

XXV.—*On the Possibility of combining two or more Probabilities of the same Event, so as to form one Definite Probability.* By the Right Rev. Bishop TERROT.

(Read 17th March 1856.)

(1.) The inquiry which, with its results, I propose to lay before the Society, was suggested by the following passage in the very popular Treatise on Logic by Dr WHATELY, now Archbishop of Dublin.

“ As in the case of two probable premises, the conclusion is not established except upon the supposition of their being *both* true, so in the case of two (and the like holds good with any number) distinct and independent indications of the truth of some proposition, unless *both* of them *fail*, the proposition must be true : we therefore multiply together the fractions indicating the probability of the failure of each—the chances against it—and the result being the total chances against the establishment of the conclusion by these arguments, this fraction being deducted from unity, the remainder gives the probability *for* it.

“ *E. g.* A certain book is conjectured to be by such and such an author, partly, 1st, from its resemblance in style to his known works ; partly, 2d, from its being attributed to him by some one likely to be pretty well informed. Let the probability of the conclusion, as deduced from one of these arguments by itself, be supposed $\frac{2}{5}$, and in the other case $\frac{3}{7}$; then the *opposite* probabilities will be respectively $\frac{3}{5}$ and $\frac{4}{7}$, which multiplied together give $\frac{12}{35}$ as the probability against the conclusion ; *i. e.*, the chance that the work may *not* be his, notwithstanding the reasons for believing that it is ; and, consequently, the probability in *favour* of the conclusion will be $\frac{23}{35}$, or nearly $\frac{2}{3}$ ” (WHATELY’S *Logic*, 8th Ed., p. 211.)

(2.) Now, this reasoning appears to me erroneous, because it can be so applied as to bring out two inconsistent conclusions. It must be observed, that there is no such generic difference between the chances *for* and *against* the truth of a proposition, as can require or justify any difference in the laws and methods applied to them. A negative can always be turned into an affirmative by a change of verbal expression, without any change of meaning. Thus the chance of not hitting a mark is the same as the chance of missing it. The chance of a life not falling before sixty, is the chance of its continuance up to sixty. The chance that A was not the author of the book, is the chance that some one else was the author.

Let us then take as the proposition whose probability is to be found, the negative—he did not write it—the partial probabilities *for* which are by the data $\frac{3}{5}$ and $\frac{4}{7}$.

The opposite probabilities are now $\frac{2}{5}$ and $\frac{3}{7}$, and their product is $\frac{6}{35}$, the probability *against* the conclusion whose probability we are now seeking. Consequently, $1 - \frac{6}{35} = \frac{29}{35}$ is the probability *for* our conclusion, namely, that he did not write the book. But by the former calculation, the probability of the same conclusion was found to be $\frac{12}{35}$: and, as these incompatible results follow from the same principle and method, the principle and method must be erroneous.

(3.) The only mathematical attempt at the solution of this problem which I have met with, is at section 15 of the Article Probability, in the *Encyclopædia Metropolitana*. It is given there as follows:—

“It is an even chance that A is B, and the same that B is C; and, therefore, 1 to 3 on these grounds alone, that A is C. But other considerations of themselves give an even chance that A is C. What is the resulting degree of evidence (or the probability) that A is C?” There is a previous solution which I omit, and then the passage proceeds as follows:—“Let us now treat the preceding question as having two contingencies, the compound argument 1 to 3 *for*, and the independent evidence an even chance. We have, therefore, four possible cases.

			PROB. A is C.
“Argument and Evidence both true,	.	.	$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$
Argument false, Evidence true,	.	.	$\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$
Argument true, Evidence false,	.	.	$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$
Argument and Evidence both false,	.	.	$= 0$

“The sum of these is $\frac{5}{8}$ as before (for the resulting probability that A is C).

The above generalized is as follows:—Let a and $(1-a)$ be the probabilities for and against the argument (the conclusion from the argument); and ϵ and $(1-\epsilon)$ be the probabilities from any other source. Then the chance that both are wrong is $(1-a) \cdot (1-\epsilon)$, and of the contradictory, namely, that (A is C) follows from the one or the other, is $1 - (1-a) \cdot (1-\epsilon) = a + \epsilon - a\epsilon$.”

This is the formula adopted by WHATELY; and it is open to the same objection, namely, that by applying it we can arrive at two contradictory conclusions. But, further than this, what is the meaning of *Argument true, Evidence true*? The argument and the evidence are here treated as two independent events having respectively the probabilities of $\frac{1}{4}$ and $\frac{1}{2}$; and their coincidence is represented by $\frac{1}{8}$. But nothing corresponding to this goes forward in the mind. The argument merely affords the information, that for every reason for believing that A is C, there are three equivalent reasons for believing that A is not C. This information we

are supposed to believe absolutely; there is no question as to the probability of its truth, or the possibility of its falsehood. The only matter in question is whether A is C, or is not C.

The falsity of the expression $\alpha + \epsilon - \alpha \epsilon$ will be evident, if we give to α and ϵ the values $\frac{2}{7}$ and $\frac{3}{7}$. Then $\alpha + \epsilon - \alpha \epsilon = \frac{2}{7} + \frac{3}{7} - \frac{6}{49} = \frac{1}{2} + \frac{10}{49}$. That is to say, while each of the independent probabilities is less than $\frac{1}{2}$, and, therefore, in favour of the negative, their compound force is much above $\frac{1}{2}$; and, therefore, in favour of the affirmative. If then we found from internal evidence and external evidence severally, that the chances were against the truth of the proposition A is C, we ought to conclude from their united force, that the chances are in favour of the proposition. But the human mind is incapable of coming to such a conclusion.

It may be well to notice in passing, that the problem under consideration is altogether different from that of finding the compound force of two identical assertions made by two witnesses, whose veracity, that is, the probability of their speaking truth, is expressed by α and ϵ . In that problem, we possess among the data the fact, that each witness *makes the same assertion*. But in the problem we have been considering, there is no such *assertion*. Neither the argument nor the evidence assert or deny that A is C. What they give as data, is merely that the reasons for believing that A is C, are in a given ratio to those for believing that A is not C. And as the data of the two problems are of totally different character, the methods to be applied must of course be different. I have mentioned this, because some good mathematicians whom I have consulted, were at first disposed to consider $\frac{\alpha \epsilon}{\alpha \epsilon + 1 - \alpha - \epsilon}$ as the proper expression for the conjoined force of the argument and evidence.

(4.) Let us now consider the Problem under the following form. *A, whose veracity is undoubted, states that, from his knowledge of the facts of the case, the probability of the event E is $\frac{p}{q}$. B, under the same conditions, states, that it is $\frac{r}{s}$. Supposing the facts known by each to be altogether distinct, what is the proper measure of the expectation formed in a third mind by these two statements?*

(5.) Before attempting to show how a solution of the problem ought to be sought, it may be well to observe, that the mind cannot admit two probabilities of the same event as co-existent probabilities. Thus, if A tells me that the probability of rain to-morrow is $\frac{2}{5}$, and B that it is $\frac{4}{7}$, I cannot admit both of these as probabilities; for that would be equivalent to believing on the authority of A, that it is *less* likely to rain than not, and at the same time to believe on the authority of B, that it is *more* likely to rain than not.

What really takes place is this. The two fractions are received as indications

of the effects reasonably produced upon the minds of the informants, by the knowledge of certain facts which they have not communicated to us. The fractions which they give are admitted as true exponents of the results of their respective partial knowledge, no doubt resting either upon their veracity, or upon the accuracy of their inferences. We admit, that each states the probability as he ought, under *his* circumstances: and the question is, how ought we to state it under *our* circumstances, knowing as we do something *more*, and also something *less* than either of our informants.

(6.) In attempting to answer this question, I shall have recourse to the ordinary illustration of an urn and balls. Let us suppose that A has seen p white and $q-p$ black balls introduced into an urn, which he believes to have been previously empty. He properly infers that the probability of drawing a white is $\frac{p}{q}$. B, under the same circumstances, has seen r white, and $s-r$ black balls introduced, and infers that the probability of drawing a white is $\frac{r}{s}$. If they communicate to each other only their *inferences*, there is an apparent contradiction, and no combination or agreement can take place. But if they communicate the *facts* from which the inferences were deduced, then each knows that the urn contains $p+r$ white, and $q+s-p-r$ black balls, and agree in making the probability of drawing a white $\frac{p+r}{q+s}$. If the number of balls whose introduction has been seen by the two observers be equal, then $\frac{p+r}{q+s} = \frac{p+r}{2q} = \frac{1}{2} \left(\frac{p}{q} + \frac{r}{q} \right) = \frac{1}{2}$, the sum of the several probabilities.

It may be observed that $\frac{p+r}{q+s}$, as the expression of the combined probabilities $\frac{p}{q}$ and $\frac{r}{s}$, is not exposed to the objection of admitting contradictory results, for, if we take the negative as the conclusion whose probability is to be found, then A gives for the probability of the conclusion $1 - \frac{p}{q} = \frac{q-p}{q}$, while B gives $1 - \frac{r}{s} = \frac{s-r}{s}$, therefore the combined probability against the event is $\frac{q-p+s-r}{q+s}$. But the combined probability for the event was $\frac{p+r}{q+s}$, and $\frac{p+r}{q+s} + \frac{q-p+s-r}{q+s} = \frac{q+s}{q+s} = 1$, as it ought to be.

(7.) But what we have to consider, is the impression made upon the mind of a third person, who is informed by A that, from his observation, the probability of drawing a white is $\frac{p}{q}$, and by B that, from his observation, it is $\frac{r}{s}$, and to whom no farther information is given, except that the observations were totally distinct. Now, as these data give only the ratio of white to black balls at each introduction, there may have been, in the first, p white and $q-p$ black, or there

may have been np white and $nq - np$ black, where n is any whole number from one to infinity. In like manner, the second may have consisted of nr white and $ns - nr$ black, where n is any number from one to infinity. Any one assumed state of the first introduction may have co-existed with any assumed state of the second; and thus assuming that the first contained p white and $q - p$ black, we have the infinite series of probabilities,

$$\frac{p+r}{q+s-p-r}, \frac{p+2r}{q+2s-p-2r} \cdot \cdot \cdot \cdot \frac{p+nr}{q+ns-p-nr}.$$

Again, assuming that the first contained $2p$ white, and $2q - 2p$ black, we have

$$\frac{2p+r}{2q+s-2p-r}, \frac{2p+2r}{2q+2s-2p-2r} \cdot \cdot \cdot \cdot \frac{2p+nr}{2q+ns-2p-nr},$$

and so on *ad infinitum*.

This infinite series of infinite series I cannot sum. If they can be summed, then their sum divided by the infinite of the second order n^2 , is the probability required.

In no case, except when $\frac{p}{q} = \frac{r}{s}$, so far as I see, can the sum of their sums, or the whole probability, be determinately expressed. When $\frac{p}{q} = \frac{r}{s}$, the fractions being in their lowest terms, $p=r$ and $q=s$. The two pieces of information are then identical; the same information is given by both observers; and the information, unaffected by the repetition, is absolutely received by the third party: and this is the result, if, in the foregoing series, we substitute p for r and q for s .

(8.) If we revert to the expression (3) given in the *Encyclopædia Metropolitana*, where the separate probabilities are a and ϵ , and their conjoint force is stated to be $a + \epsilon - a\epsilon$, it would follow that the effect produced by two observers making the same statement as to the probability of an event should be twice the asserted probability *minus* its square. Now, in the case of a repetition of the same probability by two observers, it must, I think, be allowed that my result is conformable to that of which we are all conscious. If, for example, the Northampton and the Carlisle Tables both give $\frac{1}{2}$ as the probability that a man of thirty will live to the age of fifty, and are both implicitly believed, we believe that there is an even chance of his living to fifty, and not, as would follow from the expression given in the *Encyclopædia*, that the chances are three to one in his favour.

(9.) It perhaps deserves to be noticed, that when a second series of observations or experiments is added to one previously admitted, the probability is not increased by the mere preponderance of favourable over unfavourable cases in the second series. To increase the probability, the ratio of favourable to unfavourable cases must be greater in the second series than in the first. For the first received probability is $\frac{p}{q}$, and the composite is $\frac{p+r}{q+s}$. (6.)

Now $\frac{p+r}{q+s} > \frac{p}{q}$, when $r q > s p$, or $r q - r p > s p - r p$,

$$\text{or } r(q-p) > p \cdot (s-r), \text{ or } \frac{r}{s-r} > \frac{p}{q-p}.$$

When the ratios only are given, any conceivable case of the grounds upon which the probabilities are given may be represented by $m p$, $m q$, $n r$, and $n s$. Hence the original probability is $\frac{m p}{m q}$, the composite is $\frac{m p + n r}{m q + n s}$, and this is greater than $\frac{m p}{m q}$, when

$$m^2 p q + m n q r > m^2 p q + m n p s$$

or $r q > p s$, as before.

(10.) Valid objections may, I think, be made to the last paragraph of the section in the Encyclopædia already referred to. As this is not long, I quote it entire. “The following theorem will be readily admitted on its own evidence. *If any assertion appear neither likely nor unlikely in itself, then any logical argument in its favour, however weak the premises, makes it in some degree more likely than not.* In the manner in which writers on Logic apply the calculus of probabilities, this is never the consequence of their suppositions. For what we have called a is their resulting probability of the argument. Suppose, for instance, a writer on logic presumed that the argument from analogy gave $\frac{3}{10}$ to the probability that there is vegetation in the planets, which must be regarded as a thing neither likely nor unlikely in regard of evidence from any other source, he would take $\frac{3}{10}$ to be the probability of this result, that is, *less after an argument in its favour than it was before.* We substitute $\frac{1}{2} + \frac{1}{2} \cdot \frac{3}{10} = \frac{13}{20}$.”

This numerical equation is the value of the expression $(a + \epsilon - a \epsilon)$, when $a = \frac{1}{2}$, $\epsilon = \frac{3}{10}$. I have already shown that this expression does not truly represent the composite force of the two probabilities a and ϵ . But farther than this, the argument from analogy, giving $\frac{3}{10}$ as the probability of the affirmative, is an argument, not *in favour of*, but *against* the proposition that there is vegetation in the planets. It implies that for every three reasons for believing that there *is*, there are seven for believing that there *is not*; and, consequently, the effect of the argument ought to be to diminish our disposition to believe the proposition, or, in other words, to diminish its probability.

(11.) But it may be worth while to examine whether the fraction $\frac{1}{2}$ be, after all, a true available expression for the probability of an event, which is neither likely nor unlikely to happen, or to have happened, there being no evidence, no reasons for belief, either for or against it.

Probability, as Mr BOOLE in his *Laws of Thought*, properly defines it, is “Expectation founded upon partial knowledge.” Events, therefore, of which we possess *complete* knowledge, and events of which we possess *no* knowledge, are equally, by the terms of the definition, excluded from the class of probable events, that is to say, of events to which the calculus of probabilities can be applied. If we are *certain* that an event has happened, we totally neglect and are unaffected by any subsequent information, which, but for that certainty, would have given to the event a definite probability expressed by a proper fraction; and never think of looking for a form by which to combine such fraction with the unit expressing the certainty. If, again, we derive from experience or observation a definite probability of any event, such, for example, as the probability for drawing a white ball from an urn, whose contents are given; namely, the fraction whose numerator is the number of white balls, and its denominator the total number of balls contained, we never think of combining this with the $\frac{1}{2}$ which is assumed as the probability when nothing whatever was known, except that the ball drawn must be either white or not white. *Complete* knowledge comprehends all previous partial knowledge; and, therefore, all fractional expressions for probability derived from the latter, are virtually contained in the unit, which is the expression adopted for the certainty produced by the former. On the other hand, *partial* knowledge destroys total ignorance, and any inference that may be drawn from it. It comprehends the hypothesis that the event may, and that it may not happen, with a definite probability to each, which do not supplement but supersede the probabilities of $\frac{1}{2}$ previously assumed for each. I cannot conclude without suggesting a doubt, whether $\frac{1}{2}$ be at any time the proper expression for the probability of an event which is “neither likely nor unlikely in regard of evidence.”

It seems more analogous with the practice in other cases to express such probability by the indefinite fraction $\frac{0}{0}$. If this expression be applied to either of the probabilities constituting the compound probability $\frac{p+r}{q+s}$, the compound probability will be reduced to the remaining simple probability, for $\frac{0+r}{0+s} = \frac{r}{s}$. And this agrees with the necessary action of the mind, which takes no note of its original ignorance, after it has arrived at a definite probability from partial knowledge.

(12.) Hitherto, I have been speaking of the combined result of two probabilities of the same event, derived from distinct sources of partial knowledge; and I have shown that to obtain a definite result, the mere ratio in such case is insufficient, and that the actual number of favourable and unfavourable cases in each of the data is requisite.

But when the given probabilities are of different events, and the quæsitum is the probability of their joint occurrence, the ratio alone is sufficient, because as factors $\frac{m p}{m q}$ and $\frac{n p}{n q}$ give the same results.

(13.) To sum up the propositions proved in the foregoing paper:—

1. If the *ratio only* of equally probable cases, in two or more probabilities for the same event be given, no definite probability can be derived from their composition. (7.)

2. If the two given probabilities $\frac{p}{q}$, $\frac{r}{s}$, indicate not merely the *ratio*, but also the *actual number* of favourable and unfavourable hypotheses or cases, their conjoined force is properly expressed by $\frac{p+r}{q+s}$. (6.)

3. Under both of these conditions, the second given probability increases or diminishes the force of the first, according as the fraction expressing the second is greater or less than that expressing the first. When the ratios only are given, then the extent of increase or diminution is indefinite. When the actual numbers are given, it is definite. (9.)

4. The *a priori* probability derived from absolute ignorance has no effect upon the force of a subsequently admitted probability. (11, 12.)