-2)}{2\cdot3} \ldots (9.)$$

and also that if (5.) be true, then will

$$(-1)^n 1.2.3\ldots n = x^n - n(x+1)^n + \frac{n(n-1)}{2} (x+2)^n$$

$$- \frac{n(n-1)(n-2)}{2\cdot3} (x+3)^n + \&c. \ldots (10.)$$

be equally true, $x$ and $\beta$ being any values whatever.

This theorem may be established either by common algebra, or by the first principles of the integral calculus; the latter being the shorter method, I shall employ it here.

Suppose the formula (9.) to be true for some preceding value of $m$, as $m=p$, so that we may have

$$0 = x^p - n(x+\beta)^p + \frac{n(n-1)}{2} (x+2\beta)^p - \frac{n(n-1)(n-2)}{2\cdot3} (x+3\beta)^p + \&c. ;$$

then, multiplying by $(p+1) dx$ and integrating, we have

$$C = x^{p+1} - n(x+\beta)^{p+1} + \frac{n(n-1)}{2} (x+2\beta)^{p+1} - \frac{n(n-1)(n-2)}{2\cdot3} \ldots (10.).$$
To determine the constant C, put $x=0$; and we have

$$C = \beta^{p+1} \left\{ -n + \frac{n(n-1)}{2} \beta^{p+1} - \frac{n(n-1)(n-2)}{2.3} \beta^{p+1} + \&c. \right\},$$

which, by (8.), is equal to zero, $p$ being by hypothesis less than $m$: hence if the formula hold for a value less than $m$, it holds also for the next value, and so on for all values up to $m$; and as it obviously holds for $p=0$ by (6.), the formula is general.

If, however, we suppose $p$ to be so great as $n-1$, then the expression for $C$ will be

$$C = \beta^n \left\{ -n + \frac{n(n-1)}{2} \beta^{n-1} - \frac{n(n-1)(n-2)}{2.3} \beta^{n-1} + \&c. \right\}, \quad (11.)$$

in which, if we make $\beta=1$, we have for $C$, as shown in (5.),

$$0 = \beta^{n-1} \left\{ -n + \frac{n(n-1)}{2} \beta^{n-1} - \frac{n(n-1)(n-2)}{2.3} \beta^{n-1} + \&c. \right\}, \quad (12.)$$

and therefore the general formula (10.) is also true.

It thus appears that we may give to $m$ in (9.) any positive integral value up to $n-1$; for this latter value the formula is

$$0 = x^{n-1} - n(x+\beta)^{n-1} + \frac{n(n-1)}{2} (x+2\beta)^{n-1} - \frac{n(n-1)(n-2)}{2.3} (x+3\beta)^{n-1} + \&c.;$$

and therefore, by transposing the $(n+1)$th term,

$$(-1)^{n-1} (x+n\beta)^{n-1} = x^{n-1} - n(x+\beta)^{n-1} + \frac{n(n-1)}{2} \quad (13.)$$

This expression will be useful in simplifying the investigation of the theorem of Abel already alluded to.* There is only one step in Abel's process in which the need for any such simplification can be felt; it is that in which the constant in the result of his integration is to be determined; and where, by the analytical artifice employed by Abel, there seems to be something like an anticipation of the value sought.

Abel's generalization of the binomial theorem is thus expressed:

$$(x+a)^n = x^n + na(x+\beta)^{n-1} + \frac{n(n-1)}{2} a(\alpha-2\beta)(x+2\beta)^{n-2} + \frac{n(n-1)(n-2)}{2.3} a(\alpha-3\beta)(x+3\beta)^{n-3} + \&c.;$$

$$n(\alpha - (n-1)\beta)^{n-2}(x+(n-1)\beta) + a(\alpha - n\beta)^{n-1} \quad (14.)$$

* Oeuvres Complètes, vol. i. p. 31.
where \( x, \alpha, \) and \( \beta \) are any quantities whatever, and \( n \) any positive integer.

That this is true when \( n=0 \), is obvious; and it may be shown, as follows, that if it be true for \( n=m \), it must be true for \( n=m+1 \); and thence that it is true generally*.

Let

\[
(x+\alpha)^m = x^m + mx(x+\beta)^{m-1} + \frac{m(m-1)}{2} \alpha(x-2\beta)(x+2\beta)^{m-2}
\]

\[
+ \frac{m(m-1)(m-2)}{2.3} \alpha(\alpha-3\beta)(x+3\beta)^{m-3} + \ldots .
\]

\[\ma(\alpha - (m-1)\beta)^{m-2}(x+(m-1)\beta) + \alpha(\alpha-m\beta)^{m-1};\]

then, multiplying by \((m+1)dx\), and integrating, we have

\[
(x+\alpha)^{m+1} = x^{m+1} + (m+1)x(x+\beta)^m + \frac{(m+1)m}{2} \alpha(x-2\beta)
\]

\[
(x+2\beta)^{m-1} \ldots .(m+1)x(\alpha-m\beta)^{m-1}(x+m\beta) + C.
\]

To determine \( C \), put \( x=-\alpha \); then, dividing by \(-\alpha\),

\[
0 = (-\alpha)^m - (m+1)(-\alpha+\beta)^m + \frac{(m+1)m}{2} (-\alpha+2\beta)^m - \ldots .
\]

\[
(m+1)(-\alpha+m\beta)^m - \frac{C}{\alpha};
\]

\[\therefore \frac{C}{\alpha} = (-\alpha)^m - (m+1)(-\alpha+\beta)^m + \frac{(m+1)m}{2} (-\alpha+2\beta)^m - \ldots \text{to } m+1 \text{ terms.}
\]

But, by (13.), the right-hand member of this is the development of

\[
(-1)^m(-\alpha+(m+1)\beta)^m, \text{ or } (\alpha - (m+1)\beta)^m;
\]

\[\therefore C = \alpha(\alpha - (m+1)\beta)^m.
\]

Hence the theorem has place for \( n=m+1 \), and thence generally.

The theorem (12.) is included in this of Abel, as that illustrious analyst has noticed; but the other analogous theorems in this paper, where the exponents are different from \( n=1 \), are not implied in the theorem of Abel, nor have I ever met with them.

It is probable, however, that some works on finite differences may contain these expressions; but this is a point about which I have not thought much inquiry necessary, as my chief object is to show, in detail, how a remarkable theorem in the higher analysis may be arrived at by only the simplest principles of the calculus, combined with common algebra.

* The process, up to the determination of \( C \), is Abel's.
If we interchange \(x\) and \(a\), the foregoing development will become

\[
(x + a)^n = x^n + n(x + \beta)^{n-1}x + \frac{n(n-1)}{2}(x + 2\beta)^{n-2}x(x - 2\beta) + \frac{n(n-1)(n-2)}{2.3}(x + 3\beta)^{n-3}x(x - 3\beta)^2 + \&c.
\]

Having obtained these results for the expanded binomial, in the case of \(n\) a whole number, it is natural to inquire whether the theorem does not admit of a wider application; whether indeed it is not perfectly general for every function of \((x + a)\).

Let us then assume, from the law thus suggested, that

\[
\phi(x + a) = A + Bx + Cx(x - 2\beta) + Dx(x - 3\beta)^2 + Ex(x - 4\beta)^3 + \&c.
\]

\[
\begin{align*}
\frac{d\phi(x + a)}{dx} &= B + 2C(x - \beta) + D\{2(x - 3\beta)^3 + 2x(x - 3\beta)\} + \&c. \\
\frac{d^2\phi(x + a)}{dx^2} &= 2C + 2.3D(x - 2\beta) + 2.3E \cdot (x - 2\beta) + \&c. \\
\frac{d^3\phi(x + a)}{dx^3} &= 2.3D + 2.3.4E(x - 3\beta) + \&c. \\
\frac{d^4\phi(x + a)}{dx^4} &= 2.3.4E + \&c.
\end{align*}
\]

Now the right-hand members of these equations become each reduced to its first term by putting \(x = \beta, x = 2\beta, x = 3\beta, \&c.\), and so on in succession. Hence

\[
\phi(x + a) = \phi(x) + \frac{d\phi(x + a)}{dx}x + \frac{d^2\phi(x + a)}{dx^2}x(x - 2\beta) + \frac{d^3\phi(x + a)}{dx^3}x(x - 3\beta)^2 + \&c.;
\]

or, interchanging \(a\) and \(x\), and remembering that

\[
\frac{d^n\phi(x + a)}{dx^n} = \frac{d^n\phi(x + a)}{(dp)^n},
\]

we have

\[
\phi(x + a) = \phi(x) + \frac{d\phi(x + a)}{dx}x + \frac{\phi(x) - 2\phi(x + a)}{2}x + \frac{\phi(x) - 3\phi(x + a)}{2.3}x + \&c.,
\]

\[
\phi(x + a) = \phi(x) + \frac{\phi(x) - 2\phi(x + a)}{2}x + \frac{\phi(x) - 3\phi(x + a)}{2.3}x + \&c.,
\]

\[
\phi(x + a) = \phi(x) + \frac{\phi(x) - 2\phi(x + a)}{2}x + \frac{\phi(x) - 3\phi(x + a)}{2.3}x + \&c.,
\]
which theorem is established by Abel in an elaborate memoir at page 82 of his second volume.

If we put $x-h$ for $x$, and make $\alpha = h$, and $\beta = -h$, we shall have

$$\varphi(x) = \varphi(x-h) + \frac{d\varphi(x-2h)}{dx} \frac{2h}{2} + \frac{d^2\varphi(x-3h)}{dx^2} \frac{(3h)^2}{2.3} + \&c.,$$

a theorem given by Murphy in the Philosophical Transactions for 1837, and which is usually referred to as "Murphy's theorem." It appears, however, from the above method of obtaining it, that it is a particular case of the general theorem of Abel. But as Abel's posthumous papers—among which this theorem was found—were not published till 1839, ten years after the author's death, Murphy's claim is quite independent, although not, as hitherto supposed, exclusive.

The preceding method of establishing this general theorem of Abel, by assuming a series of a particular form, and then, as in the investigation of Maclaurin's theorem, putting, in the successive differential coefficients, those values for $x$ which cause all the terms containing this variable to vanish, is very short and simple; but without the previous proof of the particular case of the binomial theorem, we should have had nothing to suggest to us the peculiar form selected. This suggestion, however, once supplied, we may, instead of examining each differential coefficient in succession, as above, and thence inferring the general uniformity of the law of the coefficients $A$, $B$, $C$, &c. by induction,—instead of this, we may proceed at once to demonstrate the general property that

$$\frac{d^n x(x-m\beta)^{m-1}}{dx^n} \ldots \ldots \ldots (A.)$$

is zero for $x=n\beta$, provided $n < m$; and that it is $1.2.3\ldots n$, for $n=m$; which property will establish the above values of $A$, $B$, $C$, &c. rigorously and universally.

To prove it, conceive $x$ in (A.) to be changed into $x+h$, the binomial to be developed, and the series multiplied by $x+h$: the coefficient of $h^n$, in the result, will, of course, when multiplied by $1.2.3\ldots n$, be the $n$th differential coefficient (A.). It is easy to see that this coefficient, that is the coefficient of $h^n$, is

$$\frac{(m-1)(m-2)(m-3)\ldots(m-n)}{1.2.3\ldots n} x(x-m\beta)^{m-n-1}$$

$$+ \frac{(m-1)(m-2)\ldots(m-n+1)}{1.2\ldots(n-1)} (x-m\beta)^{m-n},$$

* The more general form of the theorem is got by putting

$x=x-\alpha$, $\alpha=nh$, $\beta=-h$.  

Downloaded by [Universite Laval] at 15:57 25 June 2016
of which the factors involving $x$, when $n\beta$ is put for $x$, are

$$n(n-m)^{m-n-1}\beta^{m-n}, \text{ and } (n-m)^{m-n}\beta^{m-n};$$

so that the expression for the $nth$ differential coefficient, still omitting the numerical factor $1.2.3\ldots n$, is

$$\frac{(m-1)(m-2)\ldots(m-n+1)}{1.2\ldots(n-1)} \left\{ -(n-m)^{m-n} + (n-m)^{m-n} \right\} \beta^{m-n},$$

which, when $n < m$, is zero. If $n=m$, the first term of the general coefficient of $h^n$ above, is simply unit: hence, introducing the absent numerical factor, we have in this case

$$(A.) = 1.2.3\ldots n,$$

as was to be proved.

The expression under the sign of differentiation in (A.), if developed, and then the series differentiated $n$ times, $n\beta$ being afterwards put for $x$, will furnish new properties of the binomial coefficients, which need not however be here formally exhibited.

There are probably other expressions, analogous to (A.), that might suggest developments of like generality with that of Abel; but I question whether they are worth searching for. In fact, and the remark is of some importance, all such general theorems must be received with caution and qualification: they are strictly true only when the terms of the series become at length actually zero, or at least tend to zero. Without this qualification, the binomial theorem itself, even in its ordinary form, would be fallacious. Analysis, unfortunately, abounds with fallacies of this kind. Theorems are sometimes presented to us with an aspect of unlimited generality, which are not true even in a single individual case! The following is an instance: viz.

$$\phi(x) = \phi(x+\alpha) + \phi(x-\alpha) - \phi(x+2\alpha) - \phi(x-2\alpha) + \ldots,$$

the simplest case of which, viz.

$$x = x + x - x - x + x + \ldots,$$

is repugnant to common sense.

This theorem was, I believe, first given by Français in the Annales des Math., vol. iii. It was again obtained by Abel, in vol. ii. of his Œuvres, p. 85, and has been reproduced by Gregory in his Examples, p. 243. It is the tendency of analysis, if left unrestrained, to spread, in some directions, into unprofitable luxuriance, by which it is encumbered, but not advanced. A little in the way of correcting this fault, I have myself attempted in the pages of this Journal and elsewhere; but much more is required. This, however, is a task which I leave to those whose position may be less inimical to the
cultivation of science than mine happens to be. I shall merely add, in conclusion, that in order not to encroach unreasonably on these pages, I have transferred two articles connected with the present paper to the Mechanics' Magazine; they probably appear in the numbers for Sept. 16 and Sept. 23; in the latter of which I have given a short and easy investigation of the theorem of Leibnitz for the advanced differential coefficient of the product of two functions.

Belfast, Aug. 26, 1848.

XLI. Sixth Memoir on Induction. By M. Elie Wartmann, Professor of Natural Philosophy in the Academy of Geneva*.

[Continued from p. 94.]

§ XVIII. Does Induction affect the acoustic properties of Elastic Bodies?

165. I PUBLISHED two years ago† some experimental researches on the causes of the sounds produced by discontinuous electric currents in metallic wires. This phenomenon may be viewed under other aspects, and it may be asked whether a permanent electric induction determines in the molecules of sonorous bodies a change of elasticity which is rendered evident by appreciable modifications in their acoustic properties. The following experiments have been undertaken with a view to solve this question.

166. A disc was selected of 0.198 diameter, and 0.0018 thickness, forming part of a Marloye's set of plates. This disc was made with the greatest possible care: its texture is remarkably homogeneous, and the acoustic figures, formed of diameters or concentric circumferences, are produced upon it with extreme precision. Its lower surface was covered with a thick layer of gum-lac varnish; and after strongly electrizing this sort of electrophore, the charge was sustained by means of a good machine. It was however impossible to discover the least difference in its sonorous state, whether it was electrified or not.

167. A glass disc, 0.135 in diameter and 0.002 in thickness, was furnished on both its surfaces with a circular armature of tinfoil, 0.117 in diameter. This flat condenser was

* Communicated by the Author.

† Archives des Sciences Physiques et Naturelles, vol. i. p. 419. In the sitting of the 8th of May last, M. Wertheim presented to the Academy of Sciences of Paris some further researches on this subject, the conclusions of which are identical with those which I then enunciated with respect to the effects of discontinuous currents. See the Institut of the 10th of May, No. 749.