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XXXVI. *On the Elliptic Polarization produced by the Direct Transmission of a Plane Polarized Stream through a Plate of Quartz, cut in a Direction oblique to the Optic Axis, with a Method of Determining the Error of a Plate supposed to be Perpendicular to the Axis.* By JAMES WALKER, M.A., Oxford*.

1. LET the primitive stream with polarization-vector $\text{Exp}(nt)$ be polarized in an azimuth α with respect to the principal section of the quartz plate. This may be replaced by the elliptically polarized stream represented by the vector

$$(\xi_1, \eta_1) = c_1(\cos \beta, -\iota \sin \beta) \text{Exp}\{\iota(nt + \epsilon_1)\},$$

with its plane of maximum polarization (β being numerically less than $\pi/4$ and positive or negative according as the quartz is left- or right-handed) in the principal section, together with the oppositely polarized stream defined by the vector

$$(\xi_2, \eta_2) = c_2(-\sin \beta, -\iota \cos \beta) \text{Exp}\{\iota(nt + \epsilon_2)\},$$

where

$$c_1 \text{Exp}(\iota \epsilon_1) = \cos \alpha \cos \beta + \iota \sin \alpha \sin \beta,$$

$$c_2 \text{Exp}(\iota \epsilon_2) = -\cos \alpha \sin \beta + \iota \sin \alpha \cos \beta.$$

These oppositely polarized streams traverse the plate with different speeds, and on emergence the second is retarded in phase relatively to the first by an amount δ , so that the emergent light will be in general elliptically polarized and may be denoted by

$$(u, v) = c(\cos \gamma, -\iota \sin \gamma) \text{Exp}\{\iota(nt + \epsilon)\},$$

where

$$c(\cos \gamma \cos \theta + \iota \sin \gamma \sin \theta) \text{Exp}\{\iota(\epsilon + \delta/2)\}$$

$$= c_1 \cos \beta \text{Exp}\{\iota(\epsilon_1 + \delta/2)\} - c_2 \sin \beta \text{Exp}\{\iota(\epsilon_2 - \delta/2)\},$$

$$c(\cos \gamma \sin \theta - \iota \sin \gamma \cos \theta) \text{Exp}\{\iota(\epsilon + \delta/2)\}$$

$$= -c_1 \sin \beta \text{Exp}\{\iota(\epsilon_1 + \delta/2)\} - c_2 \cos \beta \text{Exp}\{\iota(\epsilon_2 - \delta/2)\},$$

θ being the angle between the principal section of the plate and the plane of maximum (or minimum) polarization of the resultant stream, according as γ is numerically less or greater than $\pi/4$.

* Read November 27, 1908.

Substituting for $c_1 \text{Exp}(\iota\epsilon_1)$ and $c_2 \text{Exp}(\iota\epsilon_2)$ and writing
 $\tan R = \sin 2\beta \tan \delta/2$, $\tan \Delta/2 = \cot 2\beta \sin R$, . . . (1)

we obtain

$$\begin{aligned} c(\cos \gamma \cos \theta + \iota \sin \gamma \sin \theta) \text{Exp}\{\iota(\epsilon + \delta/2)\} \\ = \cos(\alpha + R) \cos \Delta/2 + \iota \cos \alpha \sin \Delta/2, \\ c(\cos \gamma \sin \theta - \iota \sin \gamma \cos \theta) \text{Exp}\{\iota(\epsilon + \delta/2)\} \\ = \sin(\alpha + R) \cos \Delta/2 - \iota \sin \alpha \sin \Delta/2, \end{aligned}$$

or

$$\begin{aligned} c \left\{ \cos \gamma \cos \left(\theta - \frac{R}{2} \right) + \iota \sin \gamma \sin \left(\theta - \frac{R}{2} \right) \right\} \text{Exp} \left\{ \iota \left(\epsilon + \frac{\delta}{2} \right) \right\} \\ = \cos \left(\alpha + \frac{R}{2} \right) \left(\cos \frac{\Delta}{2} + \iota \sin \frac{\Delta}{2} \right), \\ c \left\{ \cos \gamma \sin \left(\theta - \frac{R}{2} \right) - \iota \sin \gamma \cos \left(\theta - \frac{R}{2} \right) \right\} \text{Exp} \left\{ \iota \left(\epsilon + \frac{\delta}{2} \right) \right\} \\ = \sin \left(\alpha + \frac{R}{2} \right) \left(\cos \frac{\Delta}{2} - \iota \sin \frac{\Delta}{2} \right); \end{aligned}$$

whence

$$\begin{aligned} \frac{\cos \gamma \cos \left(\theta - \frac{R}{2} \right) + \iota \sin \gamma \sin \left(\theta - \frac{R}{2} \right)}{\cos \gamma \sin \left(\theta - \frac{R}{2} \right) - \iota \sin \gamma \cos \left(\theta - \frac{R}{2} \right)} \\ = \cot \left(\alpha + \frac{R}{2} \right) (\cos \Delta + \iota \sin \Delta), \end{aligned}$$

which gives

$$\left. \begin{aligned} \frac{\cos 2\gamma \sin (2\theta - R)}{1 - \cos 2\gamma \cos (2\theta - R)} &= \cot \left(\alpha + \frac{R}{2} \right) \cos \Delta \\ \frac{\sin 2\gamma}{1 - \cos 2\gamma \cos (2\theta - R)} &= \cot \left(\alpha + \frac{R}{2} \right) \sin \Delta \end{aligned} \right\} . . . (2)$$

From these we obtain the equations

$$\left. \begin{aligned} \tan (2\theta - R) &= \tan (2\alpha + R) \cos \Delta, \\ \sin 2\gamma &= \sin (2\alpha + R) \sin \Delta, \\ \tan^2 \gamma &= \tan (\alpha + \theta) \tan (\alpha + R - \theta) \end{aligned} \right\} . . . (3)$$

that determine the character of the stream emerging from the quartz, β and δ and consequently R and Δ being supposed to be known.

2. For the discussion of the elliptic polarization it is more

convenient to express $\tan 2(\theta - \alpha)$ and $\sin 2\gamma$ in terms of $\tan R$. Solving for $\tan 2(\theta - \alpha)$ we obtain

$$\begin{aligned}\tan 2(\theta - \alpha) &= \frac{\tan(2\alpha + R) \cos \Delta - \tan(2\alpha - R)}{1 + \tan(2\alpha + R) \tan(2\alpha - R) \cos \Delta} \\ &= \frac{\sin 2R \cos^2 \Delta/2 - \sin 4\alpha \sin^2 \Delta/2}{\cos 2R \cos^2 \Delta/2 + \cos 4\alpha \sin^2 \Delta/2} \\ &= \frac{\sin 2R - \sin 4\alpha \cot^2 2\beta \sin^2 R}{\cos 2R + \cos 4\alpha \cot^2 2\beta \sin^2 R} \\ &= \frac{2 \tan R - 2 \sin 2\alpha \cos 2\alpha \cot^2 2\beta \tan^2 R}{1 + (\cos 4\alpha \cot^2 2\beta - 1) \tan^2 R}.\end{aligned}$$

Also

$$\begin{aligned}\sin 2\gamma &= \sin 2\alpha \cos R \sin \Delta + \cos 2\alpha \sin R \sin \Delta \\ &= \frac{2 \sin 2\alpha \cot 2\beta \tan R + 2 \cos 2\alpha \cot 2\beta \tan^2 R}{1 + (\cot^2 2\beta + 1) \tan^2 R},\end{aligned}$$

since

$$\sin \Delta = \frac{2 \tan \Delta/2}{1 + \tan^2 \Delta/2} = \frac{2 \cot 2\beta \sin R}{1 + \cot^2 2\beta \sin^2 R}.$$

These expressions have the form

$$y = 2(av + bx^2)/(1 + cx^2),$$

and the maximum and minimum values of y are $y = ax_m$ occurring when

$$x = x_m = (b \pm \sqrt{b^2 + a^2 c})/(ac).$$

3. First considering $\sin 2\gamma$, we see that $\sin 2\gamma = 0$, or the light is plane polarized, when $\tan R = 0$ and when $\tan R = -\tan 2\alpha$, the corresponding values of $\tan 2(\theta - \alpha)$ being 0 and $-\tan 4\alpha$. $\sin 2\gamma$ attains its maximum and minimum values $\sin 2\alpha \cot 2\beta \tan r_m$, when

$$\tan R = \tan r_m = \frac{1 \pm \sqrt{1 + \tan^2 2\alpha / \sin^2 2\beta}}{\tan 2\alpha / \sin^2 2\beta};$$

writing $\tan 2\alpha / \sin 2\beta = \tan \psi$, ψ being a positive angle less than π ,

$$\tan r_m = \sin 2\beta \cot \psi/2, \text{ or } -\sin 2\beta \tan \psi/2,$$

and since $\tan R = \sin 2\beta \tan \delta/2$, the maximum and minimum values of $\sin 2\gamma$ occur when $\delta = n\pi - \psi$.

The values of $\tan R$ corresponding to the maximum and minimum values of $\sin 2\gamma$ may be written

$$\tan R = \sin 2\beta \cdot x, \text{ where } x = \cot \psi/2 \text{ or } -\tan \psi/2,$$

$$\begin{aligned} \text{and } \tan 2\alpha &= -2 \sin 2\beta \cot \psi/2 / (1 - \cot^2 \psi/2) \\ &= 2 \sin 2\beta \tan \psi/2 / (1 - \tan^2 \psi/2) \\ &= -2 \sin 2\beta \cdot x / (1 - x^2); \end{aligned}$$

substituting these values in the expression for $\tan 2(\theta - \alpha)$, we find

$$\tan 2(\theta - \alpha) = 2 \sin 2\beta \frac{x(1+x^2)(1 - \cos 4\beta \cdot x^2)}{(1-x^2)(1+x^2)(1 - \cos 4\beta \cdot x^2)};$$

and unless $x^2 = \sec 4\beta$,

$$\tan 2(\theta - \alpha) = 2 \sin 2\beta \frac{x}{1-x^2} = -\tan 2\alpha.$$

But when $x^2 = \sec 4\beta$,

$$\tan 2\alpha = \sqrt{\cot^2 2\beta - 1}, \text{ or } \tan^2 2\beta = \cos^2 2\alpha;$$

hence corresponding to the maximum and minimum values of $\sin 2\gamma$, we have $\tan 2(\theta - \alpha) = -\tan 2\alpha$, unless $\cos 2\alpha = \pm \tan 2\beta$, a case that will be considered later.

Further, $\sin 2\gamma = \pm 1$, only if

$$\tan \psi/2 = \sin 2\alpha \cos 2\beta, \text{ or } \cot \psi/2 = \sin 2\alpha \cos 2\beta,$$

that is if $\cos 2\alpha = \pm \tan 2\beta$, and consequently, except in these cases, θ determines the plane of maximum polarization.

Hence, reserving for future consideration the cases in which $\cos 2\alpha = \pm \tan 2\beta$, we see that as R increases from 0 to π , corresponding to a change of δ from $2n\pi$ to $2(n \pm 1)\pi$ according as β is positive or negative, the light is initially plane polarized in the primitive plane of polarization; it then becomes elliptically polarized of a sign, the same as or opposite to that of the plate according as $\sin 2\alpha$ is positive or negative: the ratio of the axes of the elliptic vibration attains its maximum value, the plane of maximum polarization being then either parallel or perpendicular to the principal section of the quartz, when $\tan R = \sin 2\beta \cot \psi/2$ or $= -\sin 2\beta \tan \psi/2$ according as β is positive or negative; when $\tan R = -\tan 2\alpha$ the light is again plane polarized in

an azimuth symmetrical to the primitive plane of polarization with respect to the principal section of the plate; the sense of rotation then changes, and the ratio of the axes is again a maximum when $\tan R = -\sin 2\beta \tan \psi/2$ or $= \sin 2\beta \cot \psi/2$, the plane of maximum polarization being again either parallel or perpendicular to the principal section; and finally, when $R = \pi$, the light is plane polarized in the original plane*.

4. Turning now to $\tan 2(\theta - \alpha)$, we see that this attains its maximum and minimum values $\tan R_m$, when

$$\tan R = \tan R_m \\ = \frac{-\sin 2\alpha \cos 2\alpha \pm \sqrt{(\cos^2 2\alpha - \tan^2 2\beta)(\sin^2 2\alpha + \tan^2 2\beta)}}{\cos^2 2\alpha - \sin^2 2\alpha - \tan^2 2\beta}.$$

This, however, only gives a real value for $\tan R$, if $\cos^2 2\alpha > \tan^2 2\beta$, and when this is the case, writing

$$\cos^2 2\alpha - \tan^2 2\beta = \cos^2 \chi,$$

where χ is a positive angle less than $\pi/2$, we have

$$\tan R_m = \tan (\chi - 2\alpha) \text{ or } -\tan (\chi + 2\alpha),$$

the value of $\sin 2\gamma$ in these two cases being $\tan 2\beta / \cos 2\alpha$.

Thus if $\cos^2 2\alpha < \tan^2 2\beta$, the plane of maximum polarization rotates continuously as R increases; while if $\cos^2 2\alpha > \tan^2 2\beta$, it oscillates between two extreme positions, making an angle $\chi/2$ on either side of the principal section of the plate or the perpendicular plane according as $\cos 2\alpha$ is positive or negative†.

The following tables give some of the principal cases of the changes of the plane of maximum polarization and of $\sin 2\gamma$ as R increases from 0 to π , β being taken as positive.

* The changes in the value of $\sin 2\gamma$ may also be conveniently traced from the formula

$$\sin 2\gamma = 2 \sin 2\alpha \cos 2\beta \sin \delta/2 \sin (\psi + \delta/2) / \sin \psi.$$

† This oscillation of the plane of maximum polarization was deduced by Croullebois (*Ann. de Ch. et de Phys.* [4] xxviii. p. 382 (1873)) for the special case of light initially polarized in the principal section of the quartz. He appears to have overlooked the fact that it requires in this case that $\tan 2\beta$ should be less than unity or $\beta < \pi/8$, a condition found by Monnory (*J. de Phys.* [2] ix. p. 277 (1890)). The condition for the general case does not appear to have been previously given.

TABLE I.— $\cos^2 2\alpha > \tan^2 2\beta$.

$$0 < \alpha < \pi/4.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\sin 2\gamma.$
1.	0	0	α	0
2.	$\tan(\chi - \alpha)$	$\tan(\chi - 2\alpha)$	$\frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
3.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	α	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
4.	$\sin 2\beta \cot \frac{\psi}{2}$	$-\tan 2\alpha$	0	$\sin 2\alpha \cos 2\beta \cot \frac{\psi}{2}$
5.	$\cot 2\alpha$	$-\tan 4\alpha$	$-\alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
6.	$-\tan(\chi + 2\alpha)$	$-\tan(\chi + 2\alpha)$	$-\frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
7.	$-\tan 2\alpha$	$-\tan 4\alpha$	$-\alpha$	0
8.	$-\sin 2\beta \tan \frac{\psi}{2}$	$-\tan 2\alpha$	0	$-\sin 2\alpha \cos 2\beta \tan \frac{\psi}{2}$
9.	0	0	α	0

$$\pi/4 < \alpha < \pi/2.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\sin 2\gamma.$
1.	0	0	α	0
2.	$\sin 2\beta \cot \frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\sin 2\alpha \cos 2\beta \cot \frac{\psi}{2}$
3.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi - \alpha$	0
4.	$\tan(\chi - 2\alpha)$	$\tan(\chi - 2\alpha)$	$\frac{\pi}{2} + \frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
5.	$\cot 2\alpha$	$-\tan 4\alpha$	$\pi - \alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
6.	$-\sin 2\beta \tan \frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$-\sin 2\alpha \cos 2\beta \tan \frac{\psi}{2}$
7.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	α	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
8.	$-\tan(\chi + 2\alpha)$	$-\tan(\chi + 2\alpha)$	$\frac{\pi}{2} - \frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
9.	0	0	α	0

TABLE II.— $\cos^2 2\alpha < \tan^2 2\beta$.

$$0 < \alpha < \pi/4.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\sin 2\gamma.$
1.	0	0	α	0
2.	$\cot 2\alpha$	$-\tan 4\alpha$	$\frac{\pi}{2} - \alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
3.	$\sin 2\beta \cot \frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\sin 2\alpha \cos 2\beta \cot \frac{\psi}{2}$
4.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	$\frac{\pi}{2} + \alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
5.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi - \alpha$	0
6.	$-\sin 2\beta \tan \frac{\psi}{2}$	$-\tan 2\alpha$	π	$-\sin 2\alpha \cos 2\beta \tan \frac{\psi}{2}$
7.	0	0	$\pi + \alpha$	0

$$\pi/4 < \alpha < \pi/2.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\sin 2\gamma.$
1.	0	0	α	0
2.	$\sin 2\beta \cot \frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\sin 2\alpha \cos 2\beta \cot \frac{\psi}{2}$
3.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi - \alpha$	0
4.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	$\frac{\pi}{2} + \alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
5.	$-\sin 2\beta \tan \frac{\psi}{2}$	$-\tan 2\alpha$	π	$-\sin 2\alpha \cos 2\beta \tan \frac{\psi}{2}$
6.	$\cot 2\alpha$	$-\tan 4\alpha$	$\frac{3\pi}{2} - \alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
7.	0	0	$\pi + \alpha$	0

5. The transition from one of these cases to the other occurs when $\cos^2 2\alpha = \tan^2 2\beta$, and then

$$\tan 2(\theta - \alpha) = \frac{2 \tan R(1 - \tan 2\alpha \tan R)}{1 - \tan^2 2\alpha \tan^2 R} = \frac{2 \tan R}{1 + \tan 2\alpha \tan R}.$$

In this case θ increases from α to $\alpha + \pi/2$, as R increases from 0 to π : the light is circularly polarized when $\tan R = \cot 2\alpha$, and while for values of R less than that given by this equation θ gives the plane of maximum polarization, for greater values it determines the plane of minimum polarization*.

* When $\tan R = \cot 2\alpha$, $\tan 2(\theta - \alpha)$ of course becomes indeterminate, but it is easy to see that the change from determining the plane of maximum polarization to giving that of minimum polarization, or *vice versa*, occurs as θ passes through the values $(2n+1)\pi/4$.

The ratio of the axes of the elliptic vibrations may be easily determined from the formula

$$\tan^2 \gamma = \tan(\alpha + \theta) \tan(\alpha + R - \theta)$$

which in this case reduces to the simple form

$$\tan \gamma = \pm \tan(\theta + \alpha) \tan(\theta - \alpha),$$

the sign to be attributed to $\tan \gamma$ being determined from that of $\sin 2\gamma$.

The following table gives the principal changes relating to this case.

TABLE III.— $\cos^2 2\alpha = \tan^2 2\beta$.

$$0 < \alpha < \pi/4.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\tan \gamma.$
1.	0	0	α	0
2.	$\cot 2\alpha$	$\cot 2\alpha$	$\frac{\pi}{4}$	1
3.	$-\cot 2\alpha$	∞	$\frac{\pi}{4} + \alpha$	$\tan\left(\frac{\pi}{4} + 2\alpha\right)$
4.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\frac{\pi}{2} - \alpha$	∞
5.	$-\frac{\sin 2\alpha \cos 2\alpha}{1 + \cos^2 2\alpha}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$-\cot^2 \alpha$
6.	0	0	$\frac{\pi}{2} + \alpha$	∞

$$\pi/4 < \alpha < \pi/2.$$

	$\tan R.$	$\tan 2(\theta - \alpha)$	$\theta.$	$\tan \gamma.$
1.	0	0	α	4
2.	$-\frac{\sin 2\alpha \cos 2\alpha}{1 + \cos^2 2\alpha}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\cot^2 \alpha$
3.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi - \alpha$	0
4.	$-\cot 2\alpha$	∞	$\frac{\pi}{4} + \alpha$	$-\tan\left(\frac{\pi}{4} + 2\alpha\right)$
5.	$\cot 2\alpha$	$\cot 2\alpha$	$\frac{3\pi}{4}$	-1
6.	0	0	$\frac{\pi}{2} + \alpha$	$-\infty$

If α be between $\pi/8$ and $3\pi/8$, cases (3) and (4) in the above table are interchanged.

6. When the light is analysed in a plane making an angle ϕ with the principal section of the plate, the intensity of the radiation characterized by (θ, γ) is

$$I = c^2 \{ \cos^2 \gamma \cos^2 (\phi - \theta) + \sin^2 \gamma \sin^2 (\phi - \theta) \} \\ = \frac{1}{2} c^2 \{ 1 + \cos 2\gamma \cos 2(\phi - \theta) \}.$$

Now from (2) and (3)

$$\begin{aligned} \cos 2\gamma \sin 2\theta \cos R - \cos 2\gamma \cos 2\theta \sin R \\ &= \sin 2\gamma \cot \Delta = \sin (2\alpha + R) \cos \Delta \\ \cos 2\gamma \sin 2\theta \sin R + \cos 2\gamma \cos 2\theta \cos R \\ &= 1 - \tan \left(\alpha + \frac{R}{2} \right) \sin 2\gamma / \sin \Delta \\ &= 1 - \tan \left(\alpha + \frac{R}{2} \right) \sin (2\alpha + R) = \cos (2\alpha + R); \end{aligned}$$

whence

$$\begin{aligned} \cos 2\gamma \sin 2\theta &= \cos 2\alpha \sin R \cos R (1 + \cos \Delta) \\ &\quad + \sin 2\alpha (\cos^2 R \cos \Delta - \sin^2 R) \\ \cos 2\gamma \cos 2\theta &= \cos 2\alpha (\cos^2 R - \sin^2 R \cos \Delta) \\ &\quad - \sin 2\alpha \sin R \cos R (1 + \cos \Delta). \end{aligned}$$

Now

$$\begin{aligned} 1 + \cos \Delta &= \frac{2}{1 + \tan^2 \Delta / 2} = \frac{2}{1 + \cot^2 2\beta \sin^2 R} \\ &= \frac{2}{\cos^2 R + \sin^2 R / \sin^2 2\beta}, \end{aligned}$$

$\therefore$

$$\sin R \cos R (1 + \cos \Delta) = \frac{2 \tan R}{1 + \tan^2 R / \sin^2 2\beta} = \sin 2\beta \sin \delta$$

$$\begin{aligned} \cos^2 R \cos \Delta - \sin^2 R &= \frac{2}{1 + \tan^2 R / \sin^2 2\beta} - 1 \\ &= 2 \cos^2 \frac{\delta}{2} - 1 = \cos \delta \end{aligned}$$

$$\begin{aligned} \cos^2 R - \sin^2 R \cos \Delta &= 1 - \frac{2 \tan^2 R}{1 + \tan^2 R / \sin^2 2\beta} \\ &= 1 - 2 \sin^2 2\beta \sin^2 \frac{\delta}{2} \\ &= \cos^2 2\beta + \sin^2 2\beta \cos \delta. \end{aligned}$$

Hence

$$\begin{aligned}\cos 2\gamma \cos 2\theta &= \cos 2\alpha (\cos^2 2\beta + \sin^2 2\beta \cos \delta) - \sin 2\alpha \sin 2\beta \sin \delta \\ &= \cos 2\alpha \cos^2 2\beta + \sin 2\beta \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta} \cos(\delta + \psi) \\ \cos 2\gamma \sin 2\theta &= \cos 2\alpha \sin 2\beta \sin \delta + \sin 2\alpha \cos \delta \\ &= \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta} \sin(\delta + \psi),\end{aligned}$$

where $\tan \psi = \tan 2\alpha / \sin 2\beta$,

and

$$\begin{aligned}I &= \frac{1}{2}c^2 [1 + \cos 2\alpha \cos^2 2\beta \cos 2\phi \\ &\quad + \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta} \{ \sin 2\beta \cos 2\phi \cos(\delta + \psi) \\ &\quad + \sin 2\phi \sin(\delta + \psi) \}] \\ &= \frac{1}{2}c^2 [1 + \cos 2\alpha \cos^2 2\beta \cos 2\phi \\ &\quad + \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta} \sqrt{\sin^2 2\phi + \cos^2 2\phi \sin^2 2\beta} \\ &\quad \cos(\delta + \psi - \psi')],\end{aligned}$$

where $\tan \psi' = \tan 2\phi / \sin 2\beta$.

This formula contains the theory of the interference phenomena that are obtained when a conical pencil of polarized light traverses the plate and is subsequently analysed.

It may also be employed for the case in which the stream that passes through the polarizer, quartz plate and analyser is a parallel pencil of white light that is investigated with a spectroscop. With a sufficiently thick plate, β may be regarded as constant over a region of the spectrum corresponding to a variation of δ by several multiples of π , and we see that the minima of intensity occur for the radiations for which $\delta = (2n+1)\pi + \Psi$, where

$$\tan \Psi = \frac{\sin 2\beta (\tan 2\phi - \tan 2\alpha)}{\sin^2 2\beta + \tan 2\phi \tan 2\alpha},$$

the intensity then being

$$I = \frac{1}{2}c^2 \{ 1 + \cos 2\alpha \cos^2 2\beta \cos 2\phi - \sqrt{1 - \cos^2 2\alpha \cos^2 2\beta} \sqrt{1 - \cos^2 2\beta \cos^2 2\phi} \}.$$

Thus as the analyser is turned, the dark bands travel along the spectrum, changing their intensity as they progress: they are perfectly black when $\phi = \pi/2 \pm \alpha$, and are least marked when $\phi = 0$ and $\pi/2$.

With a plate perpendicular to the optic axis, $\beta = \pi/4$, and

$$I = c^2 \cos^2 \frac{\delta - \Psi}{2} = c^2 \cos^2 \frac{\delta - 2(\phi - \alpha)}{2};$$

hence as the analyser is turned, the bands traverse the spectrum, retaining the same intensity.

When the plate is parallel to the optic axis, $\beta = 0$ or is at any rate very small, and

$$I = \frac{1}{2} c^2 \{1 + \cos 2\alpha \cos 2\phi + \sin 2\alpha \sin 2\phi \cos \delta\};$$

when $\phi = 0$ or $\pi/2$, there are no bands, and in other cases the bands occur where $\delta = (2n+1)\pi$ or $2n\pi$, according to the sign of $\sin 2\alpha \sin 2\phi$. Thus when $0 < \alpha < \pi/2$, the bands occur at the points for which $\delta = (2n+1)\pi$ so long as $0 < \phi < \pi/2$, their intensity being zero, when $\phi = \pi/2 - \alpha$; and as ϕ increases through $\pi/2$, they change their position to that of the former maxima of intensity, becoming perfectly black when $\phi = \pi/2 + \alpha^*$.

7. Let us now suppose that the light emergent from the plate is examined with a Savart's analyser.

On leaving the quartz, the stream may be represented by the components

$$(u, v) = c(\cos \gamma, -\iota \sin \gamma) \text{Exp} \{\iota(nt + \epsilon)\}$$

polarized in planes making angles θ and $\theta + \pi/2$ with the principal section of the plate. Hence if μ be the angle that the principal section of the quartz makes with that of the first plate of the analyser, the stream emerging from the Savart's plate will have components

$$\xi = \{\cos \gamma \cos(\theta + \mu) + \iota \sin \gamma \sin(\theta + \mu)\} \text{Exp} \{\iota(nt + \epsilon' + D/2)\}$$

$$\eta = \{\cos \gamma \sin(\theta + \mu) - \iota \sin \gamma \cos(\theta + \mu)\} \text{Exp} \{\iota(nt + \epsilon' - D/2)\}$$

polarized in the principal sections of the first and second plate, where $D = D_2 - D_1$, D_1, D_2 being the relative retardations of phase introduced by the two plates.

* The above results agree with Beaulard's description of the phenomena (*J. de Phys.* (3) ii. p. 399 (1893)). Croullebois, however, gives a different account of the phenomena in the first and last cases (*loc. cit.* p. 391).

On analysis in an azimuth ϕ , these give a stream

$$\begin{aligned} \Xi &= \xi \cos \phi + \eta \sin \phi \\ &= [\{\cos \gamma \cos (\theta + \mu) + \epsilon \sin \gamma \sin (\theta + \mu)\} \cos \phi \text{Exp} (\epsilon D/2) \\ &\quad + \{\cos \gamma \sin (\theta + \mu) - \epsilon \sin \gamma \cos (\theta + \mu)\} \sin \phi \text{Exp} (-\epsilon D/2)] \\ &\quad \times \text{Exp} \{\epsilon (nt + \epsilon')\}, \end{aligned}$$

and the intensity, obtained by multiplying this by the conjugate expression, is

$$\begin{aligned} I &= \{\cos^2 \gamma \cos^2 (\theta + \mu) + \sin^2 \gamma \sin^2 (\theta + \mu)\} \cos^2 \phi \\ &\quad + \{\cos^2 \gamma \sin^2 (\theta + \mu) + \sin^2 \gamma \cos^2 (\theta + \mu)\} \sin^2 \phi \\ &\quad + \{\cos 2\gamma \sin 2(\theta + \mu) \cos D - \sin 2\gamma \sin D\} \sin \phi \cos \phi \\ &= \frac{1}{2} [1 + \cos 2\gamma \cos 2(\theta + \mu) \cos 2\phi \\ &\quad + \sin 2\phi \{\cos 2\gamma \sin 2(\theta + \mu) \cos D - \sin 2\gamma \sin D\}] \end{aligned}$$

and when, as is ordinarily the case, $\phi = \pi/4$

$$I = \frac{1}{2} \{1 + \cos 2\gamma \sin 2(\theta + \mu) \cos D - \sin 2\gamma \sin D\}.$$

In order, then, that the bands may disappear, we must have

$$\sin 2\gamma = 0 \quad \text{and} \quad \sin 2(\theta + \mu) = 0;$$

that is, the light emergent from the quartz must be plane polarized in the principal section of one of the plates of the Savart's analyser—a result that is obvious from elementary considerations.

Now the light on leaving the quartz is plane polarized (1) when $\delta = 2n\pi$, the plane of polarization being then the same as it was initially; (2) when $\tan \delta/2 = -\tan 2(\alpha - \mu)/\sin 2\beta$, α being now the azimuth of the initial plane of polarization with respect to the principal section of the first plate of the analyser, and in this case $\tan 2(\theta + \mu - \alpha) = -\tan 4(\alpha - \mu)$. Hence in the first case, in order that the bands may disappear, the light must be initially polarized in one of the principal sections of the Savart's plate; and in the second case the principal section of the quartz must be in the azimuth $\mu = k\pi/4 + \alpha/2$.

It follows, then, that if the initial polarization be such that there are no bands before the introduction of the quartz plate, there will be a disappearance of the bands whenever the

relative retardation of phase due to the quartz plate is $\delta = 2n\pi$, and if the principal section of the quartz bisect the angle between the principal sections of the Savart's plate *, there will be a further disappearance whenever $\delta = (2n+1)\pi$.

These results afford a method of setting the principal section of a quartz plate perpendicular to the axis of a spectrometer and of determining the error in the cutting of a plate that is supposed to be normal to the optic axis.

The collimator is furnished with a web, placed by the ordinary method perpendicular to the axis of the instrument. The telescope is turned to one side, and a telescope with a Savart's analyser is inserted in its place with its axis in the direction of that of the collimator, and the analyser is adjusted so that, when plane polarized light passes, the fringes are parallel to the web of the collimator. This sets the plane bisecting the angle between the principal sections of the plates of the analyser perpendicular to the turning axis of the spectrometer. This adjustment being made, the polarizer is turned until the fringes disappear exactly in the centre of the field.

The quartz plate is then placed at the centre of the table of the instrument, and by means of the telescope of the spectrometer its faces are made parallel to the turning axis, and the reading determined for which its normal is along the axis of the collimator.

With the polarizer adjusted as above, the fringes will in general be restored when the light traverses the quartz, but a series of positions can be found for which they vanish over a narrow streak extending across the field, and if the quartz plate be turned about its normal so as to bring its optic axis perpendicular to the turning axis of the spectrometer, there will be, intermediate to these, further positions at which the fringes are absent over a small area at the centre of the field.

In this way the principal section of the quartz is set perpendicular to the turning axis of the instrument, and a

* The second of the above cases becomes identical with first when $\alpha = 0$, if k be even, and when $\alpha = \pi/2$, if k be odd, since the value of δ then becomes $2n\pi$.

determination of the angles of incidence, for which the centres of the zones of disappearance of the fringes are at the centre of the field, will give by calculation or by the graphic method employed by Beaulard* the direction of the optic axis with respect to the faces of the quartz.

XXXVI. *The Use of the Potentiometer on Alternate Current Circuits.* By CHARLES V. DRYSDALE, D.Sc.†

ONE of the greatest inconveniences in connexion with alternate current measurements has always been the question of range. Owing to the fact that all instruments employed for direct indication of alternate currents or P.D. have a square law, their range is small and they become impracticably delicate when low voltages are concerned. There is therefore a very great need for some instrument which, like the direct-current potentiometer, should be capable of measuring P.D.s. and currents of any range with accuracy. It would also be a great convenience if this instrument were capable of indicating the phase as well as the magnitude of the P.D. or current tested.

In attempting to apply the potentiometer principle to alternate current measurements, two processes seem to be possible—(a) the balancing of the alternate current P.D. against an equal direct current P.D. by the employment of some differential balancing device preferably of an electrostatic character, or (b) the balancing of two alternate current P.D.s. against one another. Such propositions as have hitherto been made for an alternate-current potentiometer have, the writer believes, always been based on the former idea, owing to the fact that the latter principle would involve equality of phase as well as of P.D. in the two voltages compared. He understands that some time ago Mr. Swinburne suggested the use of a differential electrometer in connexion with a potentiometer, but has not seen any

* *Thèse de Doctorat*, Marseille, 1893.

† Read January 22, 1903.