On the Strength of Ductile Materials under Combined Stress

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XVIII. *On the Strength of Ductile Materials under Combined Stress.* By JAMES J. GUEST *. (Communicated by the Secretaries.)

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1. *LACK of knowledge of the Laws of Strength.*—From the point of view of both pure and applied science, it would be of interest to know the complete laws of the strength of materials; but although a multitude of tests have been made in certain simple modes, our knowledge of the laws of strength has been extended by few experiments exposing the material to two or more principal stresses, and thus, except in the simpler cases, the elastician is without experimental guidance as to what he should seek analytically as determining the strength of the body under consideration.

In the series of experiments herein described the materials employed have been subjected to a certain variety of simultaneous principal stresses; and the results are recorded in the hope that they will prove of service to elasticians and engineers.

The simplest, and most primitive, method of ascertaining the strength of a material is to subject a cylindrical specimen of it to a direct tension or compression, and to increase the

force until the specimen breaks or collapses; the breaking-stress, thus found, being taken as the basis for calculations of the strength of all pieces of that material used in structures or machines.

As material is frequently exposed to torsion, another frequently employed test is to break a circular cylinder of material by the application of a torque, the stress so found being used in calculations for shafts, &c.

2. *Separation of the Isotropic Materials into Ductile and Brittle.*—A consideration of the form of the fractured surface in the various types of test affords some suggestions. In the case of ductile materials such as iron and steel (structural) broken by tension, the material draws out at the point of rupture and the surfaces of fracture, plane or conical, are partly or entirely inclined to the axis of the specimen at about 45° . If, however, cast iron be broken in tension, there is no evidence of drawing out, and the fracture is practically normal to the axis of the specimen. When ductile specimens are broken in torsion, the surfaces of rupture are normal to the axis; but in the case of cast iron the surface of rupture takes a curious form, one of the edges of the fracture of a circular cylindrical specimen being a helix whose angle is about 45° . Some steels of a treacherous nature exhibit a fracture partaking of the two types, part of it being normal to the axis, but part extending along the length of the specimen and having the helical edge. Under compression ductile materials ultimately give way by lateral flow; cast-iron specimens exhibit rupture-surfaces inclined to the axis at about 30° .

It will be noticed that in all cases the ductile materials exhibit rupture-surfaces coincident to a large extent with the surfaces of maximum shearing-stress, suggesting that the final action in such cases partakes of the nature of a viscous flow. Cast iron, however, under tension and torsion fractures along the surfaces across which the tension or elongation is greatest; in compression it appears to fail by a shearing action, influenced perhaps by the friction of the crystals.

These illustrations of modes of fracture tend to show that a sharp line of demarcation may be drawn between the behaviour under stress of ductile and other materials, and this in spite

of the gradual transition from one stage to the other in manufacture and in use.

Owing to the increasing tendency to substitute steel for cast iron in structures and machines, the ductile materials were selected as the subject of experiment; steel of four grades, copper, and brass have been experimented upon.

3. *The Yield-point in Tension Tests.*—Upon observing the operation of testing more carefully and taking measurements of the extension at frequent intervals, we notice that under the lower loads the specimen stretches so very slightly as hardly to affect the beam of the testing-machine. This continues until we reach a point called the yield-point, marked by a large extension of the specimen for a slight increase of load, the lever of the testing-machine in consequence moving considerably. At the same time the scale, if there is any, on the specimen peels off, starting by cracks inclined at about 45° to the generating-lines, separating the scale into squares; or if the specimen be machined, certain characteristic changes take place upon its surface. After some time the stretching under this load ceases and the load on the specimen can be increased, the elongation also increasing until rupture takes place. The loads and the measurements of the extension they produce may be plotted into a curve of which an example is shown in fig. 1, which is reproduced from a diagram (for Swedish iron) drawn automatically by a Wicksteed testing-machine, the ordinates being loads and the abscissæ actual extensions in an original length of 7 inches. Up to the yield-point the extension is very small, being about one-thousandth part of the length of the specimen; the final elongation, however, was in this case 29 per cent. of the original length.

4. *The Yield-point, rather than the ultimate, Stress the criterion of strength.*—A closer examination into the relation of the stress and strain below the yield-point shows that they are proportional to one another (Hooke's law), and it is upon this proportionality, or rather upon a generalization of which the above is a special case, that the theory of elasticity is based. Now if over any portion of a body the yield-point of the material is exceeded, the basis of our calculations is vitiated, and the redistribution of stress and strain may be widely

Fig. 1. Typical Stress-Strain Curves.
 2. Testing Apparatus.
 3. New Pressure Gauge.

different from the proposed distribution and lead to immediate failure. For this reason the yield-point, rather than the breaking, strength should be taken as the criterion of the strength of a ductile material in simple tension.

Further arguments to the same effect arise from Wöhler's experiments, in which he showed that the frequent repetition of a stress slightly exceeding the yield-point stress will finally cause rupture. The yield-point as the criterion of strength is so far accepted in modern practice, that most specifications of steel now call for a yield-point not lower than one half the ultimate strength.

5. *Variations of the Ultimate and Yield-point Stresses: Annealing.*—Neither the ultimate nor the yield-point stress for a material is quite definite. For the former, besides depending upon the taking account of, or the neglect of, the reduction of area due to drawing out, is affected by the rate of application of the load; in fact, the phenomena displayed by ductile materials under stresses approaching their ultimate value are rather the dynamics of a semiviscous fluid than the statics of an elastic body. And the yield-point is affected by mechanical treatment.

If in a tension test, after the stress-strain curve has begun to rise after passing the yield-point, the load be removed, the specimen is to all appearance unchanged. Let it, however, be regarded as a new specimen and be tested: it will exhibit a yield-point close to the greatest stress previously applied to it; but a continuance of the test will show that its capability of extension has been diminished. Such false yield-points, due to stresses or mechanical treatment, can be removed by annealing, which is generally considered to render the material isotropic.

6. *The Elastic Limit.*—It would so far appear that the yield-point is a definitely defined point with the material in its normal condition, but a closer examination of the phenomena involved brings to light a difficulty in its accurate determination. In addition to the autographic curve in fig. 1, a highly magnified edition of the same curve is shown, the points on this curve being obtained by the use of an extensometer; the curve is referred to as the "extensometer curve." At some point D in the extensometer curve (fig. 1), before

the yield-point A, the stress-strain curve for the material ceases to be a straight line, and at the same point a time-effect manifests itself, the material yielding a little for some time after the application of the load. This time-effect renders the value of the strain just as the yield-point is reached a somewhat vague quantity; it, however, does not very materially affect the determination of the stress at the yield-point. The point at which this deviation of the straight line and time-effect appears is usually known as the elastic limit. I shall refer to the effect of time upon the strain and to the deviation of the stress-strain curve from linearity as the elastic-limit effect.

7. *The Elastic-limit effect due to the existence of Local Yield-points.*—This phenomenon may be one *sui generis*, but, partly as the departure from the straight line and the manifestation of the time-effect are simultaneous, I am inclined to think that it is due to local variations in the material, which partly by disturbing the uniform distribution of stress and strain in their neighbourhood, and partly by their nature, result in small volumes of the material reaching the yield-point prematurely, and so cause the specimen as a whole to exhibit a foreshadowing of the yield-point phenomena. On this supposition, if a specimen be once strained, as a whole, up to or beyond the yield-point, at the locations which give rise to the elastic-limit effect, the strain would be carried well beyond the yield-point; and if the load were removed, on a subsequent test they would contribute no time-effect until their new yield-point was reached, which would occur simultaneously with the yield-point of the whole specimen; hence a specimen would display no elastic-limit effect, or at any rate a very much reduced one, on a second testing. This agrees with experiment, and illustrations of it will be pointed out later. (See figs. 16, 17, & 18.)

Furthermore, although the effect is removed from a particular kind of test, it is unaffected, or even increased, for tests involving the application of stresses of a different type; examples of this may be seen in figs. 16 & 18, in which a tension and a torsion test respectively upon the material are compared, the material in the interval of the two tests having been subjected to yield-point stresses of a different type.

Not very many years ago the existence of the yield-point was not recognized, the stress-strain curve sketched as normal showing a gradual bend over (somewhat similar to the copper tension tests shown in figs. 21-24). This would be the case with imperfect material, the portions of the material in the neighbourhood of flaws and variations reaching the yield-point successively, and so masking the phenomena. There is no doubt that the materials of construction as produced to-day are much more uniform and free from defects than those of a quarter of a century ago, and thus can display phenomena perhaps hidden in the past by their imperfections.

It should be noted that the same effect (elastic-limit effect) on the stress-strain curve is caused by the non-coincidence of the axis of the specimen with the load-line.

8. *The Yield-point, in preference to the Elastic limit, selected as the criterion of strength.*—The elastic-limit effect may then be merely due to local yielding, and taking this view of it I have regarded the yield-point as the true criterion of the strength of the material, and sometimes have assumed that Hooke's law held up to it.

In all those of the experiments of this series which were made upon steel, a yield-point, sometimes much obscured by elastic-limit phenomena, was sought for and determined; and evidence will be given to show that, although there is on the whole a slight tendency for an annealed specimen, when frequently stressed, to have its yield-point rise, yet, if the material be not much strained at the yield-points, this rise is not very large, especially if the type of stress be varied. In the case of copper, the curves obtained were such as to make it difficult to determine the yield-point, if indeed one exists; several of the curves, however, have been plotted out so that the range of choice may be appreciated.

9. *Torsion, a case of Combined Stress.*—The case of torsion presents a case of combined stresses, two principal stresses being equal in amount but opposite in sign, while the third is zero. Of such tests a large number have been made on round bars, but the essentials of the phenomena involved are masked by the variation of strain from the axis outwards, which causes the outer layers to reach the yield-point first, and so produce effects corresponding to the explanation above

given of the elastic-limit effect in simple tension. As any defects in the material will contribute their effect, the result is that it is practically impossible to select the yield-point, or to determine with any degree of precision the point where the curve first deviates from the straight line.

10. *First reason for use of Thin Tubes as specimens.*—To avoid this masking of the phenomena, I conducted my experiments upon thin tubes, the walls of which were from $\frac{1}{20}$ to $\frac{1}{25}$ of the radius in thickness. I also tested two solid bars, more to ascertain, from the agreement of the elastic constants with their known values, that no large mistake had been made in the measurement of the parts of the apparatus. An inspection of the torsion stress-strain, or rather torque-twist curves (fig. 15), for these bars will show the practical impossibility of locating the yield-point (or the elastic-limit, if it be a true phenomenon), particularly if we bear in mind Mr. Love's theorem that a small round flaw will double the shearing-stress in its neighbourhood, which if the flaw were near the surface would produce the elastic-limit effect at about half the true yield-point torque.

11. *Results of previous Torsion Experiments.*—Published results of torsion experiments (see Unwin, 'Machine Design,' vol. i.) make the elastic strength of iron and bronze in torsion to be from 0.625 to 0.735 of the strength in tension: for copper the ratio is about 0.545. The ratio of the ultimate strengths is a little higher. The ultimate strength in torsion should be determined by the use of the formula $T = \frac{2}{3}\pi q r^3$, which considers the whole of the material across a normal section to be in the same state of stress, but as the employment of this formula is not universal, the ratio of the ultimate stresses is open to suspicion.

12. *Previous experiments upon the Yield-point under Combined Stress.*—Beyond the numerous experiments upon torsion, the only experiments upon the yield-point which I have been able to find are those made upon pianoforte-wire by Mr. McFarlane for Lord Kelvin's article upon Elasticity in the *Encyclopædia Britannica*. These showed that a simultaneous tension lowered the yield-point in torsion, but numerical data are not given. Lord Kelvin concludes that as a tension lowers the torsional yield-point, a compression

would raise it; but his argument does not appear to be rigorous and I venture to think that either tension or compression would lower it.

13. *Experiments upon Ultimate Strength under Combined Stress.*—Of experiments upon the ultimate strength under combined stress, there are the well-known experiments of Tresca, which show that a ductile material eventually behaves like a very viscous fluid, and a few experiments performed by Messrs. W. Voigt and L. Januszkiewicz on wax rods subjected to tension only, and to tension and external fluid pressure (compressed air) combined. In the latter case it was found that at rupture the shearing-stress in the material was the same in the two cases, thus agreeing with Tresca's experiments upon metals.

14. *Theories of Elastic Strength under Combined Stress: the Maximum Stress Theory.*—As examples of combined stress continually arise in practice, some working theory as to the criterion of strength, preferably elastic strength, must be assumed and adopted. Two theories hold the field to-day. The first is the assumption, it can hardly be called by any more ambitious name, that the material yields when one of the principal stresses reaches a certain amount, which must, by taking a special case, be the stress determined by a simple tension experiment. This was the theory adopted, in the absence of experimental data, by Rankine, and it is the one used by English and American engineers.

As all experiments upon torsion give results at wide variance with this theory, it can hardly be considered to be correct; and if it is used, a different working stress should be adopted in cases of torsion or systems of stress approximating thereto.

15. *The Maximum Strain Theory.*—The second theory is that the material yields when the greatest strain reaches a certain amount, which must, taking a special case, be the yield-point strain in simple tension. This theory was first advocated by St. Venant as fitting in with that molecular theory which leads to the uniconstant theory of elasticity. Upon this theory the value of Poisson's ratio, or the ratio of lateral contraction to axial extension of a specimen under simple tension, is 0.25; upholders of the theory maintaining that variations from this value which experiment exhibits are due

to imperfections of the material. The greatest-strain theory is that adopted upon the Continent, and is strongly upheld by many elasticians.

16. *Theorem upon the Limiting Values of σ .*—In the case of a cube of elastic material, whose elastic constants are E and σ (E being Young's modulus, or the ratio of stress to strain in a simple tension test), subjected to a uniform fluid pressure p , the linear contraction is $\frac{p}{E}(1-2\sigma)$. If the constitution of the material is to be stable, this must be a positive quantity, and hence the maximum value of σ is 0.5. This also shows that an isotropic stable material cannot decrease in volume under a simple tension (else it would decrease volumetrically under three orthogonal tensions).

It is hardly conceivable that σ should be negative, but it cannot be less than -1 ; otherwise the material would be unstable under torsion as the energy of a simple shear, $\frac{p^2}{E}(1+\sigma)$, would be negative should $1+\sigma$ be negative.

The values of σ met with in the course of experiments undertaken will be found in Table V. They are not determined by a method possessing much accuracy, being found from the values of E and C ; they are, however, sufficiently accurate for the purpose in view.

17. *The Maximum Strain Theory not disproved by published experiments.*—Upon the maximum strain theory of strength the yield-point stress under torsion to that under tension should be 0.80 for cases in which $\sigma=0.25$. The least value of this ratio will occur when σ is a maximum, *i.e.* when $\sigma=0.5$, the value then being 0.66. As the experimental results quoted above place this ratio, for steel, between 0.625 and 0.735, it will be seen that, taking into consideration the effect of flaws upon a torsion test and the difficulty of locating the yield-point, the experimental results cannot be held to disprove the theory, although they militate against it.

In discussing the phenomena of torsion, Tresca separates the state of the material into three stages: the elastic state, the plastic stage, and an intermediate condition. These correspond to the portions OA, BC, and AB respectively in a stress-strain curve such as fig. 1. St. Venant, followed by

many elasticians, does not recognize the intermediate stage, and considers that Hooke's law holds up to the point at which plasticity begins; he adopts Tresca's results that in the plastic stage the shearing-force is constant, and upholds a specific maximum strain as the condition of limiting elasticity.

I fail to understand how a material could have one condition for the commencement of plasticity and an entirely different one for its existence; perhaps users of these conditions tacitly admit the existence of the intermediate stage, but neglect it for the simplification of calculation and because the physical difference between the elastic and plastic states is so great; or perchance they do not admit the rigour of the deduction of the plastic law from Tresca's experiments. [St. Venant's proof (*Comptes Rendus*, lxx., and Todh. & Pearson's 'History of Elasticity,' vol. ii. § 236) that in the plastic stage the resistances to slide (shearing-strain) and elongation have the same value, consists in equating the work done in similar changes under a simple shear q and a combined stress $+p$ and $-p$, and hence showing that $q=p$. This is true in the elastic as well as in the plastic stage; but the combined stress $+p$ and $-p$ is merely another view of the shear, and constitutes a very different condition to a simple tensional stress, so that St. Venant's final step seems illogical.]

18. *The Maximum Shearing-Stress criterion of Elastic Strength*.—A third theory of elastic strength, mentioned by Cotterill in his 'Applied Mechanics,' is that the condition of yielding is the existence of a shearing-stress of a specific amount. Having regard to the behaviour of the material when in the plastic stage, the assumption of continuity would lead to this theory, but although the shearing-stress law of the plastic stage has been generally recognized, no formulas for elastic strength in particular cases have been based upon a similar law for the yield-point. The knowledge of Tresca's results and the observation of the phenomena presented by cold bending, tension and compression tests, punching, and the general manipulation of metal, must have urged many towards the conclusion that this is the true criterion of elastic strength in a ductile material.

19. *Further reasons for adopting Thin Tubes as specimens*.—It has been stated above, that tubular specimens were used in

this series of experiments. In addition to the reason there advanced, I was influenced by the advantage which tests made by subjecting the tubes to internal fluid-pressure would have in confirming, or not, the various theories. In such tests it is evidently an advantage to have the thickness of the tube small compared with the radius, as the stress will be more uniform, and the yield-point in consequence more sharply defined.

20. *Range of Stress covered by different types of experiment.*

—If a specimen of circular section be submitted to torsion and tension combined in various ratios, the stresses at the exterior may be made to have all combinations between the pairs $-p$, $+p$, and 0 , $+p$ [throughout I have considered tensions and elongations as positive, owing to their more frequent occurrence]; and if compression were substituted for tension the range between $-p$, $+p$, and $-p$, 0 would be included. Nothing beyond these, *i.e.* no like principal stresses can be produced by this method; these, however, can be readily impressed on the material of a hollow cylinder by the use of fluid pressure. By the combination of tension with internal pressure we can extend the range from 0 , $+p$, to $+p$, $+p$; and by combining external pressure and axial force the range from $-p$, 0 to $-p$, $-p$ could be covered, but owing to the probable instability of a tube of suitable dimensions under these conditions, experiments upon this part of the range were deferred.

21. *Method of checking the Isotropism of the Material as regards Yield-point Stresses.*—This, however, is not the only advantage of the internal pressure and tension tests, for by their use we can check the isotropism of the material within the range $+\frac{1}{2}p$, p , by having the greater of the stresses first parallel to the axis and then circumferential: to do this within the range 0 , $+\frac{1}{2}p$ it would be necessary to use a compressive load and internal pressure, as the longitudinal stress due to the internal pressure must be added to the longitudinal stress due to the externally applied load.

In addition, tests under the combination of internal pressure and torsion permit the principal axes of the stress to be inclined at varying angles to the generating lines of the specimen, and also, if a practical amount of isotropism has

already been demonstrated by the internal pressure and tension tests, afford a check upon the general accuracy of the work by comparing the results of these tests with the results of experiments involving similar stresses differently applied.

Torsion tests on thin tubes, it may be noticed, cause a tendency in the wall to buckle, while the internal-pressure and tension tests do not. No buckling occurred in the experiments.

Although combined stresses may be produced by simpler methods, such as the loading of a cranked shaft or of a helical spring, such methods, since a small portion only of the material is carried to the yield-point condition, are hardly likely to furnish decisive results. It is essential for definition, both because of the elastic-limit effect and because of the smallness of the elastic strain, that the whole of the material directly under observation should be similarly stressed. The simplest methods of attaining this appeared to me to be those indicated above, namely, the application of axial loads, torques and internal pressures, singly or in pairs, to thin-walled circular tubes. As the production of thin-walled specimens otherwise than by drawing would be difficult, I determined to use seamless drawn tubing as specimens, and to remove the state of set left by the operation of drawing by annealing.

22. *Objections to Tubes on account of want of Isotropy.*—It has frequently been urged that no such material as wires, tubing, or sheet-metal can be regarded as isotropic; but I think such criticism may fairly be met by the consideration that in the production of ductile materials, operations producing sets and internal stresses, afterwards removed by annealing, are invariable and essential to the manufacture of a uniform material, and that annealing seems usually to remove this state of set. In the results of the tests shown, it will appear, at any rate so far as the occurrence of the yield-point was concerned, that the materials were practically isotropic.

In the case of experiments upon wires, which after annealing show a marked want of isotropy, it is probably due to overdrawing in the manufacture of the wire, which produces a series of conical defects along the wire, the effect of which evidently could not be removed by annealing. Unless the

wire is wanted particularly hard for some special purpose, this defect is not likely to occur.

23. *Objection to Tubes on account of large effect of defects.*—Another objection, which can well be raised to the use of such thin-walled tubes as those which I experimented upon, is the large effect which would be produced by variations and defects in the specimen, especially in the torsion experiments. The tubes were all carefully examined before use, and none showing visible defects were employed; the greatest variation in external diameter was about 0.005 inch in 1.25 inch. This examination was intended, however, merely to avoid the waste of time which the testing of an obviously poor specimen would involve.

If any flaw existed in the material, or if the surfaces were not close approximations to concentric circular cylinders, the values of the elastic constants would be considerably changed, and always to a lower value; the density of the material would be low; and the stress-strain curve would soon cease to be straight.

Now in all the tests measurements of the distortions produced, as well as of the forces applied, were taken. The moduli of rigidity and extension calculated from these were never found to be widely different from their customary value, and especially they are never low in value. (See Table I.) Also the densities found were high. Furthermore, the stress-strain curves to within a point not far below the yield-point were always found to be straight.

24. *The System of Tests.*—The method of experimenting employed was, then, to subject the tubes to torque, to torque and tension combined, to tension only, to tension and internal pressure combined, to torsion and internal pressure combined, and to internal pressure only; and to take measurements of the axial elongation, of the twist when torque was employed, and in some cases of the circumferential strain. The range of the two principal stresses thus covered was from $-p$, $+p$ to $+p$, $+p$, the third principal stress being small, or zero.

25. *The Specimens and Holders.*—The specimens upon which the tests were made were formed of tubes (steel, copper, and brass) soldered on to holders, which served conveniently to apply the tension loads and the torques, and to introduce fluid

under pressure to the interior of the tube. A convenient size of tube appeared to be $1\frac{1}{4}$ in. diameter, with the walls from 0.025 in. to 0.036 in. in thickness. All measurements of extension and twist were made upon the same length of 8 in., but in order to provide room for the diametral extensometer which was sometimes used, and to avoid the disturbing effects of the holders upon the distribution of stress and strain, the length of tube free from the holders was a foot. A holder is shown in fig. 2. A is a screw connecting the holder to the jaw of the testing machine, it is adjusted so that the forces applying the torque have no vertical component—the specimen being vertical. B is a cross-bar serving to apply or to resist the torque. C is a passage-way for the oil used to apply the internal pressure. In the upper holder, which was the nearer to the weighing part of the testing machine, the exit was closed by a screwed cap; sometimes a piece of rubber tubing, doubled over itself to form a packing, was pushed in ahead of the closing plug, and proved to be a very efficient means of preventing leakage. The exit was not closed until fluid appeared at it, in the expectation that should a tube prove unsound, or the pressure get beyond control, no damage would then be done to the strain-measuring apparatus. The lower holder was identical with the upper one, except that, in the place of the plug, a pipe was brazed into it, there being a coupling at the further end of the pipe for connexion with the pressure system.

26. *The Tension Loads.*—The tension loads were applied to the specimen by means of a Ten-ton Wickstead machine of the screw type. The shackles of the machine were fitted with the ball-thrust, or perhaps pull, bearings shown in fig. 2. The “male cone” surfaces C were made spherical in order that the specimen could line itself up to some extent and cause the axes of pull and of the specimen to coincide, and yet allow the specimen freedom to rotate. This action was assisted by gently tapping the specimen at low loads. The method of holding the specimen was by means of split collars D, the screw A of the tube-holder being thrust through the spherical cone of the ball-bearing so far as to allow the collars to be placed round it and inserted into the spherical cone. This proved a rapid and easy method of firmly securing

the specimens. I possessed no conveniences for calibrating the testing machine by the direct application of a large dead load, but the travelling poise (500 lbs.) was removed and found to be correct, and the distance between the knife-edges was measured, with the same result. In a few of the tests there was evidence that the axes of the specimen and of the pull did not exactly coincide, causing a small bending moment on the specimen. This tends to produce an effect similar to that of the elastic limit.

27. *The Application of the Torque.*—In order to apply torque to the specimen, it was placed in the testing machine and a small load (250 lbs.) placed upon it, to line it up and steady it. The cross-bars were then placed in the holders and nipped in position by the set screws E. Steel points F were then placed in the upper cross-bar, and a pair of screws in a bracket, bolted to the casting of the testing machine, were adjusted to just touch the lower cross-bar. Finally, equal and opposite forces were applied to the steel points by means of bent pieces of iron, as shown in the left-hand top corner of fig. 2, attached to belts. These belts were belt laces, which being thin, flexible, and sufficiently strong, appeared to be the best means of transmitting round pulleys such forces as were required. The belts led off horizontally from the ends of the cross-bar, which was adjusted vertically by the screw A of the tube-holders, to pulleys, from which they went vertically downwards to knife-edges fixed in the ends of a horizontal cross lever, from a knife-edge at the centre of which hung the carrier for the weights which produced the torque. The effect of the cross lever and weight-carrier was neutralized by balance-weights hung over the pulleys. These pulleys were made by filling the rims of bicycle wheels with plaster-of-paris, and, before the plaster had set hard, rotating the wheel in contact with a former, thus producing a suitable groove true with the axis of rotation. The torque thus applied to the specimen was practically pure, and produced no bending moment on the specimen. It was resisted by the forces produced by the set screws on the lower cross-bar, there being thus little or no sideways force at the lower grip.

28. *The Friction of the Torsion Rigging.*—A fourth pulley

was used in testing the friction of the torsion apparatus. The greater part of this friction, when any considerable load was on the specimen, occurred in the ball-bearings in the shackles. To ascertain the friction two equal and opposite torques were applied to the top cross-bar by means of belts leading over each of four pulleys and terminating in a weight-carrier. The lower cross-bar was removed. The desired tension load P was then placed upon the specimen, and approximately equal additional weights placed upon a diagonally opposite pair of weight-carriers until they descended. The equal additional weights were placed upon the other pair of weight-carriers until they in turn descended. The latter pair of additional weights (shot) were then placed together and weighed. One quarter of this amount for one of the weights on the carriers represents the equivalent of the friction of the system under the load P . Now in this test four pulleys and the two shackle bearings rotate, while in the actual tests three pulleys and one shackle-bearing are rotated, thus the previous result should be halved for application to the tests. The two weights on the carriers are, however, combined into one hung at the centre of the cross-lever, so that one quarter of the weighed amount should be taken for the weight (W) at the centre of the cross-lever. The results of these friction tests are given in Table I. ; the friction given is in all cases too high, as, the masses moved being large, a small moving force produced a small acceleration which required considerable time to make its effect manifest. The corrections required for W are so small that they have been neglected in the calculations of the stresses.

TABLE I.

Friction of Torsion Apparatus.

It was found that the friction of the pulleys was so small as to be negligible in comparison with the friction of the pull-bearings, the value of W making little difference.

Mean values of δW from several experiments are given :—

P	1000	2000	3000	4000
δW	0.319	0.576	0.909	1.058

Hence $\delta W = 0.285$ lb. per 1000 lb. value of P .

And $\delta W = 0.075$ lb. in all cases for the wheels.

Hence Friction only amounts to $1\frac{1}{2}$ lb. for a load of 5000 lbs.

29. *The Application of the Internal Pressure.*—When tests involving the application of internal pressure were to be made, after a small tensional load had been placed on the specimen so as to line it up, the lower end was clamped firmly by the other screws, opposite to the first set, in the bracket (fig. 4, A). The tube brazed into the lower holder was then clamped at two points to the lower cross-bar, and finally the tube was coupled to the pressure system. The clamps were intended to prevent the production of a bending-moment on the specimen ; but in spite of these precautions, and although the pipe conveying the pressure was long and thin, evidence of a bending-moment (small, however) was evinced in some of the tests.

This could have been avoided in the tension-internal pressure tests by generating the fluid pressure in a reservoir attached to the specimen, the plunger being connected to the jaw of the testing machine, thus transmitting the pull and producing the fluid pressure at the same time. This method, however, only permits of a certain ratio of fluid pressure to tensional load for a given plunger, is inapplicable to torsion-internal pressure tests, and is subject to other objections. Accordingly the former method, despite the risk incurred of an undesired bending-moment, was adopted.

Up to 1000 lbs. per sq. in. the pressure was taken from the system which worked the hydraulic straining-gear of an Emery-Festing machine ; beyond that pressure a small intensifier was employed to further increase the pressure.

30. *Measurement of the Fluid Pressure.*—To measure the pressure I invented a special form of gauge, but although this proved useful for calibrating purposes, it was replaced in the tests by a large commercial gauge, which could be read by means of a telescope from the controlling-valve of the Emery pressure-system, which valve was at a considerable distance from the Wickstead machine. This commercial gauge was calibrated by comparison with the volume of air compressed into a uniform tube by the pressure and by comparison with the special gauge, the constant of which was determined at a low load by means of a Crosby gauge-tester. The air-gauge consisted of a glass tube of about 1 mm. diameter of bore and 1 metre long, sealed hermetically

into a glass bulb holding about 20 c.c. ; below this a tube projected downwards, and was then bent upwards for convenience in handling. The capillary tube was selected as having the most uniform bore of a large number of such tubes, and was so even as to render correction for conicallity unnecessary. The bulb of the glass apparatus was contained in a iron cylinder partially filled with mercury, to the top of which the oil of the pressure-system had access. The iron cylinder was closed with a screw plug, through which the capillary tube projected; the packing used was of the U-type. The comparison of the volume of air under pressure with the reading of the commercial gauge was conducted so slowly that the changes of volume were practically isothermal. The volume of the tube and of the bulb, and the quantity of air contained, had been carefully measured, but a large quantity of air was syphoned out by the action of the mercury; and in preference to running the risk of an unnoticeable quantity being syphoned out, the pressure at 300 lbs. per square inch was determined by the use of a Crosby gauge-tester, so that this is the ultimate basis of the measurement of the fluid pressure.

31. *A new type of Pressure-gauge.*—The special gauge (see fig. 3) consisted of a steel tube of oval section, twisted along its axis, so that the tube had a straight axis, but the extremities of the major axes of the sections perpendicular to the tube-axis lie on a pair of helices the angle of which is about 45° to 60° . One end of the tube is coupled to the pressure-system, and the free end is sealed up and a pointer or mirror, preferably the latter, mounted upon it. On the application of internal pressure the tube untwists, and the amount of angular movement is determined by the use of a telescope and scale, or in any other manner. As the strain at any part is small, it is proportional to the pressure; but the integral twist of the mirror or pointer may be made of any magnitude we please, without exceeding the elastic limit of the metal, by simply increasing the length of the gauge-tube. As, by using a tube of suitable thickness, it is easy to keep the stresses in the steel, produced by fluid pressures up to several thousand pounds per square inch, below the yield-point, and since there is no mechanism nor friction, the gauge

is at once simple and accurate. By comparing the readings of two gauges, one having thicker walls than the other, the fluid pressure corresponding to the yield-point of the lighter gauge can easily be found, as it occurs when the readings cease to be proportional. The constant of the heavy pressure-gauge may be found by comparing it with the lower gauges. Having first ascertained the permissible working pressures, a thin-walled gauge may be compared with a mercury column, and its constant thus found; this gauge may then be compared with a heavier gauge, and since the constant of the first gauge cannot alter, that of the second may be found. Thus the constants of all the gauges can be found. The axis of such a gauge-tube can evidently be coiled (as a line) and produce other desirable movements.

The gauge used in the calibration of the commercial gauge was made of tubing originally $\frac{1}{4}$ inch in diameter; it was about 8 inches long, and from my knowledge of the yield-points of thinner-walled gauges I think that its yield-point would be reached under a fluid pressure of from 4000 to 5000 lbs. per square inch.

In Table II. will be found the comparison of the commercial gauge by the two methods. They give practically the same result.

TABLE II.

Gauge Calibration : Comparison of Pressures in lbs. per sq. in., as determined by the different Gauges.

Commercial.	Air.	Special.	Commercial.	Special.
200	...	196	1100	1030
300	290	290	1200	1120
400	380	...	1300	1220
500	475	475	1400	1320
600	570	565	1500	1420
700	665	662	1600	1520
800	750	740	1700	1610
900	850	851	1800	1705
1000	...	935	1900	1795
			2000	1895

32. *Measurement of the Distortions.*—In order to determine the strains and to locate the points in the various stress-strain curves, I used instruments which I had devised with a

view to these tests. They were a combined axial extensometer and twist-meter, and a diametral extensometer used to determine the circumferential strain.

As it seems to be essential that any extensometer to be used on a part of the stress-strain curve likely to involve a time-effect should be single-reading, and should not be required to be manipulated in any way to obtain a reading, optical methods were used throughout.

33. *A New Extensometer.*—It will be convenient to describe the axial portion of the apparatus first. It consists of a lever-system and a mirror; the twist of the latter, produced by an extension of the specimen in a manner similar to the Unwin extensometer, being read by a telescope and scale. The various parts are shown in fig. 4, which is a side-elevation of the whole instrument in position on a specimen, the "adjuster" being shown on the left-hand side. The piece marked 1 is called the adjuster, and serves to adjust the grips 2 and 3 correctly on the specimen, so that they are at the correct distance apart, and so that certain points in their configuration lie on the axis of the specimen. In fig. 5 is shown a view from below the grip 2 in fig. 4, and in the broken lines is shown a view of the end of the adjuster. This adjuster consists of surfaces A_1, A_2, A_3, A_4 , of which A_1 and A_3 are parts of one plane, and A_2, A_4 are parts of another intersecting it. CD and EF are portions of kinematic slides for the grips 2 and 3, the other elements of the slides being formed by the spheres c_1, c_2, d on the grip 2, and e_1, e_2, f on the grip 3, which is shown in fig. 7. These are so arranged that the axis of the screws G, H , and the line st referred to later, will bisect the angle between the planes A_1, A_3 and A_2, A_4 , and be perpendicular to their intersection. It is evident that by this arrangement the lines GH and st will pass through the axis of the specimen (whatever be its size), which is necessary for the determination of the mean extension should the specimen stretch unequally, owing to bending-moment or other cause. At B is shown a hole, which is at the back of the adjuster, through which a rubber band is passed to support the adjuster on the specimen while the grips are being fastened. The spheres c_1, c_2, d of the grip 2 are placed in the V-groove C and on the plane D ,

Fig. 4. Extensometer, side elevation.

details.

5. " " " " "

Fig. 10. Apparatus for measuring Twist.

11. " " "

12. " " "

TABLE III.

Experiments to test Extensometer.

Swedish Iron specimen, previously tested; mean diam. 0·628 in.; length 8·00 in
Extensometer-reading multiplied by 0·000001265 = strain.

Load in lbs.	Extensometer-Readings.					
	In- creasing.	De- creasing.	In- creasing.	De- creasing.	In- creasing.	De- creasing.
500	043	045	...	265	...	264
1000	086	090	086	311	307	
1500	129	133	...	356	350	354
2000	172	176	171·5	399	393	
2500	215	220	...	443	437	441
3000	258	263	258	486	480	
3500	301	305	...	530	523	528
4000	344	347	344	573	566	
4500	387	390	...	616	610	613
5000	430	434	430	659	654	
5500	472	475	...	701	696	698
6000	515	516	515	743	738	
6500	558	...	...	785	781	
7000	...	...	599	826		
"	...	...	607			
710	...	...	617 &c.			
	In- creasing.	De- creasing.	In- creasing.	De- creasing.		
	In- creasing.	De- creasing.	In- creasing.	De- creasing.		
1800	...	-016				
1900	...	-007·5	-008			
2000	000	000·5	000			
2100	008	009				
2200	016					
3500	...	-043				
3750	...	-021	-022			
4000	...	000	-001	-002		
4250	...	...	+021	020		
4500	...	...	041			

and the grip slid along into position and the screws G, H nipped into the specimen; the screw R is then adjusted to touch the specimen. The other grip is adjusted in the same manner, and the adjuster is then withdrawn. It will be seen that the grips form one piece with the specimen, and during a test the screw-points are not employed as bearings, the friction at which would be an unknown variable quantity.

The bottom grip 2 is fitted with a trihedral hole (see fig. 9) at P, in which lies a steel ball p fastened to the length-bar 5 (fig. 5), the pair forming a universal joint. Similar universal joints are to be seen at Uu , Ss , Vv , Rr , and Ww . The length-bar connects the grip 2 with a bent lever 4 (fig. 8), which rests on the top grip 3; the length-bar being connected to the lever by the universal joint Uu , and the lever being kinematically hinged to the top grip by the universal joint Ss and a sphere t working in a V-groove T. The axis of the screws L, M intersects and bisects the line st , and the grip 2 is so adjusted that the point of intersection is on the axis of the specimen. To the other end of the lever is connected, by the universal joint Vv , the mirror-bar 6 (figs. 4 & 9), to which the mirror-frame, 7, is hinged kinematically by the universal joint Ww and the V-groove and sphere Xx . The mirror-frame 7 (figs. 4 & 7) is finally connected to the lower grip, 2, by the universal joint Rr , the sphere R being carried on the end of the calibrating-screw Q. In addition to the restraints mentioned, the mirror-bar has a wing, z , which is in contact with the (blunt) end of a screw Z, being kept in contact by the small weighted lever Y. Any extension of the specimen will result in an angular movement of the lever 4, thus raising the axis of the hinge WX, and causing the mirror to rotate about this axis. It will be noticed that the forces along all the members of the instrument are practically unaffected by the movements of the parts, being controlled by the position of the adjustable weight shown on the mirror-frame. The mirror is readily adjusted, so that the reflexion of the scale appears in the field of view of the telescope by means of the screw Z, which turns the mirror about a vertical axis, while the scale-reading can be adjusted to zero by means of the screw Q. This screw also serves, in case it should be desired to push the investigations beyond the yield-point, to readjust the reading on to the scale as often as desired, and also to ascertain directly the value of a scale-division in terms of the axial strain. To do this the change of scale-reading caused by one or more turns of the screw, the head of which is graduated and the pitch of which is known; then, by taking account of the multiplication by the levers and of the

length of the specimen, the axial strain per scale-division is easily found. The screw Q was compared directly with a Brown and Sharpe micrometer, and the two were found to agree to the $\frac{1}{10,000}$ inch in eight turns or 0.2 inch. A consideration of the behaviour of the instrument under the supposition that the specimen bends or twists will show that these do not affect the reading, which will always represent the mean axial elongation. One kinematic restraint, it will be noticed, is wanting: this, however, would be supplied by a wing attached to the length-bar and arrangements similar to YZ, but would have no effect on the working of the instrument; it is therefore omitted. Figs. 5, 6, 7, 8, and 9 represent parts of the extensometer: fig. 6 being the top grip, 7 the lever, 8 a trihedral hole, and 9 a side-view and sectional plan of the mirror and its supports, on a rather larger scale than the other parts.

The instrument proved sensitive and trustworthy; the tests of it for sensitiveness made on the Norwegian iron specimen previously referred to are given in Table IV. The scale was placed at such a distance that the extension in 8 inches corresponding to one scale-division was about the one hundred thousandth of an inch; the nearest scale-division is recorded, except when the cross-hair of the telescope was apparently midway between two divisions.

34. *Errors of Extensometer.*—In a figure formed by plotting out Table IV. it can be noticed that the hysteresis-loop, due to the yield-point having been reached previously, is well marked, and its width is considerably greater than the instrumental error; the errors, which are quite insignificant, are about the value of one scale-division, and include any error due to the want of sensibility of the testing-machine.

35. *A New Twist-measuring Apparatus.*—The apparatus for measuring the twist consisted merely of a pair of mirrors mounted on the grips 2 and 3, a scale placed horizontally being read by light reflected at each of the mirrors. Two arrangements occurred to me: one in which the mirrors had their normals perpendicular to the axis of the specimen (see fig. 12), and one in which they were inclined at 45° or so (see figs. 10 & 11). In these figures the course of the ray of light is indicated roughly by the broken line, the scale

being shown at the right-hand side, and the telescope or source of light being upon the left. The amount of twist of the specimen is determined by observing the change of reading of the scale by means of a telescope, the scale being observed by reflexion at both of the mirrors.

The second was the method selected, it being easily applicable to specimens of varying lengths.

The mirrors are mounted upon frames resting upon the spheres c_1 , c_2 , d ; e_1 , e_2 , f of the grips, which are left vacant by the withdrawal of the adjuster. The two frames are exactly similar. The angle c_1 , c_2 , d is a right angle; the contact of the frame with c_1 , b is made by means of a trihedral hole mounted on a screw, with c_2 by means of a fixed V-groove, and with d by means of a plane at the end of a screw. The normals to the mirrors can be adjusted within certain limits by the use of these screws. The various surfaces are kept in contact by the use of the springs shown in fig. 4.

Had the other method been adopted, one or both mirrors would have had to have been mounted upon sliding bars so that they could have been adjusted opposite to one another, when the grips (for torsion purposes) were placed at various distances apart. No special adjustment, however, would have been required to secure accuracy.

If, in the inclined mirror case, we consider the lower grip stationary and the top mirror alone to twist, it is very evident that the twist of the upper mirror is the same as the angle through which the reflexion of the optical axis of the reading-telescope is turned, and is therefore, to the second order of small quantities, equal to the change of scale-reading divided by the distance from the telescope to the scale. The lower grip, however, must be allowed to twist in order that the stress in the specimen between the grips may be of the normal type, uninfluenced by the proximity of the holders; in the following investigation it will be shown that, when a certain adjustment is made, the scale-reading divided by the distance from the specimen to the scale is equal to the twist of the specimen between the grips, at any rate as closely as the tangent of the angle is equal to the angle.

36. *Proof that the Torsion-Mirrors measure the Twist.*—

The light from the scale first strikes the upper mirror, and is thence reflected on to the lower mirror, whereby it is reflected to the telescope. In order, however, to ascertain the effect of a simultaneous twist of the two grips, we shall take the line in the reverse direction, and consider the optical axis of the telescope and the direction of its reflexions at each of the mirrors. The distance between the mirrors will be neglected, as it should be small compared with the distance from the specimen to the scale, or else a lens should be placed between the top mirror and the scale to render the light proceeding from the latter parallel. In the apparatus as rigged for the tests, the distance between the mirrors was 2 or 3 inches, and the distance to the scale over 200, so that this condition was satisfied.

In fig. 13, OC is the axis of the specimen, OA is perpendicular to OC and in the plane of the optical axis of the telescope, OB is perpendicular to OA and OC. The line OK, coordinates $\frac{\pi}{2} - \alpha_1, 0$ or $x_1, 0, z_1$, is the optical axis of the telescope, and cuts a sphere of unit radius whose centre is O in the point K. The normal to the mirror nearest the telescope is represented by OP, coordinates $\frac{\pi}{4} + \theta_1, \phi_1$; or ξ_1, η_1, ζ_1 . The reflexion of OK in the mirror is OL, coordinates α_2, β_2 , or x_2, y_2, z_2 , so that OL represents the path of the ray between the mirrors. The normal to the upper mirror is represented by OQ, coordinates $\frac{\pi}{4} + \theta_2, \phi_2$, or ξ_2, η_2, ζ_2 ; this line is shown drawn in the same octant as the other lines for the sake of clearness, though it should actually be reversed; but a little consideration will show that, by considering the light to come in the direction LO, it will, after reflexion in the second mirror, proceed in the direction OM, coordinates θ, ϕ or x, y, z ; while the actual ray of light will merely be in the direction KO, instead of OK. The quantities α ; the error of setting the telescope in altitude; θ_1, θ_2 the errors of setting the mirrors; and ϕ_1 and ϕ_2 , the azimuthal movements of the mirrors, are supposed small, ϕ_2 being considerably larger than any of the others.

Fig. 13. Diagram for Twist-meter.
14. Diametral Extensometer.
20. Three-dimensional view of tests.

Then $x_1 = 1$ nearly, z_1 is small;
 $z_2 = 1$ nearly, x_2 and y_2 are small;
 $x = 1$ nearly, y and z are small.

We have to find θ and ϕ or x, y, z .

Also the coordinates of

K are $x_1 y_1 z$ or $1 - \frac{\alpha_1^2}{2}$; 0; α_1 .

P are $\xi_1 \eta_1 \zeta_1$,

or $\sin \frac{\pi}{4} + \theta_1 \cos \phi_1$; $\sin \frac{\pi}{4} + \theta_1 \sin \phi_1$; $\cos \frac{\pi}{4} + \theta_1$;

or

$$\frac{1}{\sqrt{2}} \left\{ 1 + \theta_1 - \frac{\theta_1^2}{2} - \frac{\phi_1^2}{2} \right\}; \quad \frac{1}{\sqrt{2}} (1 + \theta_1) \phi_1; \quad \frac{1}{\sqrt{2}} \left\{ 1 - \theta_1 - \frac{\theta_1^2}{2} \right\}.$$

Q are $\xi_2 \eta_2 \zeta_2$,

or $\sin \frac{\pi}{4} + \theta_2 \cos \phi_2$; $\sin \frac{\pi}{4} + \theta_2 \sin \phi_2$; $\cos \frac{\pi}{4} + \theta_2$;

or

$$\frac{1}{\sqrt{2}} \left\{ 1 + \theta_2 - \frac{\theta_2^2}{2} - \frac{\phi_2^2}{2} \right\}; \quad \frac{1}{\sqrt{2}} \left(1 + \theta_2 - \frac{\theta_2^2}{2} \right) \phi_2; \quad \frac{1}{\sqrt{2}} \left(1 - \theta_2 - \frac{\theta_2^2}{2} \right).$$

To find L or $x_2 y_2 z_2$, we have:—

(1) Since $\angle LOP = \angle KOP$,

$$x_2 \xi_1 + y_2 \eta_1 + z_2 \zeta_1 = \xi_1 x_1 + \eta_1 y_1 + \zeta_1 z_1. \quad (1)$$

And (2), since LOPK is a plane,

$$\begin{vmatrix} x_2 & y_2 & z_2 \\ \xi_1 & \eta_1 & \zeta_1 \\ x_1 & y_1 & z_1 \end{vmatrix} = 0 \quad (2)$$

and

$$x_2^2 + y_2^2 + z_2^2 = 1. \quad (3)$$

Hence from (1)

$$\begin{aligned} & x_2 \left(1 + \theta_1 - \frac{\theta_1^2}{2} - \frac{\phi_1^2}{2} \right) + y_2 (1 + \theta_1) \phi_1 + z_2 \left(1 - \theta_1 - \frac{\theta_1^2}{2} \right) \\ &= \left(1 + \theta_1 - \frac{\theta_1^2}{2} - \frac{\phi_1^2}{2} \right) \left(1 - \frac{\alpha_1^2}{2} \right) + \left(1 - \theta_1 + \frac{\theta_1^2}{2} \right) \alpha_1, \quad (4) \end{aligned}$$

or to the first order

$$x_2(1 + \theta_1) + y_2\phi_1 + z_2(1 - \theta_1) = 1 + \theta_1 + \alpha; \quad \dots \quad (4a)$$

$$\therefore x_2 + z_2 = 1 \text{ to first order. } \dots \dots \dots (5)$$

From (2)

$$\begin{vmatrix} x_2 & y_2 & z_2 \\ 1 + \theta_1 - \frac{\theta_1^2}{2} - \frac{\phi_1^2}{2}, & (1 + \theta_1)\phi_1, & 1 - \theta_1 - \frac{\theta_1^2}{2} \\ 1 - \frac{\alpha_1^2}{2} & 0 & \alpha_1 \end{vmatrix} = 0;$$

or

$$x_2\phi_1\alpha_1 + y_2\left(1 - \alpha_1 - \theta_1 - \alpha_1\theta_1 - \frac{\theta_1^2}{2} - \frac{\alpha_1^2}{2}\right) - z_2(1 + \theta_1)\phi_1 = 0; \quad (6)$$

or to the first order,

$$y_2(1 - \theta_1 - \alpha_1) - z_2\phi_1 = 0; \quad \dots \dots \dots (6a)$$

hence y_2 is a small quantity.

Substituting in (3),

$$x_2^2 + z_2^2 = 1,$$

which combined with (5) gives us as values $x_2 = 0$ and $z_2 = 1$.
(The values $x_2 = 1$ and $z_2 = 0$ correspond to K.)

Also it is evident that $1 - z_2 = \lambda$ is of the second order.

[For

$$\begin{aligned} x_2^2 + y_2^2 + z_2^2 &= x_2^2 + y_2^2 + (1 - \lambda)^2 \\ &= 1 - 2\lambda + (x_2^2 + y_2^2 + \lambda^2); \end{aligned}$$

$\therefore \lambda$ is of second order.]

$$\therefore (4a) \text{ becomes } x_2 - \theta_1 = \theta_1 + \alpha_1,$$

$$\text{and } (6a) \text{ becomes } y_2 - \phi_1 = 0;$$

$$\begin{aligned} \text{hence } & \left. \begin{aligned} x_2 &= 2\theta_1 + \alpha, \\ y_2 &= \phi_1, \\ z_2 &= 1, \end{aligned} \right\} \text{ to first order.} \end{aligned}$$

Let the terms of the second order in x_2, y_2, z_2 be λ, μ, ν

respectively; then from (4)

$$\begin{aligned} & (2\theta_1 + \alpha_1 + \lambda)(1 + \theta_1) + \phi_1^2 + (1 + \nu) \left(1 - \theta_1 - \frac{\theta_1^2}{2}\right) \\ &= \left(1 + \theta_1 - \frac{\theta_1^2}{2} - \frac{\phi_1^2}{2}\right) \left(1 - \frac{\alpha_1^2}{2}\right) + \left(1 - \theta_1 - \frac{\theta_1^2}{2}\right) \alpha. \end{aligned}$$

Hence

$$\lambda + \nu = -2\theta_1^2 - 2\alpha_1\theta_1 - \frac{\alpha_1^2}{2} - \frac{3}{2}\phi_1^2. \quad (7)$$

Substituting in (3),

$$4\theta_1^2 + 4\theta_1\alpha_1 + \alpha_1^2 + \phi_1^2 + 1 + 2\nu = 1;$$

$$\therefore \nu = - \left(2\theta_1^2 + 2\theta_1\alpha_1 + \frac{\alpha_1^2}{2} + \frac{\phi_1^2}{2}\right);$$

$$\therefore \lambda = -\phi_1^2;$$

and substituting in (2),

$$\begin{vmatrix} 2\theta_1 + \alpha_1 + \lambda, & \phi_1 + \mu, & 1 + \nu \\ 1 + \theta_1 - \frac{\theta_1^2}{2} - \frac{\phi_1^2}{2}, & \phi_1 + \theta_1\phi_1, & 1 - \theta_1 - \frac{\theta_1^2}{2} \\ 1 - \frac{\alpha_1^2}{2}, & 0, & \alpha_1 \end{vmatrix} = 0.$$

From which we get

$$\mu = \phi_1(2\theta_1 + \alpha_1).$$

So that to the second order of small quantities,

$$\left. \begin{aligned} x_2 &= 2\theta_1 + \alpha - \phi_1^2, \\ y_2 &= \phi_1 + \phi_1(2\theta_1 + \alpha_1), \\ z_2 &= 1 - 2\theta_1^2 - 2\theta_1\alpha_1 - \frac{\alpha_1^2}{2} - \frac{\phi_1^2}{2}. \end{aligned} \right\}$$

We now proceed to find x, y, z from equations similar to (1), (2), and (3), namely,

$$x\xi_2 + y\eta_2 + z\zeta_2 = x_2\xi_2 + y_2\eta_2 + z_2\zeta_2, \quad (8)$$

$$\begin{vmatrix} x & y & z \\ \xi_2 & \eta_2 & \zeta_2 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0, \quad (9)$$

and $x^2 + y^2 + z^2 = 1.$

Equation (8) becomes

$$x\left(1+\theta_2-\frac{\theta_2^2}{2}-\frac{\phi_2^2}{2}\right)+y(1+\theta_2)\phi_2+z\left(1-\theta_2-\frac{\theta_2^2}{2}\right) \\ = (2\theta_1+\alpha_1-\phi_1^2)(1+\theta_2)+\{\phi_1+\phi_1\overline{2\theta_1+\alpha_1}\}(1+\theta_2)\phi_2 \\ +\left(1-2\theta_1^2-2\theta_1\alpha_1-\frac{\alpha_1^2}{2}-\frac{\phi_1^2}{2}\right)\left(1-\theta_2-\frac{\theta_2^2}{2}\right); \quad (10)$$

or to the first order,

$$x(1+\theta_2)+y\phi_2+z(1-\theta_2)=2\theta_1+\alpha_1+1-\theta_2. \quad (10a)$$

$$\therefore x+z=1.$$

Equation (9) becomes

$$\begin{vmatrix} x & y & z \\ 1+\theta_2-\frac{\theta_2^2}{2}-\frac{\phi_2^2}{2} & \phi_2(1+\theta_2) & 1-\theta_2-\frac{\theta_2^2}{2} \\ 2\theta_1+\alpha_1 & \phi_1 & 1 \end{vmatrix} = 0;$$

hence

$$x\{\phi_2(1+\theta_2)-\phi_1(1-\theta_2)\}+y\left\{2\theta_1+\alpha_1(1-\theta_2)-\left(1+\theta_2-\frac{\theta_2^2}{2}-\frac{\phi_2^2}{2}\right)\right\} \\ +z\{\phi_1(1+\theta_2)-\phi_2(2\theta_1+\alpha_1)\}=0; \quad (11)$$

or to the first order,

$$x(\phi_2-\phi_1)+y(2\theta_1+\alpha-1-\theta_2)+z\cdot\phi_1=0; \quad (11a)$$

therefore y is small,

and $x^2+z^2=1$ to the second order,

and $x+z=1$ to the first order.

$$\therefore x=1 \text{ and } z=0.$$

[The solutions $x=0$, $z=1$ refer to the point L.]

$\therefore$ as before, $x=1$ to the second order.

$$\text{From (10a)} \quad 1+\theta_2+z=2\theta_1+\alpha_1-\theta_2,$$

$$\therefore z=2\theta_1-2\theta_2+\alpha;$$

$$\text{and from (11a)} \quad \phi_2-\phi_1-y=0,$$

$$\therefore y=\phi_2-\phi_1.$$

Hence

$$x=1+\rho,$$

$$y=\phi_2-\phi_1+\sigma,$$

$$z=2(\theta_1-\theta_2)+\alpha+T,$$

where ρ , σ , and T are quantities of the second order.

To find σ consider equation (11), it becomes

$$\begin{aligned}
 1 + \rho, & \quad \phi_2 - \phi_1 + \sigma, & 2\theta_1 - 2\theta_2 + \alpha + T & = 0. \\
 1 + \theta_2 - \frac{\theta_2^2}{2} - \frac{\phi_2^2}{2}, & \quad \phi_2 + \theta_2\phi_2, & 1 - \theta_2 - \frac{\theta_2^2}{2} & \\
 2\theta_1 + \alpha_1 + \phi_1^2, & \quad \phi_1 + \overline{2\theta_2 + \alpha} \cdot \phi_1, & 1 - 2\theta_1^2 - 2\theta_1\alpha_1 - \frac{\alpha_1^2}{2} - \frac{\phi_1^2}{2} &
 \end{aligned}$$

The terms of the second order are

$$\begin{aligned}
 & \theta_2\phi_2 - \overline{2\theta_1 + \alpha}\phi_1 + \theta_2\phi_1 + \overline{\phi_2 - \phi_1}(2\theta_1 + \alpha_1 - \theta_2) - \sigma \\
 & + (2\theta_1 - 2\theta_2 + \alpha)\phi_1 = 0; \\
 \therefore \sigma & = (\phi_2 - \phi_1)(2\theta_1 + \alpha).
 \end{aligned}$$

Hence we see that the scale-reading is $(\phi_2 - \phi_1)(1 + 2\theta_1 + \alpha)$, and that the horizontal cross-hair of the eyepiece will move across the image of the scale through an angle δz or $\delta(2\theta_1 - \theta_2 + \alpha)$. This remains constant, so that the movement across the scale due to twist of the specimen is very small. Now in the use of this instrument ϕ_2 is of a larger order than any of the other quantities, but if the other quantities ϕ_1 , θ_1 , θ_2 , and α are all zero, it is evident that ϕ and ϕ_2 are exactly equal.

$$\text{Hence} \quad \phi = (\phi_2 - \phi_1)(1 + 2\theta_1 + \alpha)$$

to quantities of the orders

$$\phi_2^3, \phi_1^2, \theta_1^2, \theta_2^2, \alpha_1^2.$$

$$\text{Now put} \quad \phi_2 = \phi_1 = 0.$$

Then the ray lies in the plane COA, and

$$\begin{aligned}
 \text{COZ} &= \text{COP} - \text{LOP} = \text{COP} - \text{KOP} \\
 &= \text{COP} - (\text{AOP} - \text{AOK}) \\
 &= \frac{\pi}{4} + \theta_1 - \left(\frac{\pi}{4} - \theta_1\right) + \alpha_1 \\
 &= 2\theta_1 + \alpha_1.
 \end{aligned}$$

Hence if we make the reflexion of the optical axis in the first mirror parallel to the axis of the telescope, the value of ϕ becomes $\phi_2 - \phi_1$, or the scale-reading divided by the distance to the scale is the twist between the mirrors to the orders of small quantities stated above. This adjustment is made by the use of the adjusting-screws of the lower

mirror, the telescope being sighted so as to have the lower mirror at the centre of its field, and then the mirror adjusted so that a point similarly situated with respect to the specimen as the centre of the lower mirror appears at the centre of the field.

The adjustment was tested by turning the specimen as a whole through an angle of 0.1 , when it was found that the scale-reading changed through an angle corresponding usually to about 0.001 .

The greatest value of ϕ_2 was about 0.05 , the cube of which is 0.000125 ; the corresponding value of ϕ_1 would be about one-eighth of ϕ_2 or 0.006 , and ϕ_2^2 would be 0.0004 . The error of ϕ would hence be about 0.25 per cent., which is quite negligible.

It will be noticed that azimuthal errors in the settings of the mirrors are of no effect, being parts of ϕ_2 and ϕ_1 , changes of which alone affect the value of ϕ .

37. *The Effect of Bending.*—Unfortunately, the effect of bending* was not investigated until the series of tests had been practically completed; otherwise the alternative arrangement of the twist-mirrors, which is free from such defects, would probably have been preferred although not readily applicable to specimens of different lengths.

Bending in a plane through the telescope and specimen has the effect of causing the image of the scale to move vertically across the field of the telescope, but it does not affect the reading.

Bending in a plane at right angles to this, however, has an effect upon the reading. If the angle $CAP = \psi$ and $POA = \chi$, we have

$$\cos \theta = \sin \chi \cos \psi,$$

$$\tan \phi = \tan \chi \sin \psi,$$

and bending alters the value of ψ , so that the effect of bending is to produce

$$\delta \theta = \frac{\sin \chi \sin \psi}{\sin \theta} \delta \psi = 0 \text{ nearly.}$$

* For the suggestion that I should investigate the effect of bending, I am indebted to Mr. E. G. Coker, Assist. Prof. of Civil Engineering, McGill University.

$$\delta\phi = \frac{\tan \chi \cos \psi}{\sec^2 \phi} \delta\psi = \delta\psi \text{ nearly.}$$

So that the bending is added to the twist.

In the tests, however, a small load (250 lbs.) having been placed on the specimen, the bearings were tapped until the alignment had taken place. In pure torsion tests then the bending could have had no effect; and in the combined tests, as the torsion load was applied first and then the axial load, if the latter produced any bending it would alter the reading of the twist. On only one occasion was this of any magnitude, and usually the reading remained nearly constant until the yield-point was reached, when it increased. I therefore do not think that the source of error indicated above had an appreciable effect on the twist readings.

38. *As a Transmission Dynamometer.*—It may be noticed that in the case where the normals to the mirrors are perpendicular to the specimen, the reflected optical axis is turned through twice the angle the specimen twists through, and this relationship is exact. By using a ray of light of small cross section and allowing it, after reflexion at the two mirrors, to fall upon a scale, its position will indicate the twist between the mirrors; and since the reading can be taken when the light is intermittent, the device forms a cheap and convenient form of transmission dynamometer.

39. *The Diametral Extensometer.*—The diametral extensometer (fig. 14, p. 93) consists of a piece 1 resting in contact with the specimen at the surfaces A B C D forming parallel V's (as in the Adjuster); to this was hinged by an axis resting in two V's E, F, and a point at the end of the axis resting on a surface G, a lever 2. The lever 2 has a projection H which was kept in contact with the tube by a spring K. The piece 1 also carried the V-bearings L, M of a mirror-frame 3, the axis of the bearings being perpendicular to the axis of the specimen. The mirror-frame was connected to a screw working in the lever 2 by means of a link 4; the link having sharp knife-edges of V-form which worked on knife-edges on the screw and mirror-frame. The instrument was supported partly by springs, and partly by the friction at the surface of the tube. The distance of the knife-edge N

fastened to the mirror-frame, and upon which the link 4 worked, being small, its distance from the axis of rotation of the mirror-frame was determined by clamping the piece 1 and a micrometer-screw in such a way that turning the screw pushed, by means of a small distance-piece, the knife-edge through a definite amount. The twist of the mirror was then determined optically. The amount of the strain corresponding to any twist of the mirror can then be easily calculated ; it being remembered that, owing to the use of V's for the tube to rest in, the movement of H relative to the piece 1 is not simply the increase of the diameter, but is to be considered as the effect of an enlargement of the tube. If α is the semi-angle of the V's, the distance from the point of the V to the point of contact of the lever is $r(1 + \operatorname{cosec} \alpha)$, where r is the radius of the tube. Hence an increase of radius δr corresponds to a movement of the lever at the point of contact with the tube of $\delta r(1 + \operatorname{cosec} \alpha)$.

40. *Method of making the Tests.*—In making the tests, a tube, after its dimensions had been determined, was soldered on to the holders and then placed in the testing-machine, and a load of 250 lbs. applied, and the shackle-bearings tapped in order to make the specimen line itself up as accurately as possible. This load of 250 lbs. was employed as it pulled into contact the surface of the lower head of the testing-machine, which would not be in contact permanently until the load on the specimen exceeded the weight of all the parts between it and the straining-gear of the machine. [There was also a slight amount of backlash in the nut of the screw.] The torsion-bars and steel points were placed in their positions, and the screws in the bracket adjusted to just touch the lower cross-bar. The grips of the extensometer were then fixed to the specimen by the use of the adjuster, and the lower torsion-mirror applied. The telescope for the torsion-tests was then adjusted opposite the lower mirror with its optical axis horizontal, and the lower mirror was adjusted by the screws in its frame until the reflexion of the optical axis was parallel to the specimen. The top mirror was then fixed to the top grip and the torsion-scale placed at the same level : the screws of the top mirror were then adjusted until the image of the scale

appeared horizontal and at the centre of the field. The scale was then moved horizontally until the reading was nearly zero, the final adjustment to zero being made just before a test by the azimuth-screw of the telescope. The adjustment of the mirrors is such as to satisfy the condition found above (§ 36) for the vanishing of the terms of the second order in the expansion of the twist. The remaining parts of the extensometer were then placed in position, and the scale for axial extensions being moved so that its centre was opposite the mirror, the second screw Q was used to adjust the reading to nearly zero, the final adjustment being made by the altitude-screw of the corresponding telescope. The diametral extensometer was then adjusted on the specimen, above the other and strain apparatus.

The various forces were applied by suitable increments, and readings of the extensometer taken at each application: the time-effect at the elastic limit was looked for, and at its appearance the forces were applied by smaller amounts until it was judged that the yield-point had been reached. In tests involving tension and torsion the torque was applied first; during the application of the tension-load the first reading seldom changed appreciably, though it usually changed a few divisions. On only one occasion was the change (indicating a bending-moment) considerable. In all tests involving internal pressure, the pressure was applied last, as it was somewhat difficult to keep it constant at a desired amount. In the Tables the tension-loads are denominated by the letter P, and expressed in pounds-weight; the loads producing torque by W, so that the torques are $15 W$ inch-lbs., the cross-bar being 30 inches long between the points; the fluid pressure as recorded by the gauge by p_0 , the corresponding pressure being obtained from the table of gauge corrections. The reading of the axial extensometer is called the axial reading, the factor to reduce this is changed in the course of the tests; otherwise all scales were placed at standard distances. This factor was 0.000001286 for tubes I., II., III., X., XI., XII., XIII., and 0.000001265 for the remainder. The reading to determine the twist is called the twist-reading; the factor to reduce it to shearing-strain or slide varies with the diameter

of the tube, for a $1\frac{1}{4}$ -inch diameter tube it is 0.00000377. The diametral extensometer readings have not been tabulated, it having been deemed more reliable to deduce the strain from the elastic constants; the factor to reduce scale-readings to strains was 0.000002022 for a $1\frac{1}{4}$ -inch diameter tube. Examples of the results of this extensometer will be found plotted in some of the stress-strain curves (figs. 21, 23, & 24).

The values of P , W , p_0 , the axial and the twist readings for the yield-point in the various tests are tabulated in Tables VI., VII., and VIII. under the head of observations.

41. *Determination of the Sectional Area and Thickness of the Tubes.*—The external diameter of the specimens was determined by the use of a micrometer-caliper; they were all practically uniform. The thickness was determined in every case by calipering, and also indirectly from the area of cross-section. The cross-section was determined in cases I., III., X., XI., XII., and XIII. by weighing the tubes in air and in water, the tubes being well boiled in the water, and the specific gravity of the water at the temperature checked. The density as well as the cross-section was thus determined. The other tubes were weighed and the thickness deduced from the result: as it always closely agreed with the makers' statement and with the result of direct measurement, the values may be regarded as fairly close. It should be noticed that errors in the thickness will only very slightly affect the relative results of the tests upon a tube, as the change in the mean diameter is the only way by which the relative results would be affected.

42. *Calculation of the Stresses.*—The effect of an axial force P is to produce a stress p , equal to the force divided by the area of the specimen: the values of these stresses for the various experiments are tabulated in the column headed p .

Owing to the smallness of the friction at the bearings, it has not been thought necessary to allow for it in the value of W ; the cases which it affects most are those in which P is large and W comparatively small, the friction being practically proportional to P . The effect of a torque $15 W$ is to produce a shearing-stress q in the material of the tube varying from the inner surface to the outer proportionally to the distance

from the axis. As the thickness of the tube is only from 0.04 to 0.06 of the radius, the variation of the shearing-stress from its mean value is only 2 or 3 per cent. The value of q is taken as equal to the torque divided by the product of the area of section and the mean radius. A simple shearing-stress q is equivalent to two principal stresses $+q$ and $-q$. It is to be noticed that neither tension, torsion, nor their combination produces a third principal stress.

The effect of the fluid pressure is to produce an axial stress p_1 and a circumferential stress whose mean value is $2p_1$, where $p_1 = \frac{\pi(r-t)^2}{\text{area}} \times \text{fluid pressure}$; and a radial stress varying from the value of the fluid pressure at the interior surface to zero at the exterior surface.

Since in a cylinder under internal pressure the circumferential stress is given by the equation $p = C \left(1 + \frac{r_0^2}{r^2}\right)$, where r_0 is the external radius and r the radius at which the stress is p , the circumferential stress at the interior bears to that at the exterior the ratio $\frac{1}{2} \left(1 + \frac{r_0^2}{(r_0-t)^2}\right) = 1 + \frac{t}{r}$ nearly. Hence the variation from the mean value is about 2 or 3 per cent., as in the case of the shearing-stress due to torque.

In the calculations the mean values of the radius have been used, and no notice taken of the variation of the stress. In the hydraulic tests the actual amount of variation in pounds per sq. inch is equal to the value of the fluid pressure, since the sum of the radial and circumferential stresses is the same throughout the material of the cylinder.

The value of the axial stress due to the fluid pressure at the yield-point in the test is tabulated in the column headed p_1 . In the case of a simultaneous tension-load producing an additional stress p_0 , this is simply added on, so that the principal stresses are then $p + p_1$ axially, $2p_1$ circumferentially, and $\frac{1}{2}p_0$ radially.

To find the principal stresses when torsion is combined with tension or internal pressure, notice first that the radial stress is always perpendicular to the other components and to the plane of the shear, and hence it is a principal stress, and the

case reduces to one of two dimensions and the stress ellipsoid to

$$p_1x^2 + 2qxy + p_2y^2 = 1,$$

where p_1 , p_2 are the component axial and circumferential stresses and q the shearing-stress. The principal stresses ϖ_1 , ϖ_2 are then the roots of the discriminant

$$\begin{vmatrix} p_1 - \varpi & q \\ q & p_2 - \varpi \end{vmatrix} = 0;$$

and hence are the values of

$$\varpi = \frac{p_1 + p_2 \pm \sqrt{p_1 - p_2)^2 + 4q^2}}{2}.$$

From this equation are obtained the tabulated values of ϖ_1 and ϖ_2 . The angle at which the principal stresses are inclined to the generators may be found from its algebraic value $\tan^{-1} \frac{2q}{p_2 - p_1}$.

In the tables of results the next column contains the maximum shearing-stress; this takes place in a plane perpendicular to the axis of mean principal stress, and is that of the plane through the mean axis bisecting the angles between the other principal stresses: its value is half the difference between the greatest and least principal stresses. It will be noticed in the case of those experiments in which the stresses in the tangential plane are of opposite signs, that this tangential plane is the plane of the greatest shear, but that in the other cases the plane passes through the axis of the tube. The value of the maximum shearing-stress in cases involving torsion is, from the value of the principal stresses,

$$\sqrt{\left(\frac{p_1 - p_2}{2}\right)^2 + q^2}.$$

43. *Calculation of the Strains.*—Owing to the elastic-limit effect the measured strains vary more than the stresses, especially as the elastic-limit effect is practically absent in a test when the immediately preceding test is of the same type.

In the simple torsion tests the value of the shearing-strain or slide ϕ is obtained directly from the reading by the use of the factor. If i be the angle of twist of the specimen in a length l , r the radius of the specimen, d the distance to the

scale, measured in the same units as x the scale-reading, we have

$$\phi = \frac{r}{l} i = \frac{r}{l} \frac{x}{a}.$$

The principal component strains which produce the slide ϕ are $\eta_1 = \frac{1}{2}\phi$; $\eta_2 = -\frac{1}{2}\phi$; and $\eta_3 = 0$.

The effect of an axial stress is not only to produce an axial strain, but to produce a uniform strain of the opposite sign in all directions at right angles to it, and bearing to it, in amount, the value of Poisson's ratio (σ). The value of the axial strain was determined from the corresponding scale-reading by the use of a factor; the value of the resulting contractile strain was obtained by multiplying this by the value of σ deduced from the values of E and C for the specimen, in preference to using the results of the diametral extensometer.

In the combined tension and torsion tests, the value of the principal strains was obtained from the observed value of the axial strain ϵ_1 and of the slide ϕ , and by taking the value of the circumferential strain as $-\sigma\epsilon_1$. Since ϕ is the only slide the strain ellipsoid becomes

$$\epsilon_1 x^2 + \epsilon_2 y^2 + \epsilon_3 z^2 + \phi xy = 1,$$

where ϵ_2 and ϵ_3 are the strains perpendicular to ϵ_1 . The discriminant then becomes

$$\begin{vmatrix} \epsilon_1 - \eta & \frac{1}{2}\phi \\ \frac{1}{2}\phi & \epsilon_2 - \eta \end{vmatrix} = 0;$$

and the principal strains have the values

$$\frac{1}{2}\{\epsilon_1 + \epsilon_2 \pm \sqrt{(\epsilon_1 - \epsilon_2)^2 + \phi^2}\}; \epsilon_3.$$

The maximum shear-strain, being the difference between the greatest and least principal stresses, is

$$\sqrt{(\epsilon_1 - \epsilon_2)^2 + \phi^2}.$$

In the case of the internal pressure experiments, the strains have been calculated from the values of the principal stresses and of the elastic constants. The results so obtained are always less than the measured values of the strains would be; as in

the tension and internal pressure the axial strain is a principal one, its measured volume in the column "Axial strain" can be compared with its calculated value in the column η_1 of the principal strains.

The maximum slides or shear-strains, being in value equal to the difference between the greatest and least principal strains, have also been tabulated; these should be, but for the elastic-limit effect, directly proportional to the maximum shearing-stresses, and independent of the third stress.

44. *Maximum Shear and Slide are proportional.*—For if the principal stresses be $\varpi_1, \varpi_2, \varpi_3$, then the maximum shearing-stress perpendicular to ϖ_3 is $\frac{1}{2}(\varpi_1 - \varpi_2)$. The principal strains are $\frac{1}{E}(\varpi_1 - \sigma\varpi_2 - \sigma\varpi_3)$, $\frac{1}{E}(\varpi_2 - \sigma\varpi_3 - \sigma\varpi_1)$, $\frac{1}{E}(\varpi_3 - \sigma\varpi_1 - \sigma\varpi_2)$, and the corresponding maximum slide to $\frac{1}{2}(\varpi_1 - \varpi_2)$ is $\frac{1+\sigma}{E} \times (\varpi_1 - \varpi_2)$. So that the maximum slide directly corresponds to the maximum shearing-stress and is independent of the stress normal to its plane. Thus the comparison of the maximum shear and slide columns in the tables of results will indicate the magnitude of the elastic-limit effect.

45. *Quantities tabulated.*—Thus for the tests the values of the principal stresses, of the principal strains, of the maximum shear, and the maximum slide at the yield-point have been tabulated; a column has also been included giving the maximum principal strain as calculated from the elastic constants and stresses, this being the yield-point strain on the supposition that Hooke's law holds up to the yield-point.

The values of E , Young's modulus, and C , the modulus of rigidity, have been calculated from the elastic ratio of stress and strain in the tension and torsion experiments; they have slight variations for the same tube. The value of σ , Poisson's ratio, has been deduced from these by the formula $\sigma = \frac{E}{2C - E}$, which is a method evidently liable to large errors; these, however, I do not think are serious enough to affect the conclusions I have drawn from the tables of results, as the chief conclusions are deduced from the stresses. I had intended to deduce the value of σ directly from the internal

pressure tests; the axial elongation due to a pressure p_0 , producing stresses p_1 and $2p_1$, being $\frac{p_1}{E}(1-2\sigma)$, so that

$\sigma = 0.5 - \frac{E}{2p_1} \times (\text{axial strain})$. The method of attaching the tube conveying the oil to the specimen was not perfect, and seemed to tilt the tube slightly as a whole, so that this method was not used. Perhaps Regnault's method would have been most appropriate.

The results of the experiments are given in Tables V. (Solid Bars), VI. (Steel Tubes), VII. (Copper Tubes), and VIII. (Brass Tubes), where the observations made at the yield-point and the results will be found.

46. *Results of the Experiments.*—For the more ready presentation of the results of the experiments, diagrams of various types have been drawn, and will be described as referred to.

The Tests on Solid Bars.—In the case of the solid-bar tests it will be observed that the yield-point stresses (see Table V.) rise as the course of tests proceeds: this indicates that the yield-point of the outer layers must have been much exceeded (§ 5). A glance at the torque-twist diagram (fig. 15) for the torsion-tests will clearly demonstrate the difficulty of locating the yield-point; though the tables merely embody my estimations of its occurrence, the curves at any rate represent actual occurrences, and permit of the ready comparison of points considered to correspond. The results may be held, with previous torsion experiments, to disprove the maximum-stress theory, as the ratios of the maximum stresses in the torque and tension tests are 0.615 to 0.69 for the iron, and 0.565 for the steel. They are also at variance with the maximum-strain theory, the values of the ratio for the experimental values of σ being 0.75 and 0.71 respectively. In this and some of the succeeding figures the time-effect is shown by observations giving different strains under a constant stress; the first observations were taken immediately the loads were applied; in some cases the intervals of time are given.

SYNOPSIS OF TUBE-TESTS.

NOTATION.

The tests on any tube are numbered in order of performance.

All results refer to the yield-point.

P = actual tension-load on specimen.

W = load producing the torque 15 W.

p_0 = corrected value of internal pressure.

Axial-strain = actual reading of the axial extensometer-scale.

Twist-strain = " " twist-scale.

p = the tensile-stress due directly to P.

q = the shearing-stress due directly to W.

p_1 = the axial stress due directly to p_0 .

$\omega_1, \omega_2, \omega_3$ = the principal stresses.

Axial strain as measured = the value of the strain as deduced from the reading of the axial scale.

Twist-strain as measured = the value of the slide as deduced from the reading of the twist-scale.

η_1, η_2, η_3 are the principal strains.

TABLE IV.

Diameters, Elastic Constants, &c. of Tubes.

No. of Tube.	Material.	Diam.	Thick-ness.	Area.	E.	C.	σ .
I.	Steel	1.316	0.029	0.1175	30,400,000	10,960,000	0.39
II.	"	1.250	0.028	0.1075	29,500,000	11,450,000	0.287
III.	"	0.4987	0.0245	0.0364	28,900,000	10,660,000	0.355
IV.	"	1.25	0.025	0.0971	31,100,000	11,170,000	0.398
V.	"	"	"	"	31,700,000	11,700,000	0.355
VI.	"	"	"	"	30,900,000	11,500,000	0.344
VII.	"	"	"	"	28,800,000	10,850,000	0.328
VIII.	"	"	"	"	31,000,000	11,200,000	0.365
IX.	"	"	"	"	29,600,000	10,400,000	0.423
X.	Copper ...	1.251	0.0362	0.14113	18,100,000	6,280,000	0.43
XI.	" ...	"	"	"	17,200,000	6,000,000	0.43
XII.	Brass	1.255	0.034	0.1295	15,000,000	5,020,000	0.45*
XIII.	"	"	"	"	14,760,000	4,760,000	0.50

The units for the above table are lbs. wt. and inches.

Solid Bars, Elastic Constants, &c.

Material.	Diam.	Area.	E.	C.	σ .
Norwegian Iron ...	0.628	0.3095	29,700,000	11,160,000	0.33
Mild Steel.....	0.504	0.1992	31,800,000	11,270,000	0.40

TABLE V.—Solid Bar Tests.
Elastic Limit.

Material.	Test No.	P.	W.	<i>p</i> .	<i>q</i> .	Max. Shear.	Max. Elongation.
Norwegian Iron .	1	...	30	...	9,000	9,000	0·000403
"	2	5000	...	16,200	...	8,100	0·000554
"	3	5000	20	16,200	6,180	10,200	0·00064
"	4	6700	30	21,700	9,270	14,800	0·00088
"	5	8000	...	26,000	...	13,000	0·00083
"	6	...	48	...	14,800	14,800	0·000665
Mild Steel	1	...	22	...	13,100	13,100	0·000582
"	2	6750	...	33,900	...	16,950	0·001065
"	3						

Yield-point.

Material.	Test No.	P.	W.	<i>p</i> .	<i>q</i> .	Max. Shear.	Max. Elongation.
Norwegian Iron .	1	...	40	...	12,360	12,360	0·00055
"	2	6400+	...	20,700	...	10,350	0·000698
"	3	5000	42½	16,200	13,100	15,300	0·00087
"	4						
"	5	8200	...	26,500	...	13,250	0·000892
"	6	...	59	...	18,200	18,200	0·000815
Mild Steel	1	...	38	...	22,700	22,700	0·001007
"	2	8000	...	40,200	...	20,100	0·001265
"	3	6000	35	30,100	20,900	25,800	0·00142

The above elongations are calculated from the stresses and elastic constants, thus assuming that the linear stress-strain relation holds to the yield-point; the actual strains at the yield-point are greater: *e.g.*, the max. elongation at $W=40$ in Test 1 on the Norwegian iron was at first 0·000587 and increased to 0·000606.

TABLE VI.—Results of Tests, Steel Tubes.

OBSERVATIONS.				CALCULATED STRESSES.										CALCULATED STRAINS.						Maximum Elongation as calculated from Stresses.
Number of Tube, Test.		Stress.		Strain.		Applied.				Principal.		Max. Shear.	As measured.		Principal.		Maximum Slide.			
		P.	W.	P _s	Axial.	Twist.	p.	q.	p _s	w ₁ .	w ₂ .		w ₃ .	Axial.	Twist.	γ ₁ .		γ ₂ .	γ ₃ .	
I.	1.	5380	88	450	2281	91.5	0	17,900	4,500	17,900	17,900	0	0.00155	0.001685	0.000832	0	0.001685*	0.000817		
	2.	5300	...	680	2266	...	46,800	...	6,750	50,300	9,000	0 to -430	0.00155	...	0.00155	...	...	0.001540		
	3.	5160	...	730	2154	...	43,900	...	7,400	51,650	13,500	0 to -646	0.00154	...	0.00154	...	...	0.001510		
	4.	5800	...	...	2253	...	46,800	...	...	46,800	0	0	0.00147	...	0.00147	...	...	0.001500		
	5.	...	88	...	...	66.0	...	17,900	...	17,900	0	0	0.00154	...	0.00154	...	...	0.001540		
	6.	...	...	...	...	...	...	...	...	...	...	...	0.00197	0.00197	0.000985	0	0.001970	0.000817		
II.	1.	130	135-150	...	...	680-785	...	30,800 to	...	30,800 to	30,800	0	...	0.00263	0.00131	0.00131	3	0.00263	0.00142	
	2.	6500+	...	...	1582+	...	60,500+	34,200	...	60,500+	0	0	0.002032	0.00604	0.00152	0.000645*	0	0.00604	0.002650	
	3.	4200	130	...	1026	696	39,100	29,500	...	39,100	29,500	0	...	0.00289	0.00134	0.00134	0	0.00289	0.001290	
	4.	4200	100	...	920	527	34,500	22,500	...	45,800	11,350	0	0.00182	0.00204	0.00137	0.000377*	0	0.00182	0.001320	
	5.	3700	...	some bending	...	...	...	...	...	...	...	...	0.00182	0.00204	0.001691	0.000339*	0	0.001691	0.00168	
	6.	5950	...	...	1600	...	55,500	...	...	55,500	0	0	0.002055	...	0.002055	0.000732*	0	0.002055	0.001880	
	7.	6200	...	...	1675	...	48,500	17,100	...	57,800	0	0	0.002025	...	0.002025	0.000645*	0	0.002025	0.001960	
	8.	5200	75	...	1450	440	...	...	...	53,900	5,400	0	0.001855	0.001703	0.002135	0.000335*	0	0.001855	0.001775	
	9.	250	130	...	...	738	...	...	...	29,500	29,500	0	0.00309	0.00309	0.00154	0	0.00309	0.001280		
	10.	4200	100	...	885	560	39,100	22,800	...	49,900	10,800	0	0.001137	0.002165	0.001710	0.000326*	0	0.001137	0.001830	
	11.	4000	110	...	1074	644	37,200	25,100	...	49,800	12,600	0	0.001383	0.002490	0.001968	0.000397*	0	0.001383	0.001910	
	12.	4500	...	1050	1160	...	41,90	...	11,000	52,900	22,900	0 to 1050	0.001491	...	0.001491	0.000428	0	0.001491	0.001545	
III.	1.	50	9	...	...	948	...	15,650	...	15,650	15,650	0	...	0.001427	0.000714	0	0.001427	0.000735		
	2.	1250	...	...	946	...	34,400	...	...	34,400	0	0	0.001217	...	0.001217	0.000432*	0	0.001217	0.001190	
	3.	50	9	...	...	1039	...	15,650	...	15,650	15,650	0	0.001567	0.000753	0.000753	0	0.001567	0.000735		
	4.	750	7	...	566	752	20,600	12,150	...	26,250	5,650	0	0.000977	0.000618	0.000618	0	0.000977	0.000777		
	5.	1000	5	...	750	536	27,500	8,650	...	30,650	2,550	0	0.000342	0.000808	0.001078	0.000258	0.000342	0.001070		
	6.	1200	...	...	895	...	33,000	...	...	33,000	0	0	0.001152	...	0.001152	0.000408	0.000408	0.001140		
IV.	1.	...	90	...	...	550	...	22,500	...	22,500	22,500	0	...	0.002070	0.001035	0	0	0.002070	0.001005	
	2.	4000	...	...	1084	...	41,200	...	...	41,200	0	0	0.001385	...	0.001385	0.000545	0	0.001385	0.001325	
	3.	2750	70	...	814	450	28,300	17,500	...	38,650	8,350	0	0.001030	0.001695	0.001425	0.000545	0.000545	0.001325	0.001325	
	4.	3000	...	...	...	...	30,900	...	12,000	37,900	24,900	0 to -1050	0.001152	...	0.001075	0.000840	0.000840	0.001320	0.001320	
	5.	...	75	...	911	454	...	18,750	12,000	42,700	1,700	0 to -1050	0.001132	0.000710	0.001250	0.000545	0.000545	0.001320	0.001320	
	6.	2500	...	...	104	...	25,800	...	16,900	42,700	33,500	0 to -1500	0.001288	...	0.000940	0.000547	0.000547	0.001325	0.001325	
	7.	3400	50	...	900	903	35,000	12,500	...	41,200	0	0	0.001288	0.001140	0.001324	0.000472	0.000472	0.001325	0.001325	
	8.	4000	...	...	1097	...	41,200	...	...	41,200	0	0	0.001388	...	0.001388	0.000545	0.000545	0.001325	0.001325	
	9.	...	90	...	...	561	...	22,500	...	22,500	22,500	0	...	0.00214	0.00107	0	0	0.00214	0.001005	

V.	1...	3250	...	949	33,500	...	33,500	0	16,750	0'001200	...	0'001580	...	0'001200	...	0'000426	...	0'001626	0'001057
	2...	250	...	419	2,600	...	17,500	0	17,500	0'000790	...	0'001032	...	0'000790	...	0'000599	...	0'001580	0'000745
	3...	2500	...	775	26,800	...	30,900	4,100	17,500	0'001191	...	0'001032	...	0'000790	...	0'000599	...	0'001580	0'001020
	4...	2900	...	829	23,000	...	33,100	0	16,500	0'001135	...	0'000955	...	0'001135	...	0'000400	...	0'001425	0'000104
	5...	2800.0	...	754	30,800	...	33,100	2,300	17,700	0'001100	...	0'000912	...	0'001100	...	0'000484	...	0'001565	0'000107
	6...	2000	...	780	20,600	...	32,600	24,000	16,800	0'000760	...	0'000957	...	0'000760	...	0'000393	...	0'001383	0'000760
	7...	1150	...	150	11,250	...	30,500	5,500	13,500	0'000890	...	0'001431	...	0'000890	...	0'000403	...	0'001510	0'000900
	8...	1500	...	626	15,450	...	34,500	29,500	13,500	0'000625	...	0'000792	...	0'000625	...	0'000602	...	0'001398	0'000673
	9...	250	...	643	2,600	...	31,500	17,500	17,500	0'001212	...	0'002425	...	0'001212	...	0	...	0'002425	0'000748
	10...	2300	...	1418	34,000	...	31,000	0	17,000	0'001793	...	...	...	0'001793	...	0'000637	...	0'002425	0'001073
VI.	1...	3300	...	1440	34,000	...	34,000	0	17,000	0'001822	...	...	...	0'001822	...	0'000612	...	0'002424	0'001100
	2...	3300	...	971	34,000	...	34,000	0	17,000	0'001230	...	...	...	0'001230	...	0'000413	...	0'001632	0'001100
	3...	2900	...	866	20,600	...	32,300	24,000	17,000	0'000792	...	...	...	0'000792	...	0'000640	...	0'001432	0'000792
	4...	2500	...	747	20,800	...	30,250	18,900	15,500	0'000945	...	...	...	0'000788	...	0'000548	...	0'001316	0'000788
	5...	250	...	524	2,600	...	16,250	16,250	16,250	0'000987	...	0'001975	...	0'000987	...	0	...	0'001975	0'000705
	6...	3000	...	335	20,900	...	33,150	2,250	17,770	0'000905	...	0'000905	...	0'000905	...	0'000631	...	0'001890	0'001100
	7...	2700	...	333	21,500	...	31,800	4,000	17,900	0'001235	...	0'001235	...	0'001235	...	0'000539	...	0'001825	0'001075
	8...	2700	...	354	17,250	...	17,250	17,250	17,250	0'002055	...	0'002055	...	0'001042	...	0'001042	...	0'002055	0'000750
	9...	2550	...	714	26,300	...	31,250	4,350	18,100	0'001094	...	0'001310	...	0'001189	...	0'000596	...	0'001786	0'001065
	10...	1500	...	1125P	13,450	...	27,150	23,400	14,100	0'000792	...	...	...	0'000792	...	0'000431	...	0'001432	0'000792
VII.	1...	1500	...	1800	15,450	...	34,350	37,800	19,750	0'000625	...	...	...	0'000785	...	0'000925	...	0'001744	0'000925
	2...	2000	...	1715	20,600	...	33,650	34,900	19,300	0'000605	...	...	...	0'000926	...	0'000781	...	0'001750	0'000926
	3...	2000	...	550	20,600	...	30,900	5,000	19,300	0'000790	...	0'001785	...	0'000790	...	0'000570	...	0'002225	0'001150
	4...	60	...	100	15,000	...	39,300	5,200	20,450	0'001126	...	0'001185	...	0'001126	...	0'000500	...	0'002445	0'001501
	5...	3500	...	1216	39,300	...	39,300	39,300	19,600	0'001540	...	0'002385	...	0'001192	...	0'000605	...	0'002445	0'001382
	6...	70	...	607	26,800	...	35,500	28,500	22,500	0'001010	...	0'002385	...	0'001540	...	0'000570	...	0'002445	0'001382
	7...	2800	...	555	35,000	...	39,900	8,000	22,500	0'000755	...	0'002385	...	0'001540	...	0'000570	...	0'002445	0'001382
	8...	3400	...	370	15,450	...	34,350	4,000	21,500	0'001260	...	0'001385	...	0'001530	...	0'000670	...	0'002445	0'001382
	9...	1500	...	1012	19,900	...	34,350	39,900	20,400	0'000792	...	...	...	0'001530	...	0'000670	...	0'002445	0'001382
	10...	250	...	P 162	19,900	...	34,350	39,900	20,400	0'000792	...	...	...	0'001530	...	0'000670	...	0'002445	0'001382

TABLE VI. (continued).

[illegible]

TABLE VII.

Results of Tests.—Copper Tubes.

OBSERVATIONS.				CALCULATED STRESSES.										CALCULATED STRAINS.									
Number of Tube.	Test.	Stress.			Strain.		Applied.				Principal.			Max. Shear.	As Measured.		Maximum Principal Strain calcu- lated from Stresses.	Principal.			Max. Slide.		
		P.	W.	P ₀ .	Axial.	Tw	t	p.	q.	P ₁ .	σ.	σ ₂ .	σ ₃ .		Axial.	Twist.		η ₁ .	η ₂ .	η ₃ .			
X.	1 ...	...	35	...	...	...	378	...	6,070	...	6,070	-6,070	0	6,070	...	0.001425	0.000483	0.000712	-0.000712	0	0.001425		
	2 ...	1500	...	...	...	612	...	10,630	...	10,630	0	0	5,315	0.000786	...	0.000538	0.000786	-0.000432	-0.000432	0.000432	0.001218		
	3 ...	1300	25	...	...	530	291	9,210	4,370	10,350	-1,755	0	6,360	0.000682	0.001095	0.000647	0.000910	-0.000525	-0.000297	0.000297	0.001435		
	4 ...	900	...	700	...	438	...	6,380	...	5,120	11,500	10,240	0 to -660	0.000563	...	-0.000523	0.000393	0.00029	0.00029	-0.000523	0.000910		
	5 ...	600	...	900	...	227	...	4,250	...	6,580	10,830	13,160	0 to -850	0.000420	...	-0.000478	0.00038	0.000468	-0.000478	0.000468	0.000946		
XI.	6 ...	1700	...	...	...	662	...	12,050	...	12,050	0	0	6,025	0.000550	...	0.000665	0.000850	-0.00037	0.00037	0.00037	0.00122		
	7 ...	1800	...	...	...	606	...	12,750	...	12,750	0	0	6,375	0.000780	...	0.000705	0.000780	-0.00034	0.00034	0.00034	0.00112		
	8 ...	...	35	...	...	341	...	6,070	...	6,070	-6,070	0	6,070	...	0.001285	0.000483	0.000642	-0.000642	0	0.001285			
	9 ...	...	25	800	115	255	...	4,370	5,730	13,835	3,355	0 to -740	7,300	0.000148	0.000962	0.000685	-0.00015	0.000685	-0.000412	0.000412	0.001097		
	1 ...	...	...	825	136	...	...	...	5,925	5,925	11,850	0 to -750	6,300	0.000175	...	0.000515	0.000045	0.000515	-0.000423	0.000423	0.000938		
	2 ...	1400	...	...	464	...	9,925	...	...	9,925	0	0	4,962	0.000595	...	0.000577	0.000595	-0.00026	0.00026	0.00026	0.000855		
	3 ...	1250	...	525	525?	...	8,860	...	3,870	12,730	7,740	0 to -500	6,600	0.000675?	...	0.00052	0.00052	0.00012	-0.00049	0.00049	0.00100		
	4 ...	...	35	...	...	436	...	6,070	...	6,070	-6,070	0	6,070	...	0.00164	0.000505	0.00082	-0.00082	0	0.00082	0.00082		

TABLE VIII.
Results of Tests.—Brass Tubes.

OBSERVATIONS.				CALCULATED STRESSES.										CALCULATED STRAINS.						
Number of Tube.	Tstg.	Stress.		Strain.		Applied.			Principal.			Max. Shear.	As Measured.		Maximum Principal Strain calcu- lated from Stresses.	Principal.			Max. Slide.	
		P.	W.	p ₀	Axial.	Twist.	p.	q.	p ₁ .	σ ₁	σ ₂ .		σ ₃ .	Axial.		Twist.	η ₁ .	η ₂ .		η ₃ .
XII.	1	...	50	525	...	...	574	...	9,450	...	9,450	-9,450	0	...	0.00216	0.000940	0.00108	-0.00108	0	0.00216
	2	...	2250	...	...	1070	...	17,400	...	...	17,400	0	0	0.001375	...	0.001160	0.001375	-0.00055	-0.00055	0.001925
	3	...	2000	25	...	882	244	15,450	4,500	...	16,650	-1,215	0	0.001145	0.000918	0.001145	0.001267	-0.000582	-0.00046	0.001550
	4	...	1600	40	...	748	438	12,370	7,200	...	14,885	-2,315	0	0.000982	0.00165	0.001045	0.001353	-0.000776	-0.000385	0.00213
	5	...	1750	...	900	857	...	13,520	...	7,250	20,770	14,500	0 to -850	0.001100	...	0.001075	0.000446	-0.001013	0.00209	
	6	...	1500	...	950	763	...	11,580	...	7,680	19,260	15,360	0 to -900	0.000980	...	-0.000995	0.000945	0.000550	-0.000995	0.00194
	7	...	2600	...	...	1087	...	20,100	...	...	20,100	0	0	0.001398	...	0.00134	0.001398	-0.000655	-0.000655	0.002083
	8	...	...	57½	...	...	645	...	10,350	...	10,350	-10,350	0	...	0.00243	0.00103	0.00116	0	0.00243	
	9	...	1900	40	...	820	480	14,670	7,200	...	17,610	-2,940	0	0.001053	0.00181	0.001265	0.001485	-0.000854	-0.000422	0.00234
	10	...	1500	...	1050	661	...	11,580	...	8,400	19,980	16,800	0 to -985	0.000850	...	-0.00121	0.00078	-0.000465	-0.00121	0.00199
	11	...	1800	...	900	784	...	13,900	...	7,250	21,150	14,500	0 to -850	0.001005	...	-0.00116	0.00093	0.000272	-0.00116	0.00209
	12	...	...	45	825	118	528	...	8,100	6,520	18,520	1,040	0 to -750	0.000152	0.00199	0.00120	+0.00120	-0.000534	-0.000636	0.00184
XIII.	1	...	1500	...	950	759	...	11,580	...	7,670	19,250	15,340	0 to -900	0.000975	...	-0.00115	0.000400	0.000680	-0.00115	0.00195
	2	...	2300	...	...	1042	...	17,750	...	...	17,750	0	0	0.001340	...	0.001205	0.001340	-0.00067	-0.00067	0.00201
	3	...	350	45	...	...	480	...	8,100	...	8,100	-8,100	0	0.00181	0.000850	0.00090	-0.00090	0	0.00181	

Fig. 10.—Torsion and Shear Calculations for Brass Tubes.

47. *The Material and Shape of Tubes satisfactory.*—Turning now to the results of the tube-tests, reference to Table IV. will show that the densities found were high, and that the values of the moduli of elasticity are usual values for the various materials, and certainly not low; also inspection of various stress-strain diagrams indicates that the relation is linear up to a considerable fraction of the yield-point stress. For these reasons the tubes may be taken as of sufficient excellence in material and shape to afford results worthy of consideration.

Few torsion-internal pressure experiments have been performed, but a comparison of results obtained thus with the results of those tests when somewhat similar stresses are differently produced, will tend to cause confidence in the experiments. The torsion-internal pressure tests are Nos. IV. 5, V. 7, VII. 4, X. 9, and XII. 12.

48. *The Elastic-limit Phenomenon.*—The elastic-limit phenomenon, it will be observed, is much decreased in a repetition of a test, the observed strain at the yield-point being less in the second test than in the first, although the stress-strain curves within the elastic limit practically coincide. To illustrate this point some of the tests bearing upon it have been plotted out observation by observation; in figs. 16 and 17 are two sets of tension tests, and in fig. 18 some torsion tests.

When, however, a test imposes upon the material stresses of a different type to those of the preceding test, the elastic-limit effect, which may have temporarily disappeared owing to the similarity of the two preceding tests, is displayed to the full.

Copper seems, however, to harden so much under successive tests, that the strains appear as a whole to lessen with the amount of treatment.

The elastic-limit effect also increases with the number of times different stresses have been applied to the material, as may be seen by comparing the stress-strain curves for tests of the same type, but separated by a number of other tests. Examples are represented in figs. 16 (tension) and 18 (torsion).

This phenomenon may indicate that the cause of failure under comparatively low varying stresses, such as employed

Fig. 16.

Fig. 17.—Tension Tests on Steel.

Fig. 18.—Tension Tests on Steel.

by Wöhler in his experiments, is due to gradual increase of size of defects, initially very small; and I think tends to show that the elastic-limit phenomenon is due to the occurrence of the yield-point over small localities.

49. *Yield-point Stresses of same type nearly constant throughout series on each tube.*—It is to be noted that throughout a series of tests on any tube the yield-point stress never having been much exceeded, it has nearly the same value in tests of the same type, though a series of similar tests tend to cause it to rise.

50. *The Variation of the Maximum Stress.*—Proceeding now to a consideration of the variations of the maximum principal stress, of the maximum principal strain, and of the maximum shearing-stress at the yield-points of the different tests, we find that the first-mentioned varies very largely, its least value being found in the pure torsion tests, while in the tests by tension and internal pressure its value is practically constant, and is the same whether the greater of the principal stresses is axial or circumferential. The experiments on Tubes VII. and VIII. show the latter emphatically. We hence conclude, firstly, that in the case of two tensions (such as in a boiler) the lesser tension does not appreciably affect the yield-point; and, secondly, by the similarity of the yield-point stress, whether the greater tension be axial or circumferential, that the material is practically isotropic in respect to the occurrence of the yield-point,—at any rate as regards the tangent-plane stresses, with which we are most concerned.

51. *Variation of the Maximum Principal Strain.*—The maximum strain at the yield-point varies throughout the experiments: it is greatest in the simple tension experiments, and least either in the torsion tests or when the axial and circumferential tensions are equal. Both the strain as measured (where the axial stress is greatest) and as calculated from the stresses and elastic constants (and so not including the elastic-limit effect) are tabulated and can be compared; the calculated maximum strain is sometimes radial and negative. The variations of the maximum strain from constancy are considerable for all the materials experimented upon.

52. *The Maximum Shearing-Stress or Slide nearly constant.*—The maximum shearing-stress developed, and the corresponding maximum shearing-strain or slide (which are quantities directly proportional and independent of the intermediate principal stress within the elastic limit) are comparatively constant, although the amount is highest in the case of pure torsion and diminishes towards the tension case. It will be remembered that the plane of the maximum shear changes from being a tangential plane of the specimen in the torsion-tension cases to a plane intersecting the axis of the specimen in the tension-internal pressure tests.

The stresses at the elastic limit have been considered in several cases, and have been found to follow the same variations approximately.

53. *Graphical presentation of Variations.*—The tables and figures may be held to prove, in the case of the materials experimented upon—steel, copper and brass—that to a first approximation the shearing-stress is constant whenever the third principal stress is zero, or whenever the lesser principal stress is zero or small.

It can readily be found by trial that no other simple relation between the stresses or strains is approximately constant.

The results of the series of tests upon steel, with the exception of the unannealed specimens, are shown in fig. 19, which has been constructed as follows :— Ox is the axis of principal stress most nearly coinciding with the generators, and the abscissæ represent the relative values of the stresses in the various experiments, a mean axial yield-point stress being selected as the basis of the scale and marked 10. Relative values of the other principal stresses were then set off parallel to Oy and Oz , those parallel to Oy being the mean, and those parallel to Oz being the minimum principal stress. The paper should be now considered as folded at a right-angle along the line Ox ; and the observations recorded in the quadrant zox to lie actually on the paper, while those plotted in the quadrant xoy , though nearly in the plane of the paper, should correctly be considered as projected by lines parallel to Oz on to a plane through Ox inclined to the plane xoy at an angle whose tangent is half the ratio of the internal pressure to the circumferential stress it produces. The value of the

Fig. 19.—Synoptical diagram of Tests.

angle is about $\tan^{-1} 0.0225$, so that the points may be regarded as practically in the plane of the paper.

On the paper are drawn the lines BAC, BD representing the locus of the points had the maximum stress been true; the lines GAH, KAL, MAN, representing the locus had the maximum strain-law been true, and σ had the values 0.5, 0.25, and 0.356 respectively, the latter being the mean experimental value. The constant shearing-stress law locus is the line EFABD, and it will be seen to what extent this agrees with the mean distribution of the observations. No movement of the other lines parallel to themselves can produce any approximation to the mean of the actual distribution.

It must, however, be remembered, while inspecting this figure, that this method of setting out the points was adopted as being just to any theory of strength; had a mean basis been adopted the points could have followed the shearing-stress law much more closely.

54. *Conclusions probable for general type of Stress.*—Considering the three-dimensional view (fig. 20, p. 230), if the shearing-stress law were true the observations plotted in space between Ox , Oy , Oz , or two tensions and a third stress, would fall upon the two planes, EAB, EDB. We have in our experiments investigated such combinations of stresses as give rise to half the line AE, *i. e.* to FA, to AB, and to BD' which is half BD.

Now a similar figure may be drawn for the strains, and since the traces on the planes $\eta_2=0$ and $\eta_3=0$ in such a figure are straight lines, and since the strains are so small that no quadratic or higher power can have appreciable effect, except in the improbable case of their coefficients being extremely large, the function representing this relation is linear. Since by experiment the value of the intermediate strain has no effect on the yield-point relation, the coefficient of η_2 in the linear relation is zero. The linear relation therefore reduces to the equation of the line AE (or the plane AEB). This line, as determined by experiment, being equally inclined to the axes, the corresponding relation is that the shearing stress or strain is constant.

55. *Effect of a Uniform Volumetric Stress.*—In order to ascertain the relationship of the stresses at the yield-point

more accurately, sequences of tests in which torsion and tension combined in various amounts should be compared, and the inclination of the line AE more accurately determined. From this point of view a direct compression-test would be of much assistance. Now the linear relation may be written

$$\varpi_1 - \varpi_2 + \lambda(\varpi_1 + \varpi_2) = C,$$

since this contains two independent constants. Putting $\varpi_2 = 0$ we get the case of tension only and the yield-point stress from the equation is $\frac{C}{1+\lambda}$. Putting $\varpi_2 = -\varpi_1$ we get the case of pure torsion, and the yield-point stress is then $\frac{1}{2}C$. The mean of the experimental results gives for the ratio of these yield-point stresses the value 0.52, for which the value of λ is 0.04; and the relation becomes

$$\varpi_1 - \varpi_2 + 0.04(\varpi_1 + \varpi_2) = q_0,$$

where q_0 is the value of the yield-point shearing-stress in pure torsion.

56. *Convenient view of general type of Stress.*—Since the stress ellipsoid $\varpi_1 x^2 + \varpi_2 y^2 + \varpi_3 z^2 = 1$, where $\varpi_1, \varpi_2, \varpi_3$ are in descending order of magnitude, may be written in the form

$$\left(\frac{\varpi_1 + \varpi_2}{2}\right)(x^2 + y^2 + z^2) + \left(\varpi_3 - \frac{\varpi_1 + \varpi_2}{2}\right)z^2 + \frac{\varpi_1 - \varpi_2}{2}xy = 1$$

(since the invariants are the same), it is clear that any state of stress can be represented by the coexistence of a shearing-stress, a volumetric uniform stress, and a stress perpendicular to the plane of the shear; and the yield-point occurs when the shearing-force together with a small fraction of the volumetric stress reaches a certain amount: the volumetric stress being reckoned positive when tensile.

That the coefficient of the volumetric stress for steel is small has been shown by these tests, and that it has some value greater than zero seems probable, as volumetric tension would ultimately reduce the material to the liquid or gaseous form and cause the necessary shear to evanesce. No value can be directly deduced from such considerations owing to

lack of knowledge of ultimate molecular action. It may be noted that Bauschinger's experiments on the yield-point in tension and compression, in which the effect of λ would be most marked, would point to zero as its value.

57. *The Copper and Brass Tests, and Diagrams.*—In the case of the copper and brass tests the yield-point is more difficult to locate; the shearing-stress law still appears to be that one of the simple laws nearest to the truth. Taking into account the volumetric stress effect, the coefficient λ is larger than in the case of steel, but so few tests have been made that conclusions should be drawn with hesitation, especially as the yield-point stress tends to rise with the successive tests.

In the copper tests the yield-points were very indefinite, and I have plotted out types of the unusual tests from which the stresses and resulting strains and the mode of their variation can be seen at a glance, and their values at points considered to correspond to the elastic limit or yield-point in the various tests readily compared. These curves are given in figs. 21–24.

In fig. 21 is shown a test under internal pressure only, the stresses parallel to the axis being one-half of the simultaneous circumferential stresses; the values of the internal pressure are laid off along the central line, and the values of the corresponding axial and circumferential strains given on the right and left hand respectively of the figure. The readings of the axial and diametral (fig. 14, p. 230) extensometers are given along the base line, and the corresponding strains along the top of the diagram. The axial strain is laid off positively (extension) towards the right, and the circumferential strain (from the diametral extensometer) positively towards the left.

Fig. 22 represents a combined tension and internal pressure test; the tension load was applied first, the actual loads being marked along the central line and the corresponding stresses along the right-hand side line of the figure. The axial strains are positive and are set off to the right of the central line; the circumferential strains (diametral extensometer) are set off

Fig. 21.—Copper Test by Internal Pressure.

Fig. 22.—Copper Test by Tension and Internal Pressure.

Fig. 23.—Copper Test by Tension and Torque.

Fig. 24.—Copper Test by Torque and Internal Pressure.

Fig. 25.—Curves showing Variations of Maximum Stress, of Maximum Strain, and Maximum Shear.

positively to the left of the central line, but during the application of the tension loads this strain is negative, and the stress-strain line therefore bears to the right. After a load of 900 lbs., actual, was reached, the tension load was kept constant and the internal pressure applied; the values are marked off along the central line, the resulting circumferential stresses are given on the left-hand, and the total axial stresses on the right-hand of the diagram. Along the line of no internal pressure are given the actual extensometer readings, the corresponding strains being placed at the top.

Fig. 23 represents a combined tension and torsion test; the torque was applied first, and the values of W are set off along the central line and the corresponding stresses on the right-hand side of the diagram. The shearing-strains or slides are set off to the right of the central line. After W had reached the value of 25 lbs., it was kept constant and the tension-loads applied; it will be noticed that the initial tension-load was 150 lbs. The actual tension loads are recorded along the central line, and the corresponding stresses on the left-hand side; the resulting axial strains are positive to the left of the central line. The principal strains at any point cannot be determined at once from the diagram without a construction or calculation. The addition of a tension load should not change the twist (slide) reading, unless the elastic limit has been passed; the slight inclination of this part of the line may be due to a bending-moment, or to a slight want of isotropy introduced by the previous tests.

Fig. 24 is a similar diagram for a torque and internal-pressure test, and can be readily understood from its similarity to the preceding figures.

All these figures, and the corresponding ones for simple tension and torsion, show the stress-strain line as bending gradually over, the strongly-marked yield-point of iron and steel being absent; they also show how unsatisfactory, in such cases, any tabulation of elastic-limit or yield-point stresses and strains must be. The selection of apparently similar points on the curves, however, leads to the discovery at them of the same amount of shearing-stress.

58. *Practical Conclusions and Note.*—The result, then, of these experiments, as applicable to practice, is that the condition for initial yielding of a uniform ductile material is the existence of a specific shearing-stress; and that the intermediate principal stress is without effect.

Note.—Cases of more than one principal stress are met with most frequently in boilers, hydraulic cylinders, and crank-shafts. In the former cases the results obtained above lead to the same dimensions as are obtained by the consideration of the greatest stress only, while the maximum-strain theory would lead to the adoption of too small a thickness. In the case of crank-shafts, or “combined bending and twisting,” both the greatest stress and greatest strain theories lead to too small dimensions, the corresponding well-known formulæ for the “equivalent bending moment” being

$$\frac{1}{2}(M + \sqrt{M^2 + T^2}) \quad \text{and} \quad \frac{3}{8}M + \frac{5}{8}\sqrt{M^2 + T^2}$$

respectively, where M is the applied bending-moment and T the applied twisting-moment. The greatest shearing-stress theory leads to the (greater) value $\sqrt{M^2 + T^2}$ for the equivalent bending-moment, and where the dimensions are thus determined that formula should be used. A lower factor of safety might, however, be used in such cases, where there is a large reserve of understressed material.

It may also be noted that the specific shearing-stress at the yield-point is better determined by taking one half of the tensional yield-point stress, than from the results of torsional experiments in which the sharpness of the yield-point is masked, as explained in § 10 and shown in fig. 15.

[I have pleasure in expressing my thanks for assistance rendered in the tests by Messrs. Ashworth, Howe, Knowles, Rebboli, Nichols, Vail, and Heichert; the latter of whom also assisted me in the construction and erection of the greater part of the apparatus.]

DISCUSSION.

Dr. CHREE (communicated).—Mr. Guest, in his paper, has regarded the shearing-stress theory as a little-known one. As the shearing-stress is half the difference between the greatest and least principal stresses, this theory is the same as the maximum stress-difference theory employed by Prof. G. H. Darwin (following Tresca) in his investigations respecting the state of the earth. He (Dr. Chree) has also employed this theory, as well as the greatest strain theory, in many elastic problems. All the theories presuppose the ordinary mathematical theory of elasticity, which depends on Hooke's law, that the stress-strain relation is linear, and assumes that strains are so small that their squares and products can be neglected. Mr. Guest concludes that, in ordinary materials, the stress-strain relation is linear to the elastic limit, which answers to a stress lower than that which answers to the yield-point, and that yield-point phenomena arise locally before a general yield-point is reached. Nevertheless, he focusses attention on the yield-point as the criterion of strength.

Prof. PERRY congratulated the author upon his paper, and said there was no time left to discuss it at length. The results obtained gave information upon a subject of which very little is known experimentally.