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To cite this article: P. Givskov (1920) On investment in funds, Scandinavian Actuarial Journal, 1920:1, 207-226, DOI: [10.1080/03461238.1920.10405317](https://doi.org/10.1080/03461238.1920.10405317)

To link to this article: <http://dx.doi.org/10.1080/03461238.1920.10405317>

Published online: 22 Dec 2011.

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On Investment in Funds.

By P. Glvskov.

The purpose of this paper is to determine how large a capital risk a monetary institution is incurring by receiving capital payments under obligation to yield a fixed guaranteed interest until a prescribed date, even if the market rate of interest should fall below the rate guaranteed, and to investigate to what extent the risk is decreased if it is possible to invest the payment first received at a rate of interest exceeding the rate guaranteed.

The subject will be especially dealt with as concerning the investments made by life insurance companies and an attempt will be made to form rules applicable at the investment of capital at a given rate of interest and for a fixed period of time. In accomplishing this we can

1) regulate the interest which is earned in such a way that a maximum and a minimum rate will be provided for and losses by decreasing exchange values avoided

2) establish reserve funds equivalent to a lower rate of interest without being obliged to formally adopt a new basis of the calculation

3) decide when the increased investment connected with decreasing interest necessitates a lower rate of interest as a basis of the premiums.

We shall try to find the solution of the problem in two ways, that is to say, the general treatment (A) is connected with a mathematical and numerical treatment (B) by means of which the conclusions arrived at will be determined with further precision.

A. Assuming that a company is bound to receive arbitrary payments and to pay a certain interest (*i*) on them during

a certain period, what will, then, the value of the company's loss amount to at the beginning of the period, if the effective interest falls to r .

Let us suppose that the payments are invested in bonds O yielding a nominal interest r and maturing at a certain date coinciding with the day of reimbursement of the payments mentioned.

If a payment I is received n years before the maturity and we wish to obtain the interest (i) on it, we shall have to purchase bonds to a face value of $P = I : K_i$, K_i meaning the market price at which the bonds must be purchased, if they shall yield the effective interest i . If, however, the effective interest were only r , the price of these bonds would be $P \cdot K_r$. In this case we should thus realize a loss amounting to $P \cdot K_r - P \cdot K_i = P(K_r - K_i)$.

If the annual payments are equal, and these, bearing interest at i per cent per annum, increase to U at the expiration of the period, the amounts of payments and interest to be invested yearly will increase year by year in the same way as the instalments on a bond with a constant annual payment, the instalments on such a bond increasing with the interest on the increasing instalments.

While, as stated before, the company's obligations may be covered by purchasing bonds O with a given expiration, at the market price K , its claims may, in this case, be compared with bonds O^y with a given annual payment, the market price of which is called K^y . To the capital U correspond bonds of the first category with a face value of $U : K_i$, and of the second category with a face value of $U : K_i^y$, the effective interest being i . When the effective interest falls from i to r , the value of the former bonds will increase by $\frac{U}{K_i}(K_r - K_i)$, and the value of the latter by $\frac{U}{K_i^y}(K_r^y - K_i^y)$.

The total loss will, then, amount to $U \left(\frac{K_r}{K_i} - \frac{K_i^y}{K_i^y} \right)$. If the nominal interest on the bonds is equal to i , as it must be, if the bonds shall cover exactly the obligations of the company, we shall have $K_i = K_i^y = 1$, and the loss of the company thus amounts to $U(K_r - K_i^y)$.

The general formula

$$(1) \quad U\left(\frac{K_r}{K_i} - \frac{K_r^y}{K_i^y}\right)$$

gives the loss of the company when the payments with interest are increasing in the same way as instalments on a bond bearing an arbitrary nominal interest and with a market price K^y , nothing being assumed as to the way in which the instalments are decreasing. If we call K_a the market price of a bond (O_a) with given instalments, we shall have the formula

$$(2) \quad U\left(\frac{K_r}{K_i} - \frac{K_r^a}{K_i^a}\right).$$

We may replace different payments both increasing and decreasing by combinations of bonds with given annual payments and given annual instalments. A bond with the market price $K^a + p(K^a - K^y)$ will thus correspond to decreasing payments, if $p > 0$, and to increasing payments if $p < 0$. In such case the loss will be

$$(3) \quad U\left(\frac{K_r}{K_i} - \frac{K_r^a + p(K_r^a - K_r^y)}{K_i^a + p(K_i^a - K_i^y)}\right).$$

If some of the payments are received before interest falls, the loss will be reduced. If, for example, a payment R is received at a moment where the effective interest is e ($e > i$), we may purchase bonds to a face value of $\frac{R}{K_e}$ for it, while we only need $\frac{R}{K_i}$; we thus dispose of superfluous bonds to a face value of $R\left(\frac{1}{K_e} - \frac{1}{K_i}\right)$, the value of which, in case the interest falls to r , is $R\left(\frac{1}{K_e} - \frac{1}{K_i}\right)K_r$. If the payments with in-

terest are increasing evenly from R to U , and the interest immediately falls to r , the total loss will, then, amount to

$$(4) \quad (U - R) \left(\frac{K_r}{K_i} - \frac{K_r^a}{K_i^a} \right) - R \left(\frac{1}{K_e} - \frac{1}{K_i} \right) K_r.$$

Should we desire to establish a fund to cover such losses, we may determine each year the loss calculated by the above formula, by means of the ever effective interest e and of the payments received R . If $e > i$ the fund may be zero, in such case an eventual subsequent increase of the calculated fund, owing to a fall in the interest, can be covered by the gain realized on the market price of R , and as long as the interest is not inferior to r , we shall thus be secured against a loss on the capital U .

It is not necessary to invest the payments in bonds of the said category, it is sufficient that the market price of the bonds is always below par. If the market price rises above par, these bonds must be replaced by others bearing a lower nominal interest, so that a forced redemption at par can be avoided. In this way we shall always obtain an interest which is higher than the effective interest of the investment.

The indicated formulas (1), (2) and (3) may also be proved in the following way:

If the successive payments are due continuously, a capital 1 at the end of the period may be created by annual payments of $\frac{1}{\bar{a}_n} - \delta$, where $\bar{a}_n$ means the continuous annuity, δ the continuous rate of interest; after t years the amount paid with interest will be $1 - \frac{\bar{a}_{n-t}}{\bar{a}_n}$. As the remaining debt on a bond is $\frac{\bar{a}_{n-t}}{\bar{a}_n}$, the differences in the amounts received (with interest on precedent amount) and in the remaining debt will be the same.

The differential quotient of the amount paid is

$$\frac{dI}{dt} = d \frac{\left(1 - \frac{\bar{a}_{n-t}}{\bar{a}_n} \right)}{dt} = \frac{e^{-\delta(n-t)}}{\bar{a}_n}.$$

The amounts paid are invested in bonds with a given date of maturity and bearing a nominal continuous interest δ_2 . If the effective interest is δ_4 , their market price will be

$$K = e^{-\delta_4(n-t)} + \delta_2 \frac{1 - e^{-\delta_4(n-t)}}{\delta_4}$$

and the discounted loss on the market price

$$(K - 1) e^{-\delta_4 t} = \left(\frac{\delta_2}{\delta_4} - 1 \right) (e^{-\delta_4 t} - e^{-\delta_4 n})$$

the total loss will, then, be

$$\begin{aligned} T &= \int_0^n (K - 1) e^{-\delta_4 t} \frac{dI}{dt} dt = \int_0^n \left(\frac{\delta_2}{\delta_4} - 1 \right) (e^{-\delta_4 t} - e^{-\delta_4 n}) \frac{e^{-\delta_2(n-t)}}{\bar{a}_n^2} dt = \\ &= e^{-\delta_4 n} + \delta_2 \bar{a}_n^4 - \frac{\bar{a}_n^4}{\bar{a}_n^2} = K - K'' \end{aligned}$$

a^i a continuous annuity bearing a continuous interest δ_i

$$v_i^n = e^{-\delta_i n}.$$

If such amounts are paid that the payments with interest are increasing evenly so: with $\frac{1}{n}$ yearly, the differential quotient of the amount paid will be $\frac{1}{n} dt$, and the total loss

$$\begin{aligned} T_1 &= \int_0^n \frac{1}{n} \left(\frac{\delta_2}{\delta_4} - 1 \right) (e^{-\delta_4 t} - e^{-\delta_4 n}) dt = \\ &= e^{-\delta_4 n} + \delta_2 \bar{a}_n^4 - \left(\frac{\delta_2}{\delta_4} \left(1 - \frac{\bar{a}_n^4}{n} \right) + \frac{\bar{a}_n^4}{n} \right) = K - K^a. \end{aligned}$$

This formula may also be written

$$(5) \quad \frac{\delta_2 - \delta_4}{\delta_4} \left(\frac{1 - e^{-\delta_4 n}}{n \delta_4} - e^{-\delta_4 n} \right) = \frac{\delta_2 - \delta_4}{n \delta_4^2} - \frac{1 + n \delta_4}{n \delta_4^2} v_i^n (\delta_2 - \delta_4).$$

The same calculations are applicable to a bond which is combined as a sum or difference of these two types O^y and O^a .

Application to a Life Assurance Company.

The formulas explained here may be applied to the calculation of a security fund against a fall in the rate of interest for a life assurance company, if the annual increase of the reserve, the rest n years of the duration of the assurances and the sum U which will fall due at the end of this period can be determined.

The reserve for the most common kind of assurance, the endowment assurance, increases in a similar way as the instalments on a sinking fund with the same duration as the assurance, the difference in reserve $1 - \frac{a_{x+n, n-r}}{a_{x, r}}$ corresponding to the difference in the remaining debt $\frac{a^{n-r}}{a_n}$. In the above formula the payments represent the increases of the reserve; if these are replaced by the instalments on a sinking fund yielding the same interest as used for the basis, these instalments will be greater than the increase of the reserve at the beginning, because loans at a low interest are most rapidly amortised, and, by using the formula, we shall therefore get a higher limit for the fund. If, however, we shall determine the fund for a stock of sums assured, we must suppose that the new business has stopped, and, owing to the natural discontinuances, the increase of the reserve for a stock without new business will be quite different. If the increase in the policy value is calculated on the New Hafnia Basis and with the frequencies of discontinuance of Old Hafnia, it will be seen that the increase is decreasing in the same way as the remaining debt on a sinking fund at the market price $K^a + p(K^a - K^y)$, where p is equal to 1 for duration $n = 25$ and age at exit $w = 56$, as well as for $n = 30$ and $w = 61$, and $p = 2$ for $n = 30$ and $w = 66$.

If the fund is calculated according to the formula (page 3), which corresponds to this empiric increase of the reserve, namely, when $U = \alpha F$,

$$S = {}^{\alpha}F \left(\frac{K_r}{K_i} - \frac{K_r^a + p(K_r^a - K_r^y)}{K_i^a + p(K_i^a - K_i^y)} \right),$$

which by putting the nominal interest $= r$, at which K is simplified to

$$S = {}^{\alpha}F \left(\frac{1}{K_i} - \frac{1}{K_i^a + p(K_i^a - K_i^y)} \right),$$

we shall get the values given in the table below under A (as for the calculation see note 1)

	A			B	C
	$w = 56$	$w = 61$	$w = 66$		
$n = 25$	0,0163			0,0185	0,0163
$n = 30$		0,0160	0,0160	0,0183	0,0155

It will be seen that the values under A vary very inconsiderably and that the *fund will amount to c. 1,6 per cent of the assured sum by a fall in the interest of $\frac{1}{2}$ percent.*

[Note 1. If the nominal interest is K and the effective interest i , the formulas for the market prices are

$$K_i = (1 + i)^{-n} + r a_n^i \text{ idrt } a_n^i = \frac{1}{i} (1 - (1 + i)^{-n})$$

$$K_i^y = \frac{a_n^i}{a_n^r}, K_i^a = \frac{r}{i} \left(1 - \frac{a_n^i}{n} \right) + \frac{a_n^i}{n}.$$

$$\begin{array}{lll} \text{Putting } r = 0,03 & i = 0,035 & n = 25 \\ \text{»} & \text{»} & \text{»} & n = 30 \end{array}$$

$$\begin{array}{lll} \text{we have } K = 0,9176 & K^y = 0,9465 & K^a = 0,9513 \\ \text{»} & \text{»} & K = 0,9080 & K^y = 0,9383 & K^a = 0,9460. \end{array}$$

In the three cases treated here ${}^{\alpha}F$ has the three values: 0,3712 F , 0,2998 F and 0,2584 F . Then we shall have for the three funds in A

$$S = F \cdot 0,3712 \left(\frac{1}{0,9176} - \frac{1}{0,9513 + (0,9513 - 0,9465)} \right) = F \cdot 0,0163$$

$$S = F \cdot 0,2998 \left(\frac{1}{0,9080} - \frac{1}{0,9460 + (0,9460 - 0,9383)} \right) = F \cdot 0,0160$$

$$S = F \cdot 0,2584 \left(\frac{1}{0,9080} - \frac{1}{0,9460 + 2(0,9460 - 0,9383)} \right) = F \cdot 0,0160.$$

Assuming an even increase of the reserve, we shall have,

$$(8) \quad S = \alpha F \left(\frac{K_r}{K_i} - \frac{K_r^a}{K_i^a} \right),$$

which if $K_r = 1$ will be

$$(9) \quad S = \alpha F \left(\frac{1}{K_i} - \frac{1}{K_i^a} \right),$$

if $\alpha = 1,035^{-n}$ we shall get the values mentioned under *B* in the table on page 8 if $\alpha = 1,035^{-n}$ than mentioned under *C*. It will be seen that we may employ the simple formula for $\alpha = 1,035^{-n}$ or $1,03^{-n}$.

It will be seen by the formula on page 4 that by a fall in the interest of the basis from *i* to *r*, and when the reserve is $= R$, the security fund will amount to

$$(10) \quad (F \cdot 1,03^{-n} - R) \left(\frac{K_r}{K_i} - \frac{K_r^a}{K_i^a} \right) - R \left(\frac{1}{K_e} - \frac{1}{K_i} \right) K_r.$$

If the nominal interest is *r*, we shall have

$$(11) \quad (F \cdot 1,03^{-n} - R) \left(\frac{1}{K_i} - \frac{1}{K_i^a} \right) - R \left(\frac{1}{K_e} - \frac{1}{K_i} \right).$$

If the nominal interest is *i*, we shall have

$$(12) \quad (F \cdot 1,03^{-n} - R) (K_r - K_r^a) - R \left(\frac{1}{K_e} - 1 \right) K_r.$$

F represents the sum assured, R the reserve, K_r is the market price of a bond yielding an effective interest r and with the duration n . K_r^a represents a sinking fund, which is amortized by equal annual instalments. n is determined by putting the value of the sum assured equal to $F(1+i-0,05)^{-n}$ (see note 3).

If we choose the interest of the basis 1 percent under the actual effective interest and if the reserve has increased to about 8 percent of the sum assured (consequently after 3 years) before the interest falls, this can, as will be seen from the following example, sink $1\frac{1}{2}$ percent to $\frac{1}{2}$ percent below the interest of the basis, and although the company will obtain the interest of the basis for these assurances.

From formula (13) we may, namely, deduce the value of R , which makes the fund equal to zero,

$$R = F 1,03^{-n} \frac{\frac{1}{K_i} - \frac{1}{K_i^a}}{\frac{1}{K_e} - \frac{1}{K_i^a}},$$

if $n = 25$, $r = 0,03$, $i = 0,035$, $e = 0,045$, we shall have, K_r being $= 0,9176$, $K^a = 0,9513$ and $K_e = 0,7776$, $R = 0,078 F$.

If the actual effective interest is 5 percent, and we assume the interest of the basis to be 4 %, while the minimum interest is 3 percent, we shall have, when employing the formula (12)

$$R = F 1,03^{-n} \frac{K_3 - K_3^a}{\frac{K_3}{K_5} - K_3^a},$$

when $n = 30$, K_3 is equal to 1,1966, $K_3^a = 1,1156$, $K_5 = 0,8463$, we shall then get $R = 0,1113 F$;

when $n = 25$, K_3 is equal to 1,1740, $K_3^a = 1,1012$, $K_5 = 0,8590$, we shall get $R = 0,1130 F$.

In both cases this reserve will be accumulated after a duration of about 5 years. If no fall in the interest has taken

place in this period, we shall thus be covered against a future fall of 2 % in the interest.

It is thus certain that there will be no danger by a fall of the interest, when the rate of interest is chosen $\frac{1}{2}$ to 1 percent below the market interest, because the reserve to be invested at the high interest may be accumulated in a few years, less than 10 years, to a degree which is sufficient to guarantee against a fall of about 2 percent.]

[*Note 2.* If the sum of the quotient of mortality and the frequency of discontinuance is equal to a , the capital supposed to be paid after n years, when the sum assured is F , may be put equal to $(1-w)^n F$. If w is calculated on the basis of the mortality of New Hafnia and of the frequencies of discontinuance of Old Hafnia, we shall have the following table for w :

Age at exit	Duration	5	10	15	20	25	30
56		3,6 %	3,4	3,2	3,2	4,0	
61		4,6 %	4,0	3,6	3,5	3,4	4,1
66		5,6 %	5,0	4,5	4,2	4,0	4,1

It will thus be seen that we shall get a higher limit for the fund, if w is put equal to 0,03; in that way the remaining stock of F if $n=25$ resp. 30 will be 0,4677 F resp. 0,4000 F , while by age at exit 56 and $n=25$ it is 0,3716 F , for $n=30$ and age at exit 61 0,3000 F and for $n=30$ and age at exit 66 0,2584 F , by this process the fund will thus be 25-50 percent too high, but if, at the same time, we calculate the fund supposing that the reserve is increasing evenly, we shall get a corresponding reduction so that the total deviation will be considerably smaller.]

[*Note 3.* If we do not know n , we can get a useful determination of it by putting the value of the sum assured F equal to $F(1+i-g)^n$, where i means the interest of the basis and g , which is varying with the age at exit and the duration, may be replaced by an average value equal to 0,005; by this we shall for low ages at exit and short durations only get a value of n which is slightly too high, while for high ages at exit and long durations we shall get too small values of n .

The deviation is increasing with the mortality, the ages at exit and the duration. If assumed that the average age at exit does not exceed 65 years, and the average duration not 30 years, n will, by the above formula, not be inferior to 25 years for the most common Danish tables. (For Old Hafnia and New Hafnia we shall get $n = 28$, for the »Forsikrertavlen» of the State Institute of Assurance one shall get $n = 26$). If the security fund is calculated by the above formula (9), it will be 0,01575 resp. 0,01527 for $n = 25$ resp. $n = 30$. An increase from 0 to 0,01527 corresponds with a fall in the interest from $3\frac{1}{2}$ percent to 2,989 percent, which is practically the same.]

B. In order more closely to examine the result, arrived at on page 7, it may be assumed that there is a constant rate of mortality; in this way we can find a general formula for the security fund, by means of which we can calculate the fund for the different values of the mortality, the interest, the frequency of discontinuance and the duration.

If the probability of death is $q_x = 1 - e^{-\delta}$, the rate of interest of the calculation basis $i = e^{\delta_2} - 1$, the frequency of discontinuance $1 - e^{\delta_3}$ and the minimum rate of interest $r = e^{\delta_4} - 1$, we shall find that the reserve for an assurance, of duration m years, will t years hence be

$$1 - \frac{\bar{a}_{n-t}}{\bar{a}_{n+m}} = \frac{e^{-(\delta_1+\delta_2)n} (e^{(\delta_1+\delta_2)t} - e^{-(\delta_1+\delta_2)m})}{1 - e^{-(\delta_1+\delta_2)(n+m)}}.$$

If we keep account of the discontinuances by death and by lapses we shall have, when multiplying by $e^{-(\delta_1+\delta_3)t}$,

$$R = \frac{e^{-(\delta_1+\delta_3)n}}{1 - e^{-(\delta_1+\delta_2)(n+m)}} (e^{(\delta_1+\delta_2-\delta_1-\delta_3)t} - e^{-(\delta_1+\delta_2)m-(\delta_1+\delta_3)t}).$$

The value of the loss on the market price of a unit of investment after t years will be

$$V = \left(\frac{\delta_2}{\delta_4} - 1 \right) (e^{-\delta_4 t} - e^{-\delta_4 n}).$$

The total loss on the market price amounts then to

$$\begin{aligned}
 \int_0^n V \frac{dR}{dt} dt &= \frac{e^{-(\delta_1 + \delta_2)n} \left(\frac{\delta_2}{\delta_4} - 1 \right)}{1 - e^{-(\delta_1 + \delta_2)(n+m)}} \int_0^n (\delta_2 - \delta_3) e^{(\delta_2 - \delta_3)t} + \\
 &\quad + e^{-(\delta_1 + \delta_2)m} (\delta_1 + \delta_3) e^{-(\delta_1 + \delta_3)t} (e^{-\delta_1 t} - e^{-\delta_1 m}) dt \\
 &= \left(\frac{\delta_2 - \delta_3}{\delta_3 + \delta_4 - \delta_2} + \frac{\delta_1 + \delta_3}{\delta_1 + \delta_3 + \delta_4} e^{-(\delta_1 + \delta_2)m} - \right. \\
 (13) \quad &\quad - \frac{\delta_4}{\delta_3 + \delta_4 - \delta_2} e^{-(\delta_3 + \delta_4 - \delta_2)n} + \\
 &\quad + \frac{\delta_4}{\delta_1 + \delta_3 + \delta_4} e^{-(\delta_1 + \delta_2)m} e^{-(\delta_1 + \delta_3 + \delta_4)n} + \\
 &\quad \left. + (1 - e^{-(\delta_1 + \delta_2)m}) e^{-\delta_1 n} \right) \frac{e^{-(\delta_1 + \delta_2)n} \left(\frac{\delta_2}{\delta_4} - 1 \right)}{1 - e^{-(\delta_1 + \delta_2)(n+m)}}.
 \end{aligned}$$

If $m = 0$, we shall have:

$$\begin{aligned}
 &\frac{\delta_2 - \delta_4}{\delta_4} \cdot \frac{e^{-(\delta_1 + \delta_2)n}}{1 - e^{-(\delta_1 + \delta_2)n}} \left(\frac{\delta_2 - \delta_3}{\delta_4 + \delta_4 - \delta_2} + \right. \\
 &\quad \left. + \frac{\delta_1 + \delta_3}{\delta_1 + \delta_3 + \delta_4} - \frac{\delta_4}{\delta_3 + \delta_4 - \delta_2} e^{-(\delta_3 + \delta_4 - \delta_2)n} + \frac{\delta_4}{\delta_1 + \delta_3 + \delta_4} e^{-(\delta_1 + \delta_3 + \delta_4)n} \right).
 \end{aligned}$$

Simple figures are chosen for

$$\delta = \begin{cases} 0,005 \\ 0,01 \\ 0,015 \\ 0,20 \end{cases}, \quad \delta_2 = 0,035, \quad \delta_3 = 0,02, \quad \delta_4 = 0,03.$$

For comparison the value is calculated after formula 8, which is modified to

$$\alpha \left(\frac{\delta_2 - \delta_4}{n \delta_4^2} - \frac{1 + n \delta_4}{n \delta_4^2} (\delta_2 - \delta_4) e^{-\delta_1 n} \right) \text{ cfr. (5)}.$$

By this process we get the following values:

Table containing the values of the fund according to formula 14, when assumed that the interest falls from 0,035 to 0,03.

		Frequency of discontinuance = 0,02			
Probability of death = 0,05		0,010	0,015	0,020	$100 \frac{A-B}{A}$
Duration		<i>A</i>	<i>B</i>		
30	0,02039	0,01792	0,01574	0,01380	12,1
25	02111	01899	01705	01530	10,3
20	02101	01930	01772	01626	8,2
15	01956	01838	01725	01617	6,2
10	01613	01546	01512	01422	2,2
5	01008	00984	00984	00948	0

Frequency of discont = 0,015		The reserve is increasing evenly	
0,010	$100 \frac{A-C}{A}$	by $\frac{1}{n \cdot 1,03^n}$	by $\frac{1}{n \cdot 1,035^n}$
<i>C</i>			
0,01972	9,1	0,01739	0,01500
02053	7,5	01848	01629
02050	5,8	01880	01704
01928	4,7	01786	01660
01613	4,1	01527	01453
00984	0	00949	00926

It will be seen that the fund increases, when the frequency of discontinuance, the mortality and the interest are increasing, but that the variations are inconsiderable (see note 4).

It will further be seen that the fund does not increase with n , but has its highest limit between $n = 20$ and $n = 25$.

If therefore we choose as *average value* 0,0175 we shall have a *good approximation* except for the very short durations which are of no importance to this investigation.

Finally we see that the formula indicated on page 8, which assumes an even increase of the reserve, will give a good approximation if we put $\alpha = 1,03^{-1}$.

[Note 4. Under $100 \frac{A-B}{A}$, resp. $100 \frac{A-C}{A}$ the procentual increase of the fund for an increase of 0,005 *id* resp. δ , has been given. When children's endowment assurances are excepted, the average mortality for assurances with duration 30 years is not much under 1 percent and the frequency of

discontinuance not so low as 1,5 percent. On the other hand, the average mortality for assurances with duration 30 years does not exceed 1,5 per cent, while, as it will be seen from the table, assurances with duration under 15 years permit average mortality right up to 2 per cent before the limit of 10 per cent is surpassed.]

Let us take an assurance m years old with a remaining duration $n - m$ years and a reserve q_m per unit of sum, we shall then, by means of the formula (13), get the following values of the fund S_m , when $\delta_1 = 0,01$, $\delta_2 = 0,035$, $\delta_3 = 0,02$ and $\delta_4 = 0,03$:

$n = 30$						
$m =$	0	5	10	15	20	25
$S_m =$	0,01797	0,01531	0,01209	0,00848	0,00473	0,00149
$q_m =$		0,0884	0,1986	0,3375	0,5109	0,7280
$(S_0 - S_m) : q_m =$		0,0300	0,0296	0,0281	0,0259	0,0226
$S_m : (1 - q_m)^2 =$	0,0180	0,0184	0,0188	0,0193	0,0198	0,0201

$n = 25$					
$m =$	0	5	10	15	20
$S_m =$	0,01899	0,01489	0,01035	0,00576	0,00177
$q_m =$		0,1212	0,2732	0,4635	0,7016
$(S_0 - S_m) : q_m =$		0,0338	0,0316	0,0286	0,0246
$S_m : (1 - q_m)^2 =$	0,0190	0,0193	0,0196	0,0200	0,0199

$n = 20$				
$m =$	0	5	10	15
$S_m =$	0,01928	0,01329	0,00729	0,00226
$q_m =$		0,1730	0,3895	0,6705
$(S_0 - S_m) : q_m =$		0,0346	0,0307	0,0254
$S_m : (1 - q_m)^2 =$	0,0193	0,0194	0,0196	0,0208

The figures in the last line but one being about 1,5 δ_0 , the fund may be written $S_m = S_0(1 - 1,5 q_m)$. From the figures in the last line it will be seen that the fund may also be written $S_m = S_0(1 - q_m)^2$. In both cases δ_0 is about 0,02 by a fall in the interest of ca. $\frac{1}{2}$ per cent.

[Note 5. If $e^{-(\delta_1 + \delta_2)}$ during the first n years is equal to v_1 , during the next n_2 years equal to v_2 and during the next

n_3 years equal to v_3 , and if the security fund by a duration n and these 3 factors of discount is equal to $S_1^{(n)}$, $S_2^{(n)}$ and $S_3^{(n)}$, we shall have:

$$S = \frac{v_1^{n_1} v_2^{n_2} v_3^{n_3}}{1 - v_1^{n_1} v_2^{n_2} v_3^{n_3}} \left(\frac{1 - v_1^{n_1}}{v_1^{n_1}} S_1^{(n_1)} - \frac{1 - v_2^{n_2}}{v_2^{n_2}} S_2^{(n_2)} + \right. \\ \left. + \frac{1 - v_3^{n_3}}{v_3^{n_3}} S_3^{(n_3)} - \frac{1 - v_1^{n_1+n_2}}{v_1^{n_1+n_2}} S_2^{(n_1+n_2)} + \frac{1 - v_2^{n_2+n_3}}{v_2^{n_2+n_3}} S_3^{(n_2+n_3)} \right).$$

If $n_1=n_2=n_3=10$ and $v_1=e^{-(\delta_2+0,005)}$, $v_2=e^{-(\delta_2+0,01)}$, $v_3=e^{-(\delta_2+0,015)}$
we shall have: $S=0,01765$.

If $n_1=n_2=n_3=10$ and $v_1=e^{-(\delta_2+0,005)}$, $v_2=e^{-(\delta_2+0,01)}$, $v_3=e^{-(\delta_2+0,02)}$
we shall have: $S=0,01726$.

Thus the deviation in the table for $\delta_1=0,01$, in the table page 13 is inconsiderable.]

According to the results found on page 7, 13 and 14 the security fund can be determined at 0,035 ($F-1,5 R$) or 0,035 $\frac{(F-R)^2}{F}$ corresponding to a decrease in interest of 1%. This quantity (3,5 per Cent of the insurance sum minus 1,5 times the value of the insurance or rather 3,5 per Cent of the risk multiplied by the quotient between the risk and the amount of the insurance, since the risk is equivalent to the amount of the insurance minus the value of the insurance) expresses the loss by a decrease of the interest of 1% and may be used as the measure (module) for the monetary investment in funds.

It is possible to make different rules for these investments. The most natural is to procure such bonds as expire simultaneously with the insurances. In general one thereby obtains obligations having a shorter duration than usual bonds. It will therefore be sufficient to invest a part of the capital in ordinary bonds of long duration in such a manner that the total exchange profit in case of decreasing interest will be the same as if all the capital had been invested in bonds with the accurate shorter duration. We then obtain the following rule:

I. Such a part of the reserve is invested in funds that a decrease of 1 % in interest will result in an exchange profit equal to the module.

In this case the same result will be obtained as if the bonds expired at the same time as the insurance.

If the market rate of interest is exceptionally high and the company is developing it may be suitable to make sure of the high interest for a longer period than necessary with regard to the actual insurances by purchasing bonds of longer duration than is allowed for by rule I. It may therefore be more practical to replace it with another which allows of greater freedom in buying investments and yet provides a guarantee against a loss in exchange which will only be efficient in case of the business of a concern becoming reduced. Such a rule may be investments as follows:

II. The difference between the book value of the scrip and its real value must not be greater than α times that of the module. α can be equal to the difference between the effective rate of interest and the rate of interest which serves as the basis for the company's calculations.

When the insurances expire, the module becomes reduced to zero and the investments will therefore be noted at their real value. In other cases they cannot be noted at a higher rate of exchange than that which gives the basis rate of interest.

Should rule II show that the booked value of the investments is either too high or too low the difference must be balanced by future investments in loans of long or short duration or by making reinvestments or by the realisation of loans.

Should a company wish to guard itself against a decrease in the rate of interest to a fixed minimum rate, then besides rule II the following rule must be adopted:

II a. A guarantee fund should be established equal to the module multiplied by the difference between the basis rate of interest and the minimum rate.

This minimum rate can be put in relation to the State Bank rate, savings bank rate or the lowest rate allowed by the Insurance Inspection Board as a basis for the calculations.

The fund in question can be reduced by the difference between the investments, calculated on the rate of the basis

and its booked value just as the expression $R\left(\frac{1}{K_e}-1\right)K_r$ (see page 3) can be substituted by $\frac{R}{K_e}-R$ which is less.

Rules II and IIa provide for a decrease in the rate of exchange and in the interest only by making monetary investments consequently without any real restrictions on the company. If a company has adopted rule IIa and in consequence of a decrease in interest is unable to follow this rule when investing capital, and therefore finds it necessary to adopt a lower rate of interest in calculating premiums, the company can nevertheless obtain the basis rate for the entire insurance term for the insurance stock for which the rule has been observed; but naturally on the condition that the market rate of interest does not sink below the minimum rate.

In establishing rules similar to rules II and IIa it is possible to regulate income yielded by interest. In case the company desires to obtain a constant normal rate of interest higher than the basis rate during the entire insurance time rule IIa will then have the following appearance:

III a. A fund is established equal to the module multiplied by the difference between the normal rate of interest and the minimum rate.

III. In rule (II) α can be put equal to the difference between the effective rate and the normal rate. In this case the investments may be booked at such a rate that they yield a normal rate. Consequently it is possible to quote new investments at a higher rate than that at which they are purchased if the effective rate is higher than the normal rate, but they ought to be booked below the purchase rate if the effective rate is lower. Should the company wish to avoid the latter, then the margin for the fluctuations of the exchange rate provided for by rule III must be extended by the difference between the purchase price and the value of the investments corresponding to the normal rate of interest if the later value is lower. In this way the investments will be booked at the purchase rate but not lower than that they yield at the most the normal interest. It is clear that in continually using in this way a capitalised super rate as

income there will be no exchange profit in case of a possible decrease in the rate of interest by which to reduce the guarantee fund. The latter must be booked at its full value.

If a company has no wish to secure itself against a loss in the interest by establishing such rules, but only seeks a security against a loss on the market price by arranging that the papers shall be booked at the market price at the purchase without any margin for a deviation from the real value, a company may, even if it is retrograding, continue the investment of all its capitals in papers with a very long duration, even in papers which cannot be amortized entirely or partially and which are yielding a very low nominal interest. When the papers must be realized, those which will give the smallest losses on the market price must, moreover, be sold first, and at last it can then be necessary to realize papers at a time, where the interest is higher than at the purchase.

After the present period with a high interest there will probably come a period during which the interest is low, and in this period papers yielding a low nominal interest must be purchased at a high price. The realization of these papers may very well take place in the next period with a high interest and will then involve considerable losses, whose covering cannot be secured by means of the surplus on the loadings of the premiums, namely if there are expenses of new business which have not been amortized. In order to obtain a complete security there must, in this case, also a certain relation exist between the real and the booked value. A rule which will only be effective if the company has too many or too slowly expiring bonds, is rule II which may have the following form:

The difference between the real and the effective value of funds or similar papers must not exceed the module multiplied by the difference between the effective interest and the interest of the basis. (It may here be assumed that the effective interest is equal to the discount of the National Bank, eventually with a constant deduction. It is, namely, not necessary that the determination is exact, the essential point being that the module shall be equal to zero for an old stock.) If the difference in the value at the market

price is greater than permitted by this, the deficiency shall be regulated by means of future investments of capital.

[*Not 6.* The exchange profit or fluctuation mentioned in rules I, II and III can approximately be derived from the market price of all investments noted on the exchange, but it is of importance that companies keep accurate accounts of all loans, that is to say that one is able at any time to decide the average duration and market value. This may be done the same way as a policy valuation, but much simpler by determining once and for all for each loan a constant for the exchange value and another to decide the balance of the debt. If the nominal rate of interest is i , the annual yield y and the remainder of the time until the expiration of the bond n , then the balance of the debt will be

$$G = y \frac{1 - (1 + i)^n}{i},$$

while the exchange value for the interest $i = 0,01$ will be

$$K = y \frac{1 - (0,99 + i)^n}{i - 0,01}.$$

If the expiration year of the bond is O and the financial year is A we get

$$n = O - A, \quad G = \frac{1}{i} (y - y(1 + i)^{-O} (1 + i)^A)$$

and

$$K = \frac{1}{i - 0,01} (y - y(0,99 + i)^{-O} (0,99 + i)^A).$$

If $T_1 = y(1 + i)^{-O}$ and $T_2 = y(0,99 + i)^{-O}$ then the total balance of the debt will be

$$\Sigma G = \frac{\Sigma y - (1 + i)^A \Sigma T_1}{i}$$

and the total exchange value

$$\Sigma K = \frac{\Sigma y - (0,99 + i)^A \Sigma T_2}{i - 0,01}.$$

The actual value mentioned in II can when the effective rate of interest is i be fixed at $\Sigma G + (e - i)(\Sigma G - \Sigma K)$. A more accurate determination may be arrived at by interpolating to e in a table in which to the arguments i or $(i - 0,01)$ correspond the functional values

$$\frac{\log \Sigma T_1 - \log \Sigma y}{i} \text{ or } \log \frac{\Sigma T_2 - \log \Sigma y}{i - 0,01}$$

and thereby obtain

$$\frac{\Sigma T e - \log \Sigma y}{e}$$

and the corresponding market value. This determination is exact when the loans with allowance for the duration are distributed according to the typical curve of errors.]