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THE USE OF TANGENTIAL COORDINATES.

THERE are two reasons why the use of tangential coordinates should be introduced at an earlier stage than is customary into the teaching of analytical geometry and placed on a somewhat different footing. The educational advantages of encouraging the student to follow out a course of reasoning with the help of only a few symbols instead of grinding out hard examples by long meaningless algebra should not require much pressing, although they do not seem to be sufficiently recognised in this country; and on the other hand many theories receive their most appropriate expression in tangentials chiefly owing to the fact that the absolute has a less specialised form in tangential than in point coordinates, being (in two dimensions) the product of two distinct factors in the former and a perfect square in the latter.

The student must, of course, begin with point coordinates and will at first regard an equation in line coordinates as the condition for an envelope: but gradually the new idea of regarding a line as a fundamental geometrical entity must be introduced, and the ultimate goal is the ability to *read* a tangential equation without any preliminary translation into point coordinates.

To take an example from the elementary part of the subject, it is an easy exercise to express the general tangential equation of a circle in the form

$$k(l^2 + m^2) + (Gl + Fm + C)^2 = 0,$$

from which the whole of the interesting projective theory can clearly be deduced. Again the condition for a parabola, $C=0$, is at the same time simpler and of more interesting interpretation than the more familiar condition $ab = h^2$.

Turning now to more advanced subjects, the connection between tangential equations, complex variables, and foci is instructive. Each of the equations

$$\begin{aligned} la + mb + n &= 0 \\ a + ib - (x + iy) &= 0 \end{aligned}$$

is sufficient to define the point (a, b) . The latter is the equation of the line joining (a, b) to one of the circular points at infinity. If this line is a tangent to a real curve, the point (a, b) is a focus. We therefore can obtain the foci of any curve by substituting $l=1$, $m=i$, $-n=z=x+iy$ in the tangential equation and solving the resulting equation for the complex variable z . Conversely, for example, if

$$f(z) \equiv (z - z_1)(z - z_2)(z - z_3), \quad z_s = a_s + ib_s,$$

then the roots of $f'(z) = 0$

are the foci of the conic

$$u_2u_3 + u_3u_1 + u_1u_2 = 0, \quad u_s = la_s + mb_s + n,$$

which touches the sides of the triangle $z_1z_2z_3$ at their middle points. This theorem, of which other proofs have been given, is evidently capable of generalisation to polynomials of any degree.

An interesting theorem and one immediately proved from the tangential equation, though offering considerable difficulties when treated otherwise, is concerned with the 'centre' of an algebraic curve.* This point is best defined as the polar point of the line at infinity, and the theorem states that it is the centroid of the points of contact of parallel tangents. Let the tangential equation of a curve of class μ be

$$\phi(l, m, n) \equiv n^\mu + (al + bm)n^{\mu-1} + \dots = 0.$$

* See *Quarterly Journal*, vol. xxiv, 1890, Taylor, p. 55, and Baker, p. 338.

The polar curve of class ν of the line (l', m', n') is

$$\left(l' \frac{\partial}{\partial l} + m' \frac{\partial}{\partial m} + n' \frac{\partial}{\partial n} \right)^{\mu-\nu} \phi(l, m, n) = 0.$$

and hence the polar point of the line at infinity has for its equation

$$al + bm + \mu n = 0$$

representing the point $(a/\mu, b/\mu)$. For given values of l and m the equation $\phi=0$ gives μ values of n whose sum is $-(al+bm)$; hence the line of mean position among μ parallel tangents is

$$lx + my - \mu^{-1}(al + bm) = 0,$$

passing through the centre $(a/\mu, b/\mu)$ which is independent of l/m . From the consideration that two near tangents intersect in a point which becomes ultimately the point of contact the theorem follows as stated.

Great clearness is introduced into the theory of averages in connection with areas and volumes by the exclusive use of tangential coordinates. In two dimensions it is best to lead up to the theory of moments by beginning with the *null conic*, that is, the envelope of lines about which the area or lamina has no second moment. Leaving out the density for simplicity—the modifications due to variable density are easily introduced—the equation of the null conic is

$$\iint (lx + my + n)^2 dx dy = 0.$$

It is often desirable to omit the signs of integration, and to substitute for them the notation of averages. Thus, by a proper choice of axes the above may be written

$$l^2 \bar{x^2} + m^2 \bar{y^2} + n^2 = 0.$$

Then the *ellipse of gyration* is the confocal

$$l^2 \bar{y^2} + m^2 \bar{x^2} - n^2 = 0,$$

and the *ellipse of inertia* is the conic ‘conjugate’ to the null conic

$$l^2 \bar{x^2} + m^2 \bar{y^2} - n^2 = 0.$$

One last example will suffice to illustrate the advantage of attacking certain problems by tangentials from the beginning. The *surface of floatation* is the envelope of a plane cutting off a constant volume from a vessel. We want its approximate equation, correct as far as the curvatures. In this, as in other approximate work, it is convenient to take as the standard equation of a plane

$$z + n = lx + my,$$

for then, when x, y, l, m are small quantities of the first order, z and n can be regarded as of the second order.

Taking a vessel of any shape, the condition of cutting off a constant volume, which is itself the tangential equation of the envelope, is, as far as terms of the second order,

$$\iint z dx dy + \int \frac{1}{2} z \delta v ds = 0,$$

where z has the above value: the integrals are taken over and round the section of floatation, and at the edge, $z/\delta v$ is the tangent of the angle between the side of the vessel and the section. Let δz be any small constant and let $\delta v_1/\delta z = \delta v/z$; then

$$\int z \delta v ds = (\delta z)^{-1} \int z^2 \delta v_1 ds = \frac{\delta}{\delta z} \int \int z^2 dx dy$$

where the symbol δ refers to a change to a parallel section at distance δz . Hence the equation of the surface is

$$\iint (lx + my - n) dx dy + \frac{1}{2} \frac{\delta}{\delta z} \iint (lx + my - n)^2 dx dy = 0,$$

and when the centroid of the section is the origin, this takes the elegant form

$$2n + \frac{\delta}{\delta V} \iint (lx + my)^2 dx dy = 0,$$

V being the volume immersed. This form gives the curvature immediately, and renders clear the connection with the momental ellipse of the section and with the surface of buoyancy.*

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NOTE ON THE TREATMENT OF CONIC SECTIONS AND CONICOIDS BY PURE GEOMETRY.

The object of the following note is to derive Pascal's theorem and the Anharmonic and Harmonic properties of Conic Sections directly from the focus and directrix definition by the early introduction of an extended form of Carnot's theorem, viz. "the product of the ratios in which the sides of a closed polygon, taken in order, are cut by a conic = +1." When the above have been demonstrated the simpler properties are known to be easily deducible. Subsequently the treatment of Conicoids by Pure Geometry is introduced by means of the same theorem.

The sequence of propositions, commencing after the definition of a Conic, is as follows (proofs where necessary are very briefly given):

(1) Construct points on the curve, and observe that the curve can nowhere cease abruptly, so that any branch it may have must either be closed or extend to infinity at each end.

(2) If a conic meets a line at α and β , and P, Q are two other points on that line, the product of the ratios in which PQ is cut at α and β

$$= \frac{SP^2 - e^2 PM^2}{SQ^2 - e^2 QN^2},$$

where PM, QN are perpendicular to the directrix.

Let PQ meet the directrix at Z .

Draw $P\lambda, Q\mu$ parallel to $S\alpha$ and $P\lambda', Q\mu'$ parallel to $S\beta$ to meet SZ .

Draw aR perpendicular to directrix.

Then $\frac{P\lambda}{S\alpha} = \frac{ZP}{Z\alpha} = \frac{PM}{\alpha R}; \therefore P\lambda = ePM.$

So $P\lambda' = ePM$, and $Q\mu = eQN = Q\mu'.$

Also $\frac{Pa}{\alpha Q} \cdot \frac{P\beta}{\beta Q} = \frac{S\lambda}{S\mu} \cdot \frac{S\lambda'}{S\mu'} = \frac{SP^2 - e^2 PM^2}{SQ^2 - e^2 QN^2},$

since λ and λ' lie on a circle whose centre is P and radius ePM .

* Hudson, *Messenger of Mathematics*, vol. xxxiii, p. 50.