is intersected by any circle,

$$\sec R = lx + my + n = w,$$

in the same four points as the sphero-conic

$$aw_1^2 + (bu_1 + v_1)w_1 + v_1 = 0.]$$

ERRATA.—In § 4, read “The line of nodes in the sphero-cyclide is the polar of the centre of any (*J*) circle with respect to the polar conic of the corresponding (*F*) focal conic.”

In § 6, Cor., read “If the cyclide has the imaginary circle at infinity as a cuspidal edge, or if $v_2 \equiv c(x^2 + z^2) + dy^2$, the generated sphero-cyclide is called by Dr. Casey a Sphero-Cartesian.”

Prof. Hart has already published his Memoir on the “Five Focal Quadrics of a Cyclide,” in the *Messenger of Mathematics*, Vol. xrv., pp. 1–8. See § 5.

On the Limits of Multiple Integrals. By HUGH MACCOLL, B.A.

[Read November 13th, 1884.]

In my first paper on the “Calculus of Equivalent Statements” (*Proceedings*, Vol. ix., Nos. 124, 125), I showed how the limits of integration in a multiple integral might always be ascertained whenever we had enough data for the purpose. I now propose to show how the expression of the limits thus ascertained may often be simplified and reduced to a form more convenient for integration.

DEFINITION.—When we have an elementary statement of the form $x_{m',n}$ or $y_{m',n}$ $x_{r,s}$ or $z_{m',n}$ $y_{r,s}$ $x_{u,v}$, &c., presenting the *nearest* limits (or true limits of integration), we may, for brevity's sake, take any of these symbols to denote, *not* the statement itself, but the *integral* which has the limits of integration indicated by the statement.

Thus $y_{m',n} x_{r,s}$ will be a mere abbreviation for

$$\int_{y_n}^{y_m} dy \int_{x_s}^{x_r} dx.$$

From this definition we get the following self-evident rules :

RULE 1 :

$$x_{m',n} = -x_{n',m},$$

$$y_{m',n} x_{r',s} = -y_{n',m} x_{r',s} = y_{n',m} x_{s',r},$$

and so on.

RULE 2 :

$$x_{n',n} = 0.$$

RULE 3 :

$$x_{m',n} + x_{r',s} = x_{m',s} + x_{r',n};$$

and, generally, in adding any number of integrals of the same function and variable, the superior limits may interchange their inferior limits in any order.

RULE 4 :

$$x_{m',n} + x_{n',r} = x_{m',r},$$

and so on for any number of terms.

As an example of the application of these rules, take

$$y_{3',4} (x_{3',1} + x_{2',4}) + (y_{3',1} + y_{2',4}) x_{1',2}.$$

Applying Rule 3 to the bracket integrals, the expression becomes

$$y_{3',4} (x_{3',4} + x_{2',1}) + (y_{3',4} + y_{2',1}) x_{1',2} = y_{3',4} x_{3',4} + y_{2',1} x_{1',2};$$

for the two terms omitted cancel each other by Rule 1.

As another example, take

$$z_{1',0} y_{1',0} x_{2',0} + z_{1',0} y_{2',0} x_{1',2} + z_{2',0} y_{1',2} x_{1',2}.$$

By Rule 4, we can put this into the form

$$z_{1',0} y_{1',0} (x_{2',1} + x_{1',0}) + z_{1',0} (y_{2',1} + y_{1',0}) x_{1',2} + (z_{2',1} + z_{1',0}) y_{1',2} x_{1',2}.$$

Multiplying out and cancelling (as in the preceding example) those terms which destroy each other by Rule 1, we get

$$z_{1',0} y_{1',0} x_{1',0} - z_{1',2} y_{1',2} x_{1',2}.$$

When the data from which we have to find the limits of integration involve *literal constants*, this circumstance often gives rise to different cases, the solution being different for each case.

When we have found the limits of the *variables*, we must then apply the same process to the *constants*, and determine their limits

also. In this way we shall separate the cases. Suppose, for example, the final result to be

$$Ua_1 + Va_{1'} + W,$$

in which U , V , and W are each expressions of the form

$$y_{m'.n} x_{r'.s} + y_{m'.o} x_{s'.p},$$

while a is a constant. Multiply the above final result by $a_1 + a_{1'}$, and

we get

$$a_1(U + W) + a_{1'}(V + W).$$

This asserts that $U + W$ is the proper expression for the integration when $a > a_1$, but that $V + W$ is the proper expression when a is less than a_1 .

Next, suppose the final expression to be

$$Ua_{1'.2} + Va_2 + Wa_1 + Z.$$

Multiply by $(a_1 + a_{1'})(a_2 + a_{2'})$, and we get

$$a_{1.2}(V + W + Z) + a_{2.1}(W + Z) + a_{1'.2}(V + Z) + a_{2'.1'}(U + Z).$$

This asserts that $V + W + Z$ is the proper expression for the integration when a_1 and a_2 are both inferior limits of a ; that $W + Z$ is the proper expression when a is greater than a_1 and less than a_2 ; and so on.

The cases $a_{2.1}$ and $a_{1'.2}$ in the above example are contradictory. When mere inspection of the table of limits is not sufficient to show which case must be expunged as impossible, we must have recourse to the formula $a_{m'.n} : p(a - a_n)$.

Since the transformations and simplifications, founded on Rules 1, 2, 3, 4 of this paper, are independent of any particular table of limits, it is evident that they may often be greatly facilitated by diagrams representing merely *rectangular* areas or volumes, without having recourse to curves or representations of curved surfaces. One or two simple examples will sufficiently illustrate my meaning. Take the

expression $y_{2'.1} x_{1'.0} + 2y_{2'.1} x_{2.1} + y_{1'.0} x_{2.1}.$

Fig. 1 makes it evident that this is identical with the simpler expression

$$y_{2'.1} x_{2.0} + y_{2.0} x_{2'.1},$$

the elementary area containing the double *plus* representing the

middle term of the original expression. Suppose we had taken the

FIG. 1.

FIG. 2.

order of the limiting lines given in Fig. 2, we should have obtained our signs thus :

Observing the usual convention as to signs in regard to lines and areas, the first term $y_{2,1} x_{1,0}$ of the original expression directs us to insert a *plus* in each of the right-hand two elementary areas which make up the whole *positive* area $y_{2,1} x_{1,0}$. The second term $2y_{2,1} x_{2,1}$ directs us to insert *twice* a *minus* sign in each of the four elementary areas which make up the whole *negative* area $y_{2,1} x_{2,1}$. The third term $y_{1,0} x_{2,1}$ directs us to insert a *plus* sign in each of the two bottom elementary areas which make up the whole *positive* area $y_{1,0} x_{2,1}$. The whole collection of signs will then appear as in Fig. 2. Cancelling now the double sign $\pm$ wherever found, Fig. 2 will then assume the simpler form exhibited in Fig. 3, which makes it evident (as before) that the simplest expression for the area is the two-term expression

$$y_{2,1} x_{2,0} + y_{2,0} x_{2,1}.$$

It is also clear, from Fig. 3, that the expression may also be put into the form

$$2y_{2,0} x_{2,0} + y_{1,0} x_{0,2} + y_{2,0} x_{0,1},$$

which is not so clear from Fig. 1.

FIG. 3.

FIG. 4.

Take next the expression illustrated in Fig. 4. At first sight it

does not look as if this could be expressed in fewer than four terms; but place a double sign $\pm$ in the central blank elementary area, and it then becomes evident at once that the expression may be put into the simpler form

$$y_{2,4} x_{4,2} + y_{1,3} x_{1,3}.$$

Of course, all simplifications thus obtained by mere inspection of rectangular areas, or (without the aid of diagrams) by the preceding symbolical rules of reduction, hold good for all possible tables of limits, and therefore for limits the true representations of which would be *curves*, and not straight lines. It sometimes happens, however, when we come to actual integration, that, of two equivalent expressions, the one involves imaginary quantities, while the other does not. The final result, however, will be the same, whatever expression we take; for, in such cases, when imaginary quantities enter into our integration, they will always be found to vanish afterwards by cancelling.

In the case of three variables, as the elementary volumes which collectively represent the whole integration cannot generally be *all* exhibited to the eye, we cannot adopt the *plus* and *minus* system of marking which has just been illustrated in the case of areas. The following method is the best substitute that I can think of.

Let the symbol $m \cdot n \cdot r$ marked on the top block of a pile indicate that the m^{th} , n^{th} , and r^{th} blocks of that pile (counting upwards from the bottom) form parts of the integration, but that no other block of that pile (or column) forms part of it. A dot *above* a number asserts that the block indicated by that number is to be taken *positively*, while a dot *below* a number asserts that the block indicated by that number

is to be taken *negatively*. For example, $\overset{\cdot}{3} \cdot \overset{\cdot}{4} \cdot \underset{\cdot}{5}$ (which refers to blocks of the same column; see Fig. 5) asserts that the third block (counting upwards) is taken *twice* positively and *once* negatively; that the fourth block is taken *three* times positively and *twice* negatively; and that the fifth block is taken *once* positively and *once* negatively. Evidently, therefore, the above symbol may be replaced by $\overset{\cdot}{3} \cdot \underset{\cdot}{4}$, which asserts that the *third* and *fourth* blocks (shaded in Fig. 5) enter positively into the integration, the other blocks of the column being left out of account.

FIG. 5.

As an illustration, let it be required to simplify the expression

$$z_{2,0} y_{2,0} x_{1,3} + z_{1,0} y_{1,3} x_{1,3} + z_{3,0} y_{1,0} x_{2,0}.$$

Considering upward, eastward, and northward as the positive directions, and their opposites as the negative, we shall obtain the result given in Fig. 6. Cancelling the double-signed (and therefore zero) volumes $\dot{1} \cdot \dot{2}$ and $\dot{1}$, we shall have left the positive column $1 \cdot \dot{2}$ and the negative block 2, expressed

by $z_{3,0} y_{1,0} x_{1,0} - z_{3,1} y_{3,1} x_{3,1}$,

which is the utmost simplification of which the given expression is capable. The same result may, of course, be obtained, without the aid of a diagram, by the preceding rules, but, I think, less readily.

FIG. 6.

Note.—In evaluating multiple integrals of the form $z_{m',n} y_{r,s} x_{u,v}$ and $\phi(x, y, z) z_{m',n} y_{r,s} x_{u,v}$, the following notation would, I think, be convenient:

Let the symbols $\phi(x) x_{m',n}$ and $x_{m',n} \phi(x)$, which differ only in arrangement, have different meanings by virtue of this difference of arrangement. Let the former denote the value of the integral $\int dx \phi(x)$, when taken between the higher limit x_m and the lower limit x_n ; and let the latter denote $\phi(x_m) - \phi(x_n)$.

For example, let $\int dx \phi(x) = \psi(x)$. According to the proposed convention, we should have

$$\phi(x) x_{m',n} = x_{m',n} \psi(x) = \psi(x_m) - \psi(x_n).$$

As a simple illustration, suppose we are required to evaluate $z_{1,2} y_{1,2} x_{1,0}$ with reference to the accompanying table.

$z_1 = y$	$y_1 = x$	$x_1 = a$
$z_2 = c$	$y_2 = b$	

The full process would be as follows:

$$\begin{aligned} z_{1,2} y_{1,2} x_{1,0} &= (y-c) y_{1,2} x_{1,0} \\ &= y_{1,2} \left(\frac{1}{2} y^2 - cy \right) x_{1,0} \\ &= \left(\frac{1}{2} x^2 - cx - \frac{1}{2} b^2 + bc \right) x_{1,0} \\ &= x_{1,0} \left(\frac{1}{6} x^3 - \frac{1}{2} cx^2 - \frac{1}{2} b^2 x + bcx \right) \\ &= \frac{1}{6} a^3 - \frac{1}{2} a^2 c - \frac{1}{2} a b^2 + abc. \end{aligned}$$

The ordinary symbols of integration might thus be dispensed with, while the successive steps of the integration would be exhibited in their natural order and without a break.

*On Quadric Transformations.** By Mr. W. SPOTTISWOODE, P.R.S.

A quantic is said to be transformed when the variables originally contained in it are replaced by others, and the transformation is called linear, or quadratic, or of a higher degree, according as the equations connecting the two sets of variables are of the first, or second, or of higher degrees. The transformations usually considered are linear, whereby, for example, a rational homogeneous function U of x, y , is transformed into another of the same degree in ξ, η , either by a substitution of the form

$$\xi : \eta = lx + my : lx' + m'y,$$

or by a substitution of the form

$$x : y = \alpha\xi + \beta\eta : \alpha'\xi + \beta'\eta.$$

The theory of linear transformations has, as is well known, been investigated by Boole, by Sylvester, by Cayley, and by many others; and notably by the last-mentioned mathematician in his classical "Memoirs on Quantics" (*Phil. Trans.*, 1856, *et seqq.*).

Tschirnhausen's transformation is an instance of one of a higher degree; and the substitution is of the first form, viz., it is as follows :

$$\xi : \eta = V : V',$$

where V, V' are homogeneous functions of x, y , of a given degree n . And, if U be the given function of x, y , which is to be the subject of transformation, the process is effected by eliminating x, y from the equations

$$U = 0, \quad \eta V - \xi V' = 0;$$

* Some papers by the late Mr. W. Spottiswoode, P.R.S., which had apparently been written with a view to their being brought before this Society, were, at the instance of the Council, submitted to Prof. Cayley for him to report upon the advisability of their being printed in the *Proceedings*. In accordance with Prof. Cayley's report, the following portions, which have been kindly edited by him, are printed here.