38. The modulus equation for

$$\omega = \sqrt{-75} = 5\sqrt{-3}$$

is very readily obtained by substituting for $\lambda\lambda'$ in the modular equation of the 5th order the value $\frac{1}{4}$. The resulting equation is

$$4s^6 + 4\sqrt[3]{2}\,s^5 - 2\sqrt[3]{2}\,s + 1 = 0,$$

which breaks into the quadratics

$$2s^3 + s^2\sqrt[3]{2}\,(1 \pm \sqrt{5}) + s\sqrt[3]{4} - 1 = 0.$$

It is remarkable that the solution of these cubics gives

$$\sqrt[3]{16\kappa^2\kappa'^2} = (2 \pm \sqrt{5})\left[\pm \frac{1 \pm 3\sqrt{5}}{2}\sqrt[6]{5} - 4\sqrt[3]{2}\right],$$

corresponding to $\omega = 5\sqrt{-3}$ and $\dfrac{\sqrt{-3}}{5}$.

The modulus equation in this case suggests $\dfrac{1 - s\sqrt[3]{4} - 2s^3}{s^3\sqrt[3]{2}}$ as auxiliary function, but I have made no attempt to examine what consequences follow from such an assumption.]

On the Function which denotes the excess of the Number of Divisors of a Number which $\equiv 1$, *mod.* 3, *over the number which* $\equiv 2$, *mod.* 3. *By* J. W. L. Glaisher, Sc.D., F.R.S.

[*Read May* 8*th*, 1890.]

1. In a paper "On the Square of Euler's Series,"* use was made of the function $H(n)$ which expresses the excess of the number of divisors of n which $\equiv 1$, mod. 3, over the number of divisors which $\equiv 2$, mod. 3. In the present paper I propose to consider the properties of this function more fully than was possible in a paper in which it occupied only a subordinate place. It will be seen that the function

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* *Proc. Lond. Math. Soc.*, Vol. XXI., pp. 182–215. See especially pp. 198–201, 209–212.

$H(n)$ is very similar, as regards its properties, to the function $E(n)$ which expresses the excess of the number of divisors of n which $\equiv 1$, mod. 4, over the number of divisors which $\equiv 3$, mod. 4 (Vol. xv., pp. 104–122).

2. It is evident that, if $n = 3^{\mu}r$, where r is not $\equiv 0$, mod. 3, then

$$H(n) = H(r);$$

and that, if $n \equiv 2$, mod. 3, then

$$H(n) = 0;$$

for to every divisor $\equiv 1$, mod. 3, there must correspond a conjugate divisor $\equiv 2$, mod. 3.

Also, if $n = 3^{\mu}uv$, where all the prime factors of u are $\equiv 1$, mod. 3, and all the prime factors of v are $\equiv 2$, mod. 3, then

$$H(n) = 0,$$

unless v is a square number, in which case

$$H(n) = \nu(u),$$

where $\nu(u)$ denotes the number of divisors of u.

If a be a prime $\equiv 1$, mod. 3, then

$$H(a^{\alpha}) = \alpha + 1;$$

and if r be a prime $\equiv 2$, mod. 3, then

$$H(r^{\rho}) = 1 \text{ or } 0,$$

according as ρ is even or uneven.

In general, if $\quad n = 3^{\mu}a^{\alpha}b^{\beta}c^{\gamma} \ldots r^{\rho}s^{\sigma}t^{\tau} \ldots,$

where $\quad a, b, c, \ldots$ are primes which $\equiv 1$, mod. 3,

$\quad r, s, t, \ldots$,, ,, $\equiv 2$, mod. 3;

then $H(n)$ is zero, unless $\rho, \sigma, \tau, \ldots$ are all even, and, if they are all even, then

$$H(n) = (\alpha+1)(\beta+1)(\gamma+1) \ldots .$$

Also, if $n = n_1 n_2 n_3 \ldots$, where $n_1, n_2, n_3, \ldots$ are any relatively prime numbers, then

$$H(n) = H(n_1)\, H(n_2)\, H(n_3) \ldots ,$$

These theorems were stated in § 29, p. 199, of the paper on the square of Euler's Series, and two methods of proof were there indicated. The corresponding properties of the function $E(n)$ were proved in the first two sections of the paper in Vol. xv., which has been already referred to.

3. From Jacobi's theorem

$$\Pi_1^\infty (1-q^n) \times \sin x \Pi_1^\infty (1-2q^n \cos 2x + q^{2n})$$
$$= \Sigma_0^\infty (-1)^n q^{\frac{1}{2}n(n+1)} \sin (2n+1)x,$$

we find, by logarithmic differentiation,

$$\cot x + \frac{4q \sin 2x}{1-2q\cos 2x+q^2} + \frac{4q^2 \sin 2x}{1-2q^2\cos 2x+q^4} + \frac{4q^3 \sin 2x}{1-2q^3\cos 2x+q^6} + \&c.$$
$$= \frac{\cos x - 3q\cos 3x + 5q^3 \cos 5x - 7q^6 \cos 7x + 9q^{10}\cos 9x - \&c.}{\sin x - q\sin 3x + q^3 \sin 5x - q^6 \sin 7x + q^{10} \sin 9x - \&c.}.$$

Putting $x = \frac{1}{3}\pi$, this equation gives

$$1 + \frac{6q}{1+q+q^2} + \frac{6q^2}{1+q^2+q^4} + \frac{6q^3}{1+q^3+q^6} + \&c.$$
$$= \frac{1+6q+5q^3-7q^6-18q^{10}-11q^{15}+13q^{21}+\&c.}{1-q^3-q^6+q^{15}+q^{21}-q^{36}-q^{45}+\&c.}.$$

The left-hand member

$$= 1 + 6\frac{q-q^2}{1-q^3} + 6\frac{q^2-q^4}{1-q^6} + 6\frac{q^3-q^6}{1-q^9} + \&c.$$
$$= 1 + 6H(1)\,q + 6H(2)\,q^2 + 6H(3)\,q^3 + 6H(4)\,q^4 + \&c.,$$

whence $$\Sigma_1^\infty H(n)\,q^n = \frac{q+q^3-q^6-3q^{10}-2q^{15}+2q^{21}+5q^{28}+\&c.}{1-q^3-q^6+q^{15}+q^{21}-q^{36}-q^{45}+\&c.}.$$

Multiplying up by the denominator, and observing that $H(n) = 0$, if $n \equiv 2$, mod. 3, we find

$$q - 3q^{10} + 5q^{28} - 7q^{55} + \&c. + q^3 - q^6 - 2q^{15} + 2q^{21} + 3q^{36} - 3q^{45} + \&c.$$
$$= (1-q^3-q^6+q^{15}+q^{21}-\&c.)$$
$$\times \{H(1)\,q + H(3)\,q^3 + H(4)\,q^4 + H(6)\,q^6 + H(7)\,q^7 + \&c.\},$$

which breaks up into the two equalities

$$1-3q^9+5q^{27}-7q^{54}+9q^{90}-\&c.$$
$$=(1-q^3-q^6+q^{15}+q^{21}-\&c.)\,\{H(1)+H(4)\,q^3+H(7)\,q^6+\&c.\},$$

$$q^3-q^6-2q^{15}+2q^{21}+3q^{36}-3q^{45}-\&c.$$
$$=(1-q^3-q^6+q^{15}+q^{21}-\&c.)\,\{H(3)\,q^3+H(6)\,q^6+H(9)\,q^9+\&c.\},$$

giving the formulæ

$$H(1)+H(4)\,q+H(7)\,q^2+H(10)\,q^3+\&c.$$
$$=\frac{1-3q^3+5q^9-7q^{18}+9q^{30}-\&c.}{1-q-q^2+q^5+q^7-\&c.},$$

$$H(1)\,q+H(2)\,q^2+H(3)\,q^3+H(4)\,q^4+\&c.$$
$$=\frac{q-q^2-2q^5+2q^7+3q^{12}-3q^{15}+\&c.}{1-q-q^2+q^5+q^7-\&c.}.$$

4. By equating coefficients, we obtain the following recurring formulæ :—

(i.) If $n\equiv 1$, mod. 3, then

$$H(n)-H(n-3)-H(n-6)+H(n-15)+H(n-21)-\&c.$$

$=0$ or $(-1)^r(2r+1)$, according as n is not of the form $\tfrac{1}{2}(3r+1)(3r+2)$, or is equal to $\tfrac{1}{2}(3r+1)(3r+2)$.

(ii.) For all values of n,

$$H(n)-H(n-1)-H(n-2)+H(n-5)+H(n-7)-\&c.$$

$=0$, if n is not of the form $\tfrac{1}{2}r\,(3r\pm 1)$,

$=(-1)^{r-1}\,r$, if $n=\tfrac{1}{2}r\,(3r-1)$,

$=(-1)^r\,r$, if $n=\tfrac{1}{2}r\,(3r+1)$.

The numbers $\tfrac{1}{2}(3r+1)(3r+2)$ which occur in the first formula are the triangular numbers which are not divisible by 3.

The numbers $\tfrac{1}{2}r\,(3r\pm1)$ which occur in the second formula are the pentagonal numbers, *i.e.*, the thirds of the triangular numbers which are divisible by 3.

5. The final results in § 3 bear a close analogy to the following formulæ relating to the function $E(n)$, which were given in §§ 7 and 9 of the paper in Vol. xv. (pp. 109, 110):

$$E(1)+E(5)\,q+E(9)\,q^2+E(13)\,q^3+\&c. = \frac{1-3q^2+5q^6-7q^{12}+\&c.}{1-2q+2q^4-2q^9+\&c.},$$

$$E(1)\,q+E(2)\,q^2+E(3)\,q^3+E(4)\,q^4+\&c. = \frac{q-q^3-2q^6+2q^{10}-\&c.}{1-q-q^3+q^6+q^{10}-\&c.}.$$

6. The last formula may be written

$$1-4E(1)\,q+4E(2)\,q^2-4E(3)\,q^3+\&c. = \frac{1-3q+5q^3-7q^6+9q^{10}-\&c.}{1+q+q^3+q^6+q^{10}+\&c.};$$

so that we have

$$1-3q+5q^3-7q^6+\&c. = (1+q+q^3+q^6+q^{10}+\&c.)$$
$$\times\{1-4E(1)\,q+4E(2)\,q^2-4E(3)\,q^3+\&c.\},$$

$$1-3q^2+5q^6-7q^{12}+\&c. = (1-2q+2q^4-2q^9+\&c.)$$
$$\times\{E(1)+E(5)\,q+E(9)\,q^2+E(13)\,q^3+\&c.\},$$

$$1-3q^3+5q^9-7q^{18}+\&c. = (1-q-q^2+q^5+q^7-\&c.)$$
$$\times\{H(1)+H(4)\,q+H(7)\,q^2+H(10)\,q^3+\&c.\}.$$

Expressing the left-hand members of these equations as products, we have also

$$1-4E(1)\,q+4E(2)\,q^2-4E(3)\,q^3+\&c. = \left\{\frac{(1-q)(1-q^2)(1-q^3)\dots}{(1+q)(1+q^2)(1+q^3)\dots}\right\}^2,$$

$$E(1)+E(5)\,q+E(9)\,q^2+E(13)\,q^3+\&c. = \{(1+q)(1+q^2)(1+q^3)\dots\}^4$$
$$\times\{(1-q)(1-q^2)(1-q^3)\dots\}^2,$$

$$H(1)+H(4)\,q+H(7)\,q^2+H(10)\,q^3+\&c.$$
$$= \frac{\{(1-q^3)(1-q^6)(1-q^9)\dots\}^3}{(1-q)(1-q^2)(1-q^3)\dots}.$$

7. It was shown in the paper on the square of Euler's Series (Vol. xxi., p. 198) that

$$\Sigma_0^\infty\, H(4n+1)\,q^{4n+1} = \Sigma_0^\infty\, q^{m^2}\times\Sigma_{-\infty}^{\infty}\, q^{12n^2},$$

$$\Sigma_0^\infty\, H(4n+3)\,q^{4n+3} = \Sigma_{-\infty}^{\infty}\, q^{4n^2}\times\Sigma_0^\infty\, q^{3m^2},$$

m denoting any uneven number; and also (p. 211) that

$$\Sigma_0^\infty H(12n+1)\,q^{12n+1} = \Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2}.$$

From the last equation we may deduce that

$$\Sigma_0^\infty H(24n+1)\,q^{24n+1} = \Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{48n^2},$$

$$\Sigma_0^\infty H(24n+13)\,q^{24n+13} = \Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12m^2}.$$

These equations express the theorem:—The number of representations of a number n as the sum of an uneven square not divisible by 3, and twelve times a square is $4H(n)$. If the latter square is even, n must $\equiv 1$, mod. 24; and if uneven, n must $\equiv 13$, mod. 24.*

8. From the formulæ in the preceding article, we find

$$H(1)+H(5)\,q+H(9)\,q^2+\&c.$$
$$= (1+q^2+q^6+q^{12}+\&c.)(1+2q^3+2q^{12}+2q^{27}+\&c.),$$

$$H(3)+H(7)\,q+H(11)\,q^2+\&c.$$
$$= (1+q^6+q^{18}+q^{30}+\&c.)(1+2q+2q^4+2q^9+\&c.),$$

$$H(1)+H(13)\,q+H(25)\,q^2+\&c.$$
$$= (1+q^2+q^4+q^{10}+\&c.)(1+2q+2q^4+2q^9+\&c.),$$

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* In the same manner we may deduce from the same section (p. 211) of the paper on the square of Euler's Series the results

$$\Sigma_0^\infty G(24n+1)\,q^{24n+1} = \Sigma_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{48n^2},$$

$$-\Sigma_0^\infty G(24n+13)\,q^{24n+13} = \Sigma_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12m^2}.$$

By combining these formulæ with those in the text, we find

$$\text{(i.)}\quad \Sigma_{-\infty}^{\infty} q^{(12n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12m^2} = \tfrac{1}{2}\Sigma_0^\infty \left\{H(24n+13)-G(24n+13)\right\} q^{24n+13},$$

$$\text{(ii.)}\quad \Sigma_{-\infty}^{\infty} q^{(12n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{48n^2} = \tfrac{1}{2}\Sigma_0^\infty \left\{H(24n+1)+G(24n+1)\right\} q^{24n+1},$$

$$\text{(iii.)}\quad \Sigma_{-\infty}^{\infty} q^{(12n+5)^2} \times \Sigma_{-\infty}^{\infty} q^{12m^2} = \tfrac{1}{2}\Sigma_0^\infty \left\{H(24n+13)+G(24n+13)\right\} q^{24n+13},$$

$$\text{(iv.)}\quad \Sigma_{-\infty}^{\infty} q^{(12n+5)^2} \times \Sigma_{-\infty}^{\infty} q^{48n^2} = \tfrac{1}{2}\Sigma_0^\infty \left\{H(24n+1)-G(24n+1)\right\} q^{24n+1}.$$

These formulæ show that the number of representations of a number r as the sum of

(i.) the square of a number $\equiv 1$ or 11, mod. 12, and 12 times an uneven square;

(ii.) the square of a number $\equiv 1$ or 11, mod. 12, and 12 times an even square;

(iii.) the square of a number $\equiv 5$ or 7, mod. 12, and 12 times an uneven square;

(iv.) the square of a number $\equiv 5$ or 7, mod. 12, and 12 times an even square,

is $2\{H(n)-G(n)\}$ in (i.) and (iv.), and $2\{H(n)+G(n)\}$ in (ii.) and (iii.). Only numbers $\equiv 13$, mod. 12, can be expressed by the forms (i.) and (iii.), and only numbers $\equiv 1$, mod. 12, by the forms (ii.) and (iv.). These results are equivalent to the theorem given in Vol. xxi., p. 212 (§ 50).

$$H(1)+H(25)\,q+H(49)\,q^2+\&c.$$
$$=(1+q+q^2+q^6+\&c.)(1+2q^2+2q^8+2q^{18}+\&c.),$$

$$H(13)+H(37)\,q+H(61)\,q^2+\&c.$$
$$=2\,(1+q+q^2+q^6+\&c.)(1+q^4+q^{12}+q^{24}+\&c.).$$

Since

$$1+q+q^2+q^5+q^7+\&c.$$
$$=(1+q)(1+q^2)(1-q^3)(1+q^4)(1+q^5)(1-q^6)\,...,$$

$$1+q+q^3+q^6+q^{10}+\&c.$$
$$=(1+q)(1+q^2)(1+q^3)...(1-q^2)(1-q^4)(1-q^6)...$$
$$=(1+q)(1+q^3)(1-q^4)(1+q^5)(1+q^7)(1-q^8)...,$$

$$1+2q+2q^4+2q^9+2q^{16}+\&c.$$
$$=\frac{(1+q)(1-q^2)(1+q^3)(1-q^4)...}{(1-q)(1+q^2)(1-q^3)(1+q^4)...}$$
$$=(1+q)^2\,(1-q^2)(1+q^3)^2\,(1-q^4)(1+q^5)^2\,(1-q^6)...,$$

we may at once express the above five H-series as products of binomial factors.

9. The series in which the coefficient of the general term is $H(m)$, m denoting any uneven number, may be expressed in terms of elliptic functions by the formulæ

$$k\rho\,\mathrm{cd}\,\tfrac{1}{3}K = k\rho\,\mathrm{sn}\,\tfrac{2}{3}K \;= 2\sqrt{3}\,.\,\Sigma_1^\infty\,H(m)\,q^{\frac{1}{4}m},$$

$$k\rho\,\mathrm{cn}\,\tfrac{1}{3}K = kk'\rho\,\mathrm{sd}\,\tfrac{2}{3}K = 2\sqrt{3}\,.\,\Sigma_1^\infty\,(-1)^{\frac{1}{2}(m-1)}\,H(m)\,q^{\frac{1}{4}m}\,;$$

and the series in which the coefficient of the general term is $H(n)$ may be expressed in terms of Zeta functions by the formula

$$-\rho\,\mathrm{zc}\,\tfrac{1}{3}K = \rho\,\mathrm{zs}\,\tfrac{2}{3}K = \frac{1}{\sqrt{3}}\left\{1+6\,\Sigma_1^\infty\,H(n)\,q^{2n}\right\},$$

where
$$\mathrm{zc}\,x = \frac{i\pi}{2K}+\mathrm{zn}\,(x+K+iK') = \mathrm{zn}\,x-\frac{\mathrm{sn}\,x\,\mathrm{dn}\,x}{\mathrm{cn}\,x},$$

$$\mathrm{zs}\,x = \frac{i\pi}{2K}+\mathrm{zn}\,(x+iK') \qquad = \mathrm{zn}\,x+\frac{\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{sn}\,x},$$

zn x being the same as Jacobi's Zeta function $Z(x)$.

10. I give below a table of the values of $\Pi(n)$ as far as $n = 100$. The values of $E(n)$ are added for the sake of comparison.

n	$\Pi(n)$	$E(n)$	n	$\Pi(n)$	$E(n)$	n	$\Pi(n)$	$E(n)$
1	1	1	35	0	0	69	0	0
2	0	1	36	1	1	70	0	0
3	1	0	37	2	2	71	0	0
4	1	1	38	0	0	72	0	1
5	0	2	39	2	0	73	2	2
6	0	0	40	0	2	74	0	2
7	2	0	41	0	2	75	1	0
8	0	1	42	0	0	76	2	0
9	1	1	43	2	0	77	0	0
10	0	2	44	0	0	78	0	0
11	0	0	45	0	2	79	2	0
12	1	0	46	0	0	80	0	2
13	2	2	47	0	0	81	1	1
14	0	0	48	1	0	82	0	2
15	0	0	49	3	1	83	0	0
16	1	1	50	0	3	84	2	0
17	0	2	51	0	0	85	0	4
18	0	1	52	2	2	86	0	0
19	2	0	53	0	2	87	0	0
20	0	2	54	0	0	88	0	0
21	2	0	55	0	0	89	0	2
22	0	0	56	0	0	90	0	2
23	0	0	57	2	0	91	4	0
24	0	0	58	0	2	92	0	0
25	1	3	59	0	0	93	2	0
26	0	2	60	0	0	94	0	0
27	1	0	61	2	2	95	0	0
28	2	0	62	0	0	96	0	0
29	0	2	63	2	0	97	2	2
30	0	0	64	1	1	98	0	1
31	2	0	65	0	4	99	0	0
32	0	1	66	0	0	100	1	3
33	0	0	67	2	0			
34	0	2	68	0	2			