ART. XXXIV.—*On the Deviation Produced by Prisms*; by
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By consulting all available standard reference books and scientific journals, the writer has been unable to find a discussion of the general case of refraction of oblique rays by a single prism which will enable the reader to form a clear or adequate conception of the dependence of the total deviation upon the angles defining the direction of the incident ray. The reference books give but little more than a formula connecting the deviation (D) of the oblique ray with the deviation (E) of the projection of the ray on a principal section of the prism, and with the angle (η_1) which the incident ray makes with such a section. Furthermore, the formulæ to be found in the books are not equivalent, most authors write

$$\cos \frac{1}{2} D = \cos \frac{1}{2} E \cos \eta_1,^*$$

(apparently following Heath), and one author, at least, writes

$$\sin \frac{1}{2} D = \sin \frac{1}{2} E \cos \eta_1.^\dagger$$

Neither equation throws much light on the subject because E is a function of η_1 and of another angle, so that D is an implicit function of two independent variables. On the other hand, the journal articles treat, in a fragmentary manner, of certain special problems connected with the general case of the deviation of oblique rays, consequently it requires a disproportionate amount of time to find out what has already been done. Unfortunately not all of these articles are trustworthy. Since this state of affairs exists, and since the question is not without interest to students of geometrical optics, it seems desirable to collect in one place the correct results hitherto obtained, to supplement these results by original investigation, when necessary, and to present in graphical form all the facts amenable to such treatment. In other words, the endeavor will be made in the present paper to treat the problem of prismatic deviation in as complete and thorough a manner as possible.

The entire discussion will have reference to the case of a single prism with unlimited plane faces, and surrounded by one medium whose absolute index of refraction is less than the

* S. Czapski: Winkelmann's "Handbuch der Physik," vol. vi, p. 189, 1906; R. S. Heath: "Geometrical Optics," p. 32, 1887; H. Koenen: Kayser's "Handbuch der Spectroscopie," vol. i, p. 262, 1900; F. Löwe: M. von Rohr's "Die Theorie der optischen Instrumente," vol. i, p. 428, 1904; J. P. C. Southall: "The Principles and Methods of Geometrical Optics," p. 126, 1910.

† J. Walker: "The Analytical Theory of Light," p. 12, 1904.

absolute index of the material of the prism. The relative index of the prism will be denoted by n . For the sake of uniformity, the definitions of the quantities involved will be the same as given in Southall's "Geometrical Optics." Also, with few exceptions, the notation of this author will be followed.

In order to fix the ideas and to make the definitions of the symbols used in the following pages as clear as possible, great-

FIG. 1.

circle and other arcs were drawn to scale on a hemispherical blackboard, the points of intersection were lettered, and the whole was then photographed. In figure 1, which is a reproduction of this photograph, let the center of the sphere be denoted by O . The great-circle N_1N_2 corresponds to a principal plane of the prism. One pole of this circle is shown at H . The normals to the incidence and emergence faces are represented by ON_1 and ON_2 , in the order named. With N_1 and N_2 as centers let arcs of small-circles, AC and BC , be described having such spherical radii as to subtend at the center of the sphere angles equal to the critical angle, $c = \sin^{-1} \frac{1}{n}$.

[Only half of each arc is shown in the figure. The points homologous to C and H will be denoted by C' and H' .] These small-circles must intersect if transmission is to be possible. The extremity (S) of the ray inside the prism must not lie outside of the smaller region bounded by the two small-circle arcs, since total reflection is to be excluded.

$$\begin{aligned}
 \angle N_1ON_2 &= \beta = \text{angle of prism,} \\
 \angle N_1OP &= a_1 = \text{“ incidence, 1st face,} \\
 \angle N_1OS &= a_1' = \text{“ “ refraction, “ “} \\
 \angle N_2OS &= a_2 = \text{“ “ incidence, 2d “} \\
 \angle N_2OQ &= a_2' = \text{“ “ emergence, “ “} \\
 \angle QOP &= D = \text{deviation of oblique ray,} \\
 \angle qOp &= E = \text{“ “ projected ray,} \\
 \angle pOP &= \eta_1, \\
 \angle sOS &= \eta_1', \\
 \angle qOQ &= \eta_2, \quad \left. \vphantom{\begin{matrix} \angle pOP \\ \angle sOS \\ \angle qOQ \end{matrix}} \right\} = \text{altitudes,} \\
 \angle N_1Op &= \gamma_1, \\
 \angle N_1Os &= \gamma_1', \\
 \angle N_2Os &= \gamma_2, \\
 \angle N_2Oq &= \gamma_2', \quad \left. \vphantom{\begin{matrix} \angle N_1Op \\ \angle N_1Os \\ \angle N_2Os \\ \angle N_2Oq \end{matrix}} \right\} = \text{azimuths.}
 \end{aligned}$$

Angles will be counted positive when generated by a right-handed rotation about the axis $\vec{OH}$. Also, the only angles in the above list which may exceed $\frac{1}{2}\pi$ and still correspond to physical reality are β , D and E . [For further details of fig. 1, see p. 407.]

$$\begin{aligned}
 \beta &= \gamma_1' - \gamma_2 & (1) \\
 E &= \gamma_1 - \gamma_2' - \beta & (2)
 \end{aligned}$$

Since the prism is surrounded by the same medium

$$\eta_2' = \eta_1.$$

This fact was noted by Bravais as early as 1845. He says: “... ce qui nous donne cette *première loi de la réfraction oblique dans les prismes* ‘Le rayon émergent et le rayon immergent sont également inclinés sur le plan de la section principale.’” Consequently the triangle QPH is isosceles, and hence

$$\sin \frac{1}{2}D = \sin \frac{1}{2}E \cos \eta_1 \quad (3)$$

This relation was used by Bravais for the case of minimum deviation. He gave

$$\sin \frac{1}{2}\Delta = \sin \frac{1}{2}D \cos H$$

on page 83, *loc. cit.* Since, in general, $\cos \eta_1 < 1$ relation (3) shows that

$$D < E.$$

In spite of this obvious inequality, several modern authors write, with Heath,

$$\cos \frac{1}{2}D = \cos \frac{1}{2}E \cos \eta_1.$$

* A. Bravais: Journal de l'Éc. Polytechn. 18, 30^e Cahier, p. 79, 1845.

[See prefatory remarks.] Apparently, the only other fact which flows directly from (3) alone is this: if η_1 is kept constant while the azimuths are allowed to vary, D and E will decrease or increase together, so that if either passes through a stationary value so also will the other pass through the same kind of critical value simultaneously.

In addition to the relations given above, the following equations will be found useful:

$$\sin \eta_1 = n \sin \eta_1' \quad (4)$$

$$\sin \gamma_1 = \nu \sin \gamma_1' \quad (5)$$

$$\sin \gamma_2' = \nu \sin \gamma_2 \quad (6)$$

where $\nu = n \cos \eta_1' \sec \eta_1 = \sqrt{n^2 + (n^2 - 1) \tan^2 \eta_1}$.

Since, as will be shown later, the general deviation may be expressed as an explicit function either of γ_1' and η_1' or of γ_1 and η_1 (c or n and β being supposed given), two methods, at least, for exhibiting all possible variations of D suggest themselves. One plan would be to construct a surface outside of the sphere of figure 1 which would be the locus of the extremity of the extension either of the radius OS or of OP, the length of the extended segment to be taken proportional to the value of the deviation at the points S or P respectively. This process would be analogous to the scheme often used for showing the distribution of static electricity over the surface of an insulated conductor. For various reasons the model with extended radii would be inconvenient for publication. The other method is to construct a surface with rectangular coördinates, considering D as dependent variable, (axis of z), with γ_1' and η_1' or with γ_1 and η_1 as independent variables, (axes of x and y). In the following pages the properties of the surfaces referred to rectangular frames will be discussed, and diagrams of certain plane sections of these surfaces will be given.

Instead of deriving an explicit function of D in terms of γ_1' and η_1' it will be found preferable to replace the coördinate γ_1' by λ , where $\lambda = \gamma_1' - \frac{1}{2}\beta$. Furthermore, all square-root operators will be understood as meaning only the positive root. From (2)

$$\begin{aligned} \cos E = & \cos \beta \sin \gamma_1 \sin \gamma_2' + \sin \beta \sin \gamma_1 \cos \gamma_2' - \sin \beta \cos \gamma_1 \sin \gamma_2' \\ & + \cos \beta \cos \gamma_1 \cos \gamma_2'. \end{aligned}$$

From (5)

$$\sin \gamma_1 = \nu \sin (\lambda + \tfrac{1}{2}\beta) ;$$

from (1) and (6)

$$\sin \gamma_2' = \nu \sin (\lambda - \tfrac{1}{2}\beta)$$

therefore, since $\gamma_1 \times \frac{1}{2}\pi$ and $\gamma_2' \times \frac{1}{2}\pi$,

$$\left. \begin{aligned} \cos E = & \nu^2 \cos \beta \sin (\lambda + \tfrac{1}{2}\beta) \sin (\lambda - \tfrac{1}{2}\beta) \\ & + \nu \sin \beta \sin (\lambda + \tfrac{1}{2}\beta) \sqrt{1 - \nu^2 \sin^2 (\lambda - \tfrac{1}{2}\beta)} \\ & - \nu \sin \beta \sin (\lambda - \tfrac{1}{2}\beta) \sqrt{1 - \nu^2 \sin^2 (\lambda + \tfrac{1}{2}\beta)} \\ & + \cos \beta \sqrt{[1 - \nu^2 \sin^2 (\lambda + \tfrac{1}{2}\beta)][1 - \nu^2 \sin^2 (\lambda - \tfrac{1}{2}\beta)]} \end{aligned} \right\} \quad (7)$$

From (3) and (4)

$$\sin^2 c \cos D = \sin^2 \eta_1' + (\sin^2 c - \sin^2 \eta_1') \cos E,$$

hence

$$\left. \begin{aligned} \sin^2 c \cos D = & \sin^2 \eta_1' + \cos \beta \sin (\lambda + \tfrac{1}{2}\beta) \sin (\lambda - \tfrac{1}{2}\beta) \cos^2 \eta_1' \\ & + \sin \beta \sin (\lambda + \tfrac{1}{2}\beta) \cos \eta_1' \sqrt{\cos^2 (\lambda - \tfrac{1}{2}\beta) \cos^2 \eta_1' - \cos^2 c} \\ & - \sin \beta \sin (\lambda - \tfrac{1}{2}\beta) \cos \eta_1' \sqrt{\cos^2 (\lambda + \tfrac{1}{2}\beta) \cos^2 \eta_1' - \cos^2 c} \\ & + \cos \beta \sqrt{[\cos^2 (\lambda - \tfrac{1}{2}\beta) \cos^2 \eta_1' - \cos^2 c][\cos^2 (\lambda + \tfrac{1}{2}\beta) \cos^2 \eta_1' - \cos^2 c]} \end{aligned} \right\} \quad (8)$$

This is the equation of the $\lambda \eta_1' D$ surface, in the sense that all values of D which correspond to physical reality must satisfy relation (8). Inspection of (7) and (8) shows that, when η_1' is kept constant, E and D are symmetrical with respect to the locus $\lambda = 0$, i. e., the great-circle H' LH of fig. 1. [This property may also be deduced by simple geometrical considerations.] This symmetry shows that E and D have stationary values for $\lambda = 0$. That these critical values are minima is an immediate consequence of the following premises: (a) the form of (7) does not alter when ν assumes different values, such as $\nu = n$ and $\nu > n$; (b) a prism of index n and angle β with a ray of obliquity η_1' is equivalent to a prism of greater effective index ν and angle β with a ray of zero altitude; (c) when $\lambda = 0$, ($\gamma_1' = -\gamma_2 = \frac{1}{2}\beta$), the ray in a principal plane experiences minimum deviation, as is very well known. In fig. 1, $Q_0 P_0$ represents the minimum of all deviations along the small-circle UV, whose altitude, LM, has any constant value. The greatest deviations corresponding to this altitude and having physical meaning pertain to the points U and V.

To find out how the minima of D vary with η_1' , (along the arc H' LH), it is only necessary to put $\lambda = 0$ in equation (8) and to differentiate D_0 with respect to η_1' , where D_0 denotes the value of D when $\lambda = 0$. Then

$$\frac{1}{\sin \eta_1'} \cdot \frac{dD_0}{d\eta_1'} = \frac{2 \sin \beta \sin \tfrac{1}{2}\beta (\cos \tfrac{1}{2}\beta \cos \eta_1' - \sqrt{\cos^2 \tfrac{1}{2}\beta \cos^2 \eta_1' - \cos^2 c})^2}{\sin^2 c \sin D_0 \sqrt{\cos^2 \tfrac{1}{2}\beta \cos^2 \eta_1' - \cos^2 c}}$$

Consequently

$$\frac{1}{\sin \eta_1'} \cdot \frac{dD_0}{d\eta_1'}$$

is always positive, so that the minimum deviation increases with the obliquity of the ray. This includes the theorem that for symmetric passage in a principal plane the deviation is an absolute minimum. Since D_0 increases with η_1' , or η_1 , the equation

$$\sin \frac{1}{2}E_0 = \sin \frac{1}{2}D_0 \sec \eta_1$$

shows that E_0 increases with η_1 . [The same conclusion follows at once from the usual relation $\sin \frac{1}{2}(E_0 + \beta) = \nu \sin \frac{1}{2}\beta$, since ν increases with η_1 and $\frac{1}{2}(E_0 + \beta)$ is acute.]

The first valid proof of the existence of the absolute minimum seems to have been given by Sir Joseph Larmor,* although the truth involved was generally recognized for a number of years before.† Nevertheless, Larmor's article has been overlooked by later writers because, apparently without exception, they have continued to give a fallacious demonstration based on the incorrect formula

$$\cos \frac{1}{2}D = \cos \frac{1}{2}E \cos \eta_1.$$

[See foot-note page 389.] In the year 1909 the present writer published a rigorous proof of the theorem of the absolute minimum.‡ He desires to state that, at the time, he too was not aware of the existence of Larmor's article. However, the force of the later paper is not entirely lost because (a) the demonstration is altogether different from Larmor's, (b) its main object was to direct attention to the fallacy of the cosine relation, and (c) it gives a table of minimum deviations for the special case of $\beta = \frac{1}{3}\pi$ and $n = 1.65$.

Keeping η_1' constant, as λ increases from zero, $\cos(\lambda + \frac{1}{2}\beta) \cos \eta_1' - \cos c$ decreases until it passes through zero and eventually becomes negative. In equation (8), the second and third radicals become imaginary when

$$\cos(\lambda + \frac{1}{2}\beta) \cos \eta_1' < \cos c,$$

hence the equation of the small-circle boundary C'AC, of the region of real D , may be written

$$\cos(\lambda + \frac{1}{2}\beta) \cos \eta_1' = \cos c \quad (9)$$

In like manner, the equation of the small-circle C'BC is found to be

$$\cos(\lambda - \frac{1}{2}\beta) \cos \eta_1' = \cos c. \quad (10)$$

* J. Larmor: Proc. Camb. Phil. Soc., vol. ix, p. 109, 1896.

† R. S. Heath: A Treatise on Geometrical Optics, Cambridge, 1887, p. 32.

‡ H. S. Uhler: This Journal, vol. xxvii, p. 223, 1909.

[Obviously, formulæ (9) and (10) may be obtained directly from the relations $\cos c = \cos \gamma_1' \cos \eta_1'$ and $\cos c = \cos \gamma_2 \cos \eta_1'$.] The equation of the $\lambda \eta_1'$ boundary locus may also be written

$$\sin \lambda = \pm [\cos \tfrac{1}{2}\beta \sqrt{\sin(c + \eta_1') \sin(c - \eta_1')} - \sin \tfrac{1}{2}\beta \cos c] \sec \eta_1', \quad (11)$$

the upper or lower sign to be taken according as points on C'AC or C'BC are desired respectively. When referred to the rectangular coördinate frame $\lambda \eta_1' D$, the four acute angles which these boundary loci make with the η_1' -axis are each numerically equal to

$$\tan^{-1} [\sec c \cot \tfrac{1}{2}\beta \sqrt{\sin(c + \tfrac{1}{2}\beta) \sin(c - \tfrac{1}{2}\beta)}] \quad (12)$$

On the other hand, the λ -axis is normal to the small-circle boundaries. The coördinates of the points A and B are respectively

$$\left. \begin{aligned} \lambda &= +(c - \tfrac{1}{2}\beta), & \eta_1' &= 0, \\ \lambda &= -(c - \tfrac{1}{2}\beta), & \eta_1' &= 0. \end{aligned} \right\} \quad (13)$$

and

At C and C',

$$\left. \begin{aligned} \lambda &= 0 \\ \eta_1' &= \pm \cos^{-1}(\cos c \sec \tfrac{1}{2}\beta). \end{aligned} \right\} \quad (14)$$

For simultaneous grazing incidence and emergence, formula (8) reduces to

$$\sin \tfrac{1}{2}D = \cot c \sin \tfrac{1}{2}\beta. \quad (15)$$

For grazing incidence or grazing emergence in a principal plane equation (8) leads to

$$\sin \tfrac{1}{2}D = [\sqrt{\cos \tfrac{1}{2}\beta \cos(c - \tfrac{1}{2}\beta)} - \sqrt{\sin \tfrac{1}{2}\beta \sin(c - \tfrac{1}{2}\beta)}] \sqrt{\tfrac{1}{2} \csc c \sin \beta} \quad (16)$$

Equation (16) is equivalent to the following formulæ, which are suitable for logarithmic computation, namely

$$\left. \begin{aligned} \cos(D + \beta) &= \csc c \sin(c - \beta), \\ \sin \tfrac{1}{2}(D + \beta) &= \sqrt{\csc c \sin \tfrac{1}{2}\beta \cos(c - \tfrac{1}{2}\beta)} \end{aligned} \right\} \quad (17)$$

Finally, for the absolute minimum of deviation, ϵ_0 , (8) produces

$$\sin \tfrac{1}{2}\epsilon_0 = [\cos \tfrac{1}{2}\beta - \sqrt{\sin(c + \tfrac{1}{2}\beta) \sin(c - \tfrac{1}{2}\beta)}] \csc c \sin \tfrac{1}{2}\beta, \quad (18)$$

which is equivalent to the familiar formula

$$n = \frac{\sin \tfrac{1}{2}(\epsilon_0 + \beta)}{\sin \tfrac{1}{2}\beta} \quad (19)$$

Of the various plane sections of the $\lambda \eta_1' D$ surface which may be imagined, the curves of constant deviation seem best

adapted to showing the general contour of this surface. The next problem is, therefore, to derive a formula for λ in terms of β , ν , and E . Division of (6) by (5) gives

$$\text{hence} \quad \csc \gamma_1' \sin \gamma_2 = \csc \gamma_1 \sin \gamma_2', \\ (\sin \gamma_2' \pm \sin \gamma_1) \csc \gamma_1 = (\sin \gamma_2 \pm \sin \gamma_1') \csc \gamma_1',$$

whence

$$\sin \frac{1}{2}(\gamma_2' + \gamma_1) \cos \frac{1}{2}(\gamma_2' - \gamma_1) = \nu \sin \frac{1}{2}(\gamma_2 + \gamma_1') \cos \frac{1}{2}(\gamma_2 - \gamma_1') \\ \text{and} \quad \cos \frac{1}{2}(\gamma_2' + \gamma_1) \sin \frac{1}{2}(\gamma_2' - \gamma_1) = \nu \cos \frac{1}{2}(\gamma_2 + \gamma_1') \sin \frac{1}{2}(\gamma_2 - \gamma_1').$$

Introducing relations (1) and (2) it results that

$$\sin \frac{1}{2}(\gamma_2' + \gamma_1) = \frac{\nu \cos \frac{1}{2}\beta \sin \lambda}{\cos \frac{1}{2}(E + \beta)}$$

and

$$\cos \frac{1}{2}(\gamma_2' + \gamma_1) = \frac{\nu \sin \frac{1}{2}\beta \cos \lambda}{\sin \frac{1}{2}(E + \beta)}$$

$$\text{so that} \quad \frac{\nu^2 \cos^2 \frac{1}{2}\beta}{\cos^2 \frac{1}{2}(E + \beta)} \sin^2 \lambda + \frac{\nu^2 \sin^2 \frac{1}{2}\beta}{\sin^2 \frac{1}{2}(E + \beta)} \cos^2 \lambda = 1 \quad (20)$$

Consequently

$$\cos 2\lambda = \frac{\nu^2 [\sin^2 (\frac{1}{2}E + \beta) + \sin^2 \frac{1}{2}E] - \sin^2 (E + \beta)}{2\nu^2 \sin \frac{1}{2}E \sin (\frac{1}{2}E + \beta)},$$

which is rational, or

$$\tan \lambda = \pm \cot \frac{1}{2}(E + \beta) \sqrt{\frac{\sin^2 \frac{1}{2}(E + \beta) - \nu^2 \sin^2 \frac{1}{2}\beta}{\nu^2 \cos^2 \frac{1}{2}\beta - \cos^2 \frac{1}{2}(E + \beta)}} \quad (21)$$

For $\lambda = 0$ the last equation reduces to the well-known form

$$\sin \frac{1}{2}(E_0 + \beta) = \nu \sin \frac{1}{2}\beta \quad (22)$$

Formula (21) may be adapted to logarithmic computation by the introduction of certain auxiliary angles. Assuming a numerical value for η_1' , η_1 can be found at once from (4), since c or n is supposed given. Then define an angle ψ by the relation

$$\sin \psi = n \cos \eta_1' \sec \eta_1 \sin \frac{1}{2}\beta, \\ \text{i. e.,} \quad \sin \psi = \nu \sin \frac{1}{2}\beta. \quad (23)$$

Also let $\sin \phi$ be determined by means of

$$\tan \phi = \tan c \cos \eta_1 \sqrt{\sin (\frac{1}{2}E + \frac{1}{2}\beta + \psi) \sin (\frac{1}{2}E + \frac{1}{2}\beta - \psi)}, \quad (24)$$

which is possible because β and D are given constants and hence $\frac{1}{2}E$ is obtainable from (3). Lastly, since

$$\nu^2 - 1 = \cot^2 c \sec^2 \eta_1,$$

formula (21) reduces to

$$\tan \lambda = \pm \sin \phi \cot (\frac{1}{2}E + \frac{1}{2}\beta). \quad (25)$$

In plotting the loci of constant deviation it is necessary to know the coördinates of the points of intersection with the $\lambda\eta_1'$ boundary locus. Since

$$\gamma_1 = \frac{1}{2}\pi \text{ or } \gamma_2' = -\frac{1}{2}\pi,$$

and

$$\sqrt{v^2 - 1} = \cot c \sec \eta_1,$$

elimination of the azimuths from equations (1), (2), (5), and (6) leads to

$$\cot c \sin \beta = 2 \cos \eta_1 \sin \frac{1}{2}E \sin (\beta + \frac{1}{2}E).$$

Combining this result with relation (3), it will be found that

$$\cos \eta_1 = \frac{2 \csc \beta \sin^2 \frac{1}{2}D}{\cot c \cos \beta \pm \sqrt{4 \sin^2 \frac{1}{2}D - \cot^2 c \sin^2 \beta}} \quad (26)$$

Although this equation pertains directly to the $\gamma_1\eta_1 D$ surface, it is also appropriate in this place because formula (4) gives η_1' unambiguously as soon as n and η_1 are known. Equation (26) indicates that, under suitable conditions, the loci of constant deviation will not intersect the $\lambda\eta_1'$ boundary loci or will meet each of them in two coincident points, or will intersect each in two discrete points, according as $\sin \frac{1}{2}D$ is less than, or equal to, or greater than $\frac{1}{2}\cot c \sin \beta$ respectively. Reference is made here to either half of the small-circle arcs C'AC and C'BC since everything is symmetrical with respect to the principal plane BLA. Moreover, since $\eta_1 \times \frac{1}{2}\pi$ and $\cos \eta_1 \times 1$, it does not follow that both roots of equation (26) always have physical significance. For instance, writing the radical as

$$\pm \sqrt{\cot^2 c \cos^2 \beta - (\cot^2 c - 4 \sin^2 \frac{1}{2}D)}$$

it is seen that the negative sign is inadmissible for

$$\cot c \leq 2 \sin \frac{1}{2}D.$$

The question of what constitutes the "suitable conditions" is both important and interesting, and hence it will be discussed later. Suffice it to say, in this connection, that the case of equal roots suggests minima along the loci of greatest deviation, that is, along the $\lambda\eta_1'$ and $\gamma_1\eta_1$ boundary curves. As far as the coördinates of the points of intersection are concerned, after having calculated η_1' , (indirectly), from (26) the abscissæ λ may be obtained from (9) and (10) or from (11). In the special case of equal roots

$$\sin \frac{1}{2}D = \frac{1}{2}\cot c \sin \beta, \quad (27)$$

$$\cos \eta_1 = \frac{1}{2}\cot c \tan \beta. \quad (28)$$

The intercepts of a constant deviation locus on the axes of λ and η_1' are given respectively by (21) or (25) and

$$\cos \eta_1' = (\cos^2 c \sin^2 \frac{1}{2}\beta + \sin^2 c \sin^2 \frac{1}{2}D) \csc c \csc \beta \csc \frac{1}{2}D. \quad (29)$$

Defining ξ by the equation

$$\tan \xi = \tan c \csc \frac{1}{2}\beta \sin \frac{1}{2}D, \quad (30)$$

the last equation changes to the logarithmic form

$$\cos \eta_1' = \cos c \sec \frac{1}{2}\beta \csc 2\xi. \quad (31)$$

Finally, the loci of constant deviation intersect both coördinate axes at right angles.

Before passing to the discussion of the properties of the γ, η, D surface, a proof of a theorem relating to the internal ray (OS in figure 1) will be outlined. This theorem seems to be new, and it may be stated thus: "*When the deviation of the projected ray is kept constant, the internal ray generates a right cone with an elliptical base.*" In figure 1, let a plane be drawn tangent to the sphere of reference at the point L. Also, let L be taken as origin of a system of rectangular coördinates. The axis of x shall be the intersection of the principal plane with the tangent plane. The y -axis shall be the line common to the plane of minima, LOH, and the tangent plane. Finally, let a denote the radius of the sphere. It can now be shown that

$$v^2 \sin^2 \lambda = \frac{x^2}{\delta^2}$$

and

$$v^2 \cos^2 \lambda = \frac{a^2}{\delta^2},$$

where

$$\delta^2 = (x^2 + a^2) \sin^2 c - y^2 \cos^2 c.$$

Substituting in equation (20), and performing elementary transformations, it results that

$$\begin{aligned} & \sin^2 \frac{1}{2}(E + \beta) [\cos^2 c \cos^2 \frac{1}{2}(E + \beta) + \sin^2 \frac{1}{2}E \sin(\beta + \frac{1}{2}E)] x^2 \\ & + \cos^2 c \sin^2 \frac{1}{2}(E + \beta) \cos^2 \frac{1}{2}(E + \beta) y^2 = \\ & [\sin^2 c \sin^2 \frac{1}{2}(E + \beta) - \sin^2 \frac{1}{2}\beta] \cos^2 \frac{1}{2}(E + \beta) a^2 \end{aligned} \quad (32)$$

Now

$$E + \beta = \gamma_1 - \gamma_2'$$

and

$$|\gamma_1| < \frac{1}{2}\pi, |\gamma_2'| < \frac{1}{2}\pi, \gamma_1 - \gamma_2' < \pi$$

hence $E + \beta < \pi$,

for all values of E having physical significance. *A fortiori*

$$\beta + \frac{1}{2}E < \pi,$$

consequently the coefficient of x^2 is essentially positive, as well as the coefficient of y^2 . When $E = \epsilon_0$,

$$\sin^2 c \sin^2 \frac{1}{2}(E + \beta) - \sin^2 \frac{1}{2}\beta = 0,$$

by formula (19), hence the conic degenerates to the point $x = 0, y = 0$. For larger permissible values of E ,

$$\sin^2 c \sin^2 \frac{1}{2}(E + \beta) - \sin^2 \frac{1}{2}\beta$$

is positive, thus giving a true ellipse and establishing the theorem stated above. Furthermore, the coefficient of x^2 is obviously greater than the coefficient of y^2 , therefore the longer axis of the ellipse lies along the y -axis, *i. e.*, in the plane of minima. A consideration of the way in which E increases along a small-circle MV gives rise to the expectation of a cone of a general oval base, but the simplicity of the result was not anticipated. Conic sections were not found, by the writer, to pertain to the other rays under the various conditions which were tried.

Now that the more important properties of the $\lambda\eta_1' D$ surface have been presented, it is appropriate to discuss the γ, η, D locus. The problem of the internal ray was taken up first because of the simplicity and symmetry of the results obtained. In spite of the comparatively complicated nature of the γ, η, D surface it is the more interesting because it shows the dependence of the general deviation upon the azimuth and altitude of the incident ray. This added interest rests partly on the fact that it seems natural to consider the incident ray in all permissible positions, around a fixed point of incidence, and to investigate the corresponding values of the deviation.

From (2),

$$\cos E = \cos (\gamma_1 - \beta) \cos \gamma_2' + \sin (\gamma_1 - \beta) \sin \gamma_2'.$$

From (1) and (6),

$$\sin \gamma_2' = v \sin (\gamma_1' - \beta),$$

hence, using (5) and observing that $\gamma_1' \gtrless \frac{1}{2}\pi$, it will be found that

$$\sin \gamma_2' = \cos \beta \sin \gamma_1 - \sin \beta \sqrt{v^2 - \sin^2 \gamma_1}.$$

Since $\gamma_2' \gtrless \frac{1}{2}\pi$,

$$\cos \gamma_2' = + \sqrt{1 - \sin^2 \gamma_2'},$$

therefore

$$\begin{aligned} \cos E &= \sin (\gamma_1 - \beta) [\cos \beta \sin \gamma_1 - \sin \beta \sqrt{v^2 - \sin^2 \gamma_1}] \\ &+ \cos (\gamma_1 - \beta) \sqrt{1 - [\cos \beta \sin \gamma_1 - \sin \beta \sqrt{v^2 - \sin^2 \gamma_1}]^2} \quad (33) \end{aligned}$$

From (3),

$$\cos D = \sin^2 \eta_1 + \cos^2 \eta_1 \cos E,$$

so that substitution in (33) with due attention to (4), gives

$$\begin{aligned} \cos D = \sin^2 \eta_1 + \cos \eta_1 \sin (\gamma_1 - \beta) [\cos \beta \sin \gamma_1 \cos \eta_1 \\ - \sin \beta \sqrt{\cot^2 c + \cos^2 \gamma_1 \cos^2 \eta_1}] + \cos \eta_1 \cos (\gamma_1 - \beta) \times \\ \sqrt{\cos^2 \eta_1 - [\cos \beta \sin \gamma_1 \cos \eta_1 - \sin \beta \sqrt{\cot^2 c + \cos^2 \gamma_1 \cos^2 \eta_1}]^2} \quad (34) \end{aligned}$$

This is the general formula for the $\gamma_1 \eta_1 D$ surface, it being understood that D is always positive. The last equation may also be written

$$\begin{aligned} \sin \frac{1}{2} D = \\ \frac{1}{2} \left\{ \sqrt{\cos \eta_1 [1 + \sin (\gamma_1 - \beta)] \{ [1 - \sin (\gamma_1 - \beta)] \cos \eta_1 + u \sin \beta \}} \right. \\ \left. - \sqrt{\cos \eta_1 [1 - \sin (\gamma_1 - \beta)] \{ [1 + \sin (\gamma_1 - \beta)] \cos \eta_1 - u \sin \beta \}} \right\} \quad (35) \end{aligned}$$

where

$$u = \sqrt{\cot^2 c + \cos^2 \gamma_1 \cos^2 \eta_1} - \cos \gamma_1 \cos \eta_1.$$

Since $\eta_1 \nless \frac{1}{2}\pi$, the expression under the first long radical is always positive. The second long radical will become imaginary when

$$(1 + \cos \beta \sin \gamma_1) \cos \eta_1 < \sin \beta \sqrt{\cot^2 c + \cos^2 \gamma_1 \cos^2 \eta_1},$$

therefore the equation of the left boundary of the $\gamma_1 \eta_1 D$ surface may be written

$$\sin \gamma_1 \cos \eta_1 + \cos \beta \cos \eta_1 = \cot c \sin \beta \quad (36)$$

[The unrationalized form of (36) can be obtained directly by combining the condition $\gamma_2' = -\frac{1}{2}\pi$ with (1), (5), and (6). Also, along the left boundary locus formulæ (33) and (35) reduce to $E = \frac{1}{2}\pi + \gamma_1 - \beta$ and $\sin \frac{1}{2} D = \sin \frac{1}{2} (\frac{1}{2}\pi + \gamma_1 - \beta) \cos \eta_1$, as they should do, conformably to (2) and (3).] Combining the last equation with (28) it will be found that

$$\gamma_1 = \frac{1}{2}\pi - \beta$$

and therefore

$$E = \pi - 2\beta.$$

Assuming, for the time being, (since it will be formally demonstrated later), that the case of equal roots, characterized by (27) and (28), corresponds to discrete minima on the boundary loci, the following theorem, which seems to be new, may

now be stated: "At the discrete minima on the boundary loci of the $\gamma, \eta_1 D$ surface the deviation of the projected ray is the supplement of twice the prism angle." In calculating the coördinates of points on the left boundary of the $\gamma, \eta_1 D$ surface, it is convenient to assume γ_1 and to derive η_1 from the following modified form of (36), namely:

$$\cos \eta_1 = \frac{1}{2} \cot c \sin \beta \csc \frac{1}{2}(\frac{1}{2}\pi - \beta + \gamma_1) \sec \frac{1}{2}(\frac{1}{2}\pi - \beta - \gamma_1) \quad (37)$$

The right boundary is simply $\gamma_1 = \frac{1}{2}\pi$.

The following special cases will be found useful. For grazing incidence or grazing emergence formula (35) reduces to

$$\sin \frac{1}{2}D = \frac{1}{2} \sin \beta \sqrt{\cos \eta_1} \left\{ \sqrt{\cos \eta_1 + \cot c \cot \frac{1}{2}\beta} - \sqrt{\cos \eta_1 - \cot c \tan \frac{1}{2}\beta} \right\} \quad (38)$$

For $\eta_1 = 0$ formula (38) reduces to (16). For grazing emergence in a principal plane (36) gives

$$\sin \gamma_1 = \csc c \sin (\beta - c) \quad (39)$$

For simultaneous grazing incidence and emergence (36) leads to

$$\left. \begin{aligned} \cos \eta_1 &= \cot c \tan \frac{1}{2}\beta \\ \text{or} \quad \tan \frac{1}{2}\eta_1 &= \pm \sqrt{\csc (c + \frac{1}{2}\beta) \sin (c - \frac{1}{2}\beta)}, \end{aligned} \right\} \quad (40)$$

and (38) becomes (15). Anywhere along the left boundary

$$\frac{d\eta_1}{d\gamma_1} = \tan c \csc \beta \cos \gamma_1 \cos \eta_1 \cot \eta_1,$$

hence this locus crosses the γ_1 -axis, and also intersects the right boundary, at right angles. The intercepts on the η_1 -axis of the left boundary may be derived at once from (36). They are given by

$$\left. \begin{aligned} \cos \eta_1 &= \cot c \tan \beta \\ \text{or} \quad \tan \frac{1}{2}\eta_1 &= \pm \sqrt{\csc (c + \beta) \sin (c - \beta)}. \end{aligned} \right\} \quad (41)$$

Then (35) simplifies to

$$\sin \frac{1}{2}D = \cot c \tan \beta \sin \frac{1}{2}(\frac{1}{2}\pi - \beta) \quad (42)$$

In order that these intersections may be real and discrete, and that γ_1 may have negative values, the necessary and sufficient condition is $\beta < c$.

For normal incidence or emergence formulæ (8) and (34) each lead to

$$\sin (D + \beta) = \csc c \sin \beta \quad (43)$$

In the case of the $\lambda\eta_1'D$ surface, equation (8) was not used to calculate the coördinates of the points of the constant deviation loci because of the involved nature of (8). Instead,

formula (25) was derived, and the manner of using it for numerical calculations was explained. For the same reason equations (34) and (35) are not convenient for calculating the constant deviation sections of the γ, η, D surface. Furthermore, it is not desirable to attempt to transform (21) or (25) from the λ, η_1' coördinates to the γ, η_1 system, since the equations of transformation suggest that a very complicated function of γ_1 in terms of η_1 and D would result. For, the equations of transformation may be written,

$$\sin \lambda = [\cos \tfrac{1}{2}\beta \sin \gamma_1 \cos \eta_1 - \sin \tfrac{1}{2}\beta \sqrt{\cot^2 c + \cos^2 \gamma_1 \cos^2 \eta_1}] \times (\cot^2 c + \cos^2 \eta_1)^{-\frac{1}{2}}$$

and

$$\sin \eta_1' = \sin c \sin \eta_1.$$

A much simpler plan is to assume values for D and η_1 , and then calculate E and η_1' by means of (3) and (4) respectively. Next evaluate λ and hence γ_1' from (25), and finally get γ_1 from (5).

Although it can be established by analysis that each locus of constant deviation has an algebraic maximum, (η_1 positive), and a like minimum, (η_1 negative), at points on the γ, η_1 plane of projection which correspond to $\lambda = 0$, ($\gamma_1' = \tfrac{1}{2}\beta$), nevertheless this property seems to be self-evident, and therefore, the formal demonstration will be omitted. The equation of the projection on the γ, η_1 plane of the locus of these stationary points may be written

$$\sin^2 \gamma_1 \cos^2 \eta_1 - \sin^2 \tfrac{1}{2}\beta \cos^2 \eta_1 = \cot^2 c \sin^2 \tfrac{1}{2}\beta \quad (44)$$

Hence, the coördinates of points on this curve may be readily obtained by assuming values for γ_1 and calculating η_1 from

$$\cos \eta_1 = \cot c \sin \tfrac{1}{2}\beta \sqrt{\csc (\gamma_1 + \tfrac{1}{2}\beta) \csc (\gamma_1 - \tfrac{1}{2}\beta)} \quad (45)$$

Along the projected locus,

$$\frac{d\eta_1}{d\gamma_1} = \sec c \tan c \csc \tfrac{1}{2}\beta \cos \gamma_1 \cos \eta_1 \cot \eta_1 \cos \eta_1'$$

hence it intersects both the γ_1 -coördinate axis and the right boundary locus normally. The absolute minimum, ϵ_o , falls at the point

$$\left. \begin{aligned} \gamma_1 &= \sin^{-1} (\csc c \sin \tfrac{1}{2}\beta), \\ \eta_1 &= 0. \end{aligned} \right\} \quad (46)$$

For $\lambda = 0$ formula (8) may be transformed into

$$\begin{aligned} \sin \tfrac{1}{2}D &= \{ \sqrt{(\cot^2 c + \cos^2 \eta_1) \cos^2 \tfrac{1}{2}\beta} \\ &\quad - \sqrt{(\cot^2 c + \cos^2 \eta_1) \cos^2 \tfrac{1}{2}\beta - \cot^2 c} \} \sin \tfrac{1}{2}\beta^* \end{aligned} \quad (47)$$

* H. S. Uhler, this Journal, vol. xxvii, p. 227, 1909.

Attention will now be directed to the interesting problem of the variation of the deviation at grazing incidence, that is, along the right boundary locus. Formula (38) may be written

$$\sin \frac{1}{2}D = \frac{1}{2} \sin \beta (R_1 - R_2) \sqrt{\cos \eta_1},$$

where

$$R_1 = \sqrt{\cos \eta_1 + \cot c \cot \frac{1}{2}\beta}$$

and

$$R_2 = \sqrt{\cos \eta_1 - \cot c \tan \frac{1}{2}\beta}.$$

It follows that

$$\frac{dD}{d\eta_1} = (R_1 R_2)^{-1} \tan \frac{1}{2}D \tan \eta_1 (\cos \eta_1 - R_1 R_2).$$

In general, therefore, stationary values of D may correspond to

$$\eta_1 = 0$$

and

$$\cos \eta_1 - R_1 R_2 = 0,$$

the latter being equivalent to

$$\cos \eta_1 = \frac{1}{2} \cot c \tan \beta,$$

which is relation (28). It is easy to show that

$$\begin{aligned} \frac{d^2 D}{d\eta_1^2} = & \frac{1}{2} (R_1 R_2)^{-3} \csc^2 \beta \tan \frac{1}{2}D \{ 4 \sin \eta_1 \tan \eta_1 \sin^2 \frac{1}{2}D + \\ & R_1 R_2 \ominus [\ominus \sec^2 \frac{1}{2}D \tan^2 \eta_1 + 2 \sin \eta_1 \tan \eta_1 + 2 R_1 R_2 \sec^2 \eta_1] \sin^2 \beta \} \end{aligned}$$

where $\ominus = \cos \eta_1 - R_1 R_2$, therefore

$$\begin{aligned} \left(\frac{d^2 D}{d\eta_1^2} \right)_{\eta_1=0} = & \{ [1 + (2 \tan c - \tan \beta) \cot^2 c \cot \beta]^{-\frac{1}{2}} - 1 \} \tan \frac{1}{2}D = \\ & [\sin c \sqrt{\sin \beta \csc (2c - \beta)} - 1] \tan \frac{1}{2}D, \end{aligned}$$

and

$$\left(\frac{d^2 D}{d\eta_1^2} \right)_{\Theta=0} = 2 \sin^2 \frac{1}{2}D \tan \frac{1}{2}D \csc^2 \beta \sec^2 \eta_1 \tan^2 \eta_1.$$

Assuming β acute, $\left(\frac{d^2 D}{d\eta_1^2} \right)_{\eta_1=0}$ is positive or negative, according

as $\tan \beta > 2 \tan c$ or $\tan \beta < 2 \tan c$, hence the deviation at grazing incidence in a principal plane is a maximum when $\tan \beta < 2 \tan c$, and a minimum when $\tan \beta > 2 \tan c$. Again, when $\cos \eta_1 = \frac{1}{2} \cot c \tan \beta$ and $\eta_1 \leq 0$, $\left(\frac{d^2 D}{d\eta_1^2} \right)_{\Theta=0}$ is essentially

positive so that the deviation experiences a minimum value. Since $\cos(+\eta_1) = \cos(-\eta_1)$ there will be two minima at grazing incidence corresponding to $\Theta = 0$. Moreover, since the cosine of any angle may not exceed unity in magnitude it is evident that these minima will exist when, and only when, $\tan \beta < 2 \tan c$, that is, simultaneously with the aforementioned maximum in the principal plane. The case where $\tan \beta = 2 \tan c$ requires

special investigation, for then $\left(\frac{d^2 D}{d\eta_1^2}\right)_{\Theta=\eta_1=0}$ vanishes. Keeping c or

n constant and allowing β to vary in the equation $\cos \eta_1 = \frac{1}{2} \cot c \tan \beta$ it is seen that when $\beta = 0$, $\eta_1 = \frac{1}{2} \pi$, and that as β increases, η_1 decreases and approaches the limit 0 as β approaches the value $\tan^{-1}(2 \tan c)$. Since *two* minima are moving in symmetrically towards *one* maximum of fixed position, it seems reasonable to expect a resultant minimum, of higher order than the first, when the limiting condition $\tan \beta = 2 \tan c$ is fulfilled. By comparatively laborious but elementary analysis it can be

shown that $\frac{d^3 D}{d\eta_1^3}$ vanishes for $\eta_1 = 0$, and that

$$\frac{d^3 D}{d\eta_1^3} = 12 \cos^2 \beta \cot^2 \beta \csc D$$

when

$$\begin{aligned} \Theta = \eta_1 = 0, \text{ i. e.,} \\ \tan \beta = 2 \tan c \end{aligned} \quad (48)$$

This result being positive the expectation of a minimum, of higher order than the first, is realized. The expression under

the radical of $\left(\frac{d^2 D}{d\eta_1^2}\right)_{\eta_1=0}$ may also be written

$$1 - \cot^2 c + 2 \cot c \cot \beta$$

hence, (assuming $2c$ obtuse so as to permit β to become obtuse and still satisfy the necessary condition $\beta < 2c$), as β increases from a very small value up to $\tan^{-1}(2 \tan c)$, exclusive of this

limit, $\left(\frac{d^2 D}{d\eta_1^2}\right)_{\eta_1=0}$ maintains a negative value so that a maximum of

deviation persists. When $\beta = \tan^{-1}(2 \tan c)$, (which implies β acute), it has been shown that the deviation is a minimum.

The instant β exceeds $\tan^{-1}(2 \tan c)$, $\left(\frac{d^2 D}{d\eta_1^2}\right)_{\eta_1=0}$ becomes positive

and remains so while β increases to $\frac{1}{2}\pi$, passes through $\frac{1}{2}\pi$, and continues obtuse up to the limit $2c$. To sum up, *there is one minimum for $\eta_1 = \frac{1}{2}\pi$ when $2c > \beta > \tan^{-1}(2 \tan c)$, whereas there are discrete minima and one maximum when*

$$\tan^{-1} (2 \tan c) > \beta > 0.$$

For a given value of η , the deviation on the left boundary is obviously the same as the deviation on the right boundary. On the other hand, the corresponding abscissæ are

$$\begin{aligned} \gamma_1 &= \sin^{-1} (\cot c \sin \beta \sec \eta_1 - \cos \beta), \\ \text{and} \quad \gamma_1 &= \frac{1}{2}\pi. \end{aligned} \quad [\text{See formula (36)}]$$

At the discrete minima

$$\sec \eta_1 = 2 \tan c \cot \beta$$

hence

$$\gamma_1 = \frac{1}{2}\pi - \beta,$$

as was obtained before in the case of equal roots of equation (26). Thus, as expected, the equal roots pertain to the discrete minima, four in all. Also, *the azimuth of the incident ray is the complement of the prism angle for the two discrete minima of deviation corresponding to grazing emergence.* [At the four minima it was shown earlier that $E = \pi - 2\beta$.]

The value of the deviation at the discrete minima is given by formula (27). When $\tan \beta = 2 \tan c$, that is, when the discrete minima coalesce with their respective maxima, the deviation too is the supplement of twice the prism angle.

The case of two discrete minima and one maximum suggests that it may be possible for the deviation at grazing incidence in a principal plane to equal the deviation at simultaneous grazing incidence and emergence. Moreover, since, for a given altitude η , the greatest deviations correspond to the small-circle boundaries (U and V of fig. 1), the problem just suggested is very closely related to the question of the location of the greatest deviation which can be produced by a prism of given angle and relative index of refraction. A very incomplete attempt to investigate the greatest deviation has been made by A. Anderson in a paper* entitled: "On the Maximum Deviation of a Ray of Light by a Prism."

Let D_1 denote the deviation for grazing incidence in a principal plane and D_2 the deviation for simultaneous grazing incidence and emergence. A fruitful formula involving $D_1 - D_2$ will now be derived. Equation (17) may be written

$$\cos (D_1 + \beta) = \cos \beta - 2 \cot c \sin \frac{1}{2}\beta \cos \frac{1}{2}\beta.$$

The formula

$$\cos (D_2 + \beta) = \cos \beta \cos D_2 - \sin \beta \sin D_2$$

becomes

$$\begin{aligned} \cos (D_2 + \beta) &= \cos \beta - 2 \cot^2 c \cos \beta \sin^2 \frac{1}{2}\beta \\ &\quad - 2 \cot c \sin \beta \sin \frac{1}{2}\beta \sqrt{1 - \cot^2 c \sin^2 \frac{1}{2}\beta}, \end{aligned}$$

* Proc. Camb. Phil. Soc., vol. ix, p. 195, 1898.

since, by (15),

$$\sin \frac{1}{2}D_2 = \cot c \sin \frac{1}{2}\beta.$$

Hence

$$\begin{aligned} \cos (D_1 + \beta) - \cos (D_2 + \beta) &= 2 \cot c \sin \frac{1}{2}\beta \times \\ &\quad \{\cot c \cos \beta \sin \frac{1}{2}\beta - \cos \frac{1}{2}\beta + \sin \beta \sqrt{1 - \cot^2 c \sin^2 \frac{1}{2}\beta}\} \end{aligned}$$

or

$$\begin{aligned} \sin \frac{1}{2}(D_1 - D_2) &= \cos c \sin \frac{1}{2}\beta \{\sin (c - \frac{1}{2}\beta) + 2 \cos c \sin^3 \frac{1}{2}\beta \\ &\quad - \sin \beta \sqrt{\sin^2 c \sin^2 \frac{1}{2}\beta + \sin^2 c - \sin^2 \frac{1}{2}\beta}\} \div \\ &\quad \sin^2 c \sin [\frac{1}{2}(D_1 + D_2) + \beta] \end{aligned}$$

Obviously, the quantity within the braces may be rationalized by multiplying by

$$\begin{aligned} \sin (c - \frac{1}{2}\beta) + 2 \cos c \sin^3 \frac{1}{2}\beta \\ + \sin \beta \sqrt{\sin^2 c \sin^2 \frac{1}{2}\beta + (\sin^2 c - \sin^2 \frac{1}{2}\beta)}. \end{aligned}$$

Since c cannot be less than $\frac{1}{2}\beta$ each term of the last expression is real and positive, so that the fraction which equals $\sin \frac{1}{2}(D_1 - D_2)$ will not be made indeterminate or ambiguous by multiplying both numerator and denominator by the rationalizing factor. Denoting the reciprocal of this factor by κ^2 , for brevity, it will be found that

$$\begin{aligned} \sin \frac{1}{2}(D_1 - D_2) &= \kappa^2 \cot c \sin \frac{1}{2}\beta \sin (c - \frac{1}{2}\beta) \times \\ &\quad \csc [\frac{1}{2}(D_1 + D_2) + \beta] (\cos \frac{3}{2}\beta - \cot c \sin \frac{1}{2}\beta) \quad (49) \end{aligned}$$

Now

$$\pi - \beta > D_1 > 0$$

and

$$\pi - \beta > D_2 > 0,$$

hence

$$\frac{1}{2}(D_1 + D_2) + \beta < \pi$$

or

$$\sin [\frac{1}{2}(D_1 + D_2) + \beta] > 0,$$

for all cases except the trivial, extreme case where $\beta = 2c$. Hence, in general,

$$\sin (c - \frac{1}{2}\beta) \csc [\frac{1}{2}(D_1 + D_2) + \beta]$$

is positive and determinate. Therefore, *the deviation at grazing incidence in a principal plane (D_1) is greater than, or equal to, or less than, the deviation at simultaneous grazing incidence and emergence (D_2), according as $\cos \frac{3}{2}\beta$ is greater than, or equal to, or less than, $\cot c \sin \frac{1}{2}\beta$, respectively.* For all practical purposes, this answers the question concerning the

location of the greatest deviation produced by a prism of specified constants. The equation

$$\cos \frac{3}{2}\beta = \cot c \sin \frac{1}{2}\beta, \quad (50)$$

which is the necessary and sufficient condition for $D_1 = D_2$, shows that *the deviation at grazing incidence in a principal plane can never equal the deviation at simultaneous grazing incidence and emergence for a prism having an angle greater than $\frac{1}{3}\pi$, regardless of the value of the index of refraction.* For, let the curves whose equations are $y = \cos \frac{3}{2}\beta$ and $y = \cot c \sin \frac{1}{2}\beta$ be plotted on the same diagram, with the values of β as abscissæ. As β increases from zero, the curve $y = \cos \frac{3}{2}\beta$ will commence with the intercept 1 on the y -axis and will have decreasing positive ordinates until it crosses the β -axis at the intercept $\beta = \frac{1}{3}\pi$. On the other hand, assuming c given, the curve $y = \cot c \sin \frac{1}{2}\beta$ will begin at the origin and have increasing positive ordinates, as β increases, until $\beta = \pi$. Therefore, the abscissa of the point of intersection of these curves must be less than $\frac{1}{3}\pi$, save in the trivial case where $\cot c = 0$ or $n = 1$. In this extreme case the point of intersection is $(\frac{1}{3}\pi, 0)$. It is also evident that the larger the value of $\cot c$, (or n), the smaller will be the value of β corresponding to $D_1 = D_2$. When c is given and the prism angle which will make the deviations D_1 and D_2 equal is required, it will be found convenient to transform equation (50) to a function of $\frac{1}{2}\beta$ alone. Denoting $\tan \frac{1}{2}\beta$ by t , the last equation may be reduced to the following cubic

$$\cot c \cdot t^3 + 3t^2 + \cot c \cdot t - 1 = 0 \quad (51)$$

Equation (51) always has one positive real root because $\cot c$ is positive and the constant term is negative. Moreover, it can only have one positive root in accordance with the "Rule of Signs" of Descartes. Since $\pi > \beta > 0$, t must be positive so that there is no ambiguity in the solution of the cubic. Writing the left-hand member of (51) as

$$t(1 + t^2) \cot c - (1 - 3t^2),$$

and observing that t must be positive, it becomes clear that $1 - 3t^2$ must be positive to make the expression vanish. This amounts to a second proof that β must not exceed $\frac{1}{3}\pi$ in order to make possible the equality of D_1 and D_2 .

It seems appropriate, at this stage of the discussion, to direct attention to the illustrative figures and numerical data. The chalk lines reproduced in fig. 1 represent roughly the following theoretical values: $\beta = 20^\circ$, $n = 1.5$, $c = \cot^{-1} \sqrt{n^2 - 1} = \angle N_1OA = \angle N_2OB = 41^\circ 48' 37''$, $a_1 = 55^\circ 57' 45''$, $a'_1 = 33^\circ$

FIGS. 2-6.

FIG. 2.

FIG. 3.

FIG. 4.

FIG. 5.

FIG. 6.

$32' 6''$, $\alpha_2 = 21^\circ 18' 17''$, and $\alpha_2' = 33^\circ 1' 26''$. The α 's may be calculated by the aid of the following relations:

$$\cos \alpha_1 = \cos \gamma_1 \cos \eta_1 \quad (52)$$

$$\cos \alpha_2' = \cos \gamma_2' \cos \eta_1 \quad (53)$$

$$\sin \alpha_1' = \sin c \sin \alpha_1 \quad (54)$$

$$\sin \alpha_2 = \sin c \sin \alpha_2' \quad (55)$$

$\eta_1 = \eta_2' = 30^\circ 51' 57''$, $\eta_1' = 20^\circ$, $\gamma_1 = 49^\circ 18' 2''$, $\gamma_1' = 27^\circ 29' 46''$, $\gamma_2 = 7^\circ 29' 46''$, $\gamma_2' = 12^\circ 22' 12''$, $D = 14^\circ 31' 7''$, and $E = 16^\circ 55' 50''$. The absolute minimum of deviation, which is attained when S coincides with L, $\epsilon_0 = 10^\circ 11' 46''$. The partial minimum corresponding to the altitude LM, when S falls on M, $D_0 = \angle Q_0OP_0 = 11^\circ 16' 7''$. The associated deviation for the projected ray, $E_0 = 13^\circ 8' 8''$. The deviation at grazing incidence or emergence in a principal plane, when P coincides with K or R, $D_1 = 36^\circ 7' 50''$. $\angle N_1OK = \angle N_1OT = 90^\circ$, and $\angle N_1OR = -33^\circ 52' 10''$. The deviation for simultaneous grazing incidence and emergence, when P and S coincide respectively with T and C, $D_2 = 22^\circ 23' 22''$. $\angle KOT = 78^\circ 37' 49''$. When referred to rectangular frames the small-circle arcs C'AC and C'BC have been called respectively the right and left boundary loci of the $\lambda\eta_1'D$ surface. In like manner, the great-circle arc T'KT and the curve T'RT have been designated as the right and left boundary loci of the $\gamma\eta_1D$ surface, in the order named. T' being the point homologous to T. The course of the curve TR does not show very well in the photograph. When viewed from above, in the direction $\vec{HO}$, it is seen to recede more and more from H as the tracing point starts from T and moves towards R.

Figures 2 to 5, inclusive, show the projections on the $\lambda\eta_1'$ coördinate plane of certain constant deviation sections of the $\lambda\eta_1'D$ surfaces. Only the first quadrant is plotted for each locus since all of the sections are symmetrical with respect to both of the axes $O\lambda$ and $O\eta_1'$. The complete boundary loci of the $\lambda\eta_1'D$ surfaces are cylinders whose generating lines are parallel to the axes $\vec{OD}$ and hence perpendicular to the plane of the paper. For this reason any one quadrant of each boundary locus projects on the $\lambda\eta_1'$ plane as a single curve which may be intersected by any number of loci of constant deviation. The boundary loci arise from the physical condition that total reflection is to be excluded. There is nothing inherent in the mathematical functions to prevent the loci of constant deviation, or any other sections of the $\lambda\eta_1'D$ surfaces which intersect the boundary loci, from being plotted continuously so as to include points in the region outside of the

bounding cylinders, that is, beyond the convex sides of these cylinders. However, in figures 2 to 5 the loci of constant deviation have not been plotted beyond the boundary loci. The relative index of refraction has been given the value 1.5 throughout. Each figure corresponds to a different value of the prism angle, (β), so that the group of figures taken collectively shows how the dimensions of the $\lambda\eta_1'D$ surface depend upon the angle of the prism. To facilitate comparison, figures 2 to 5 are drawn to the same scale. The values of β pertaining to figures 2, 3, 4, and 5 are, respectively, 20° , $43^\circ 37' 58''$, $60^\circ 47' 39''$, and 75° . The angles 20° and 75° were chosen arbitrarily as being relatively small and large. On the other hand, with $n = 1.5$, $\beta = 43^\circ 37' 58''$ illustrates the case of equality between the deviation at grazing incidence in a principal plane and the deviation at simultaneous grazing incidence and emergence. This value of β was derived from the cubic equation (51). Figure 4 illustrates the special case in which the two minima of deviation at grazing incidence coalesce with the maximum of deviation at grazing incidence in a principal plane. The angle of the prism was calculated from condition (48), namely, $\tan \beta = 2 \tan c$. The acute angles which the boundary loci make with the $\eta_1'D$ coördinate planes equal $78^\circ 27' 34''$, $61^\circ 40' 18''$, $44^\circ 47' 29''$, and $25^\circ 24' 55''$ for figures 2, 3, 4, and 5, respectively. The successive increments of deviation from one locus of constant deviation to the next are kept constant for each diagram, but the value of this constant difference is not the same for all four figures. More precisely, for each of the figures, 2, 4, and 5, the value of ϵ_0 was subtracted from the value of the greatest possible deviation, (D_1 or D_2), and this difference was divided by 6. This accounts for the presence of minutes and seconds of arc in the deviations in these families. In figure 3 the five increments of deviation are practically constant. They have the values $4^\circ 52' 25''$, $5^\circ 0' 0''$, $5^\circ 0' 0''$, $5^\circ 0' 0''$, and $5^\circ 6' 6''$. Hence, for each family of curves, the relatively great distance of the innermost locus of constant deviation from the origin and the crowding together of the constant deviation loci near the boundary curve show how the steepness of the $\lambda\eta_1'D$ surface increases as the deviation exceeds the value of the absolute minimum by ever increasing amounts. The values of the most important quantities pertaining to the four families of constant deviation loci are collected in Table I.

A few additional points deserve notice. For $\beta = 20^\circ$ the minimum of all deviations along the boundary locus has the value $22^\circ 2' 42''$, and the coördinates $\lambda = 0^\circ 18' 51''$ and $\eta_1' = 40^\circ 44' 50''$. Hence, only within the small interval $(22^\circ 23' 22'' - (22^\circ 2' 42'') = 0^\circ 20' 40'')$ is it possible for the curves of con-

TABLE I.

β	Families, D	Intercept, λ -axis.	Intercept, η_1' -axis.	Boundary intersections		Extreme values.
				λ	η_1'	
$0^\circ 0' 0''$	$14^\circ 31' 7''$	$22^\circ 5' 3''$	$34^\circ 8' 1''$			$\epsilon_0 = 10^\circ 11' 46''$
	$18 50 27$	$27 15 1$	$39 50 1$			$\{ D_1 = 36 7 50$
	$23 9 48$	$29 46 12$	beyond boundary	$2^\circ 47' 45''$	$40^\circ 9' 6''$	$\lambda = 31 48 37$
	$27 29 9$	$31 2 58$	"	$11 0 31$	$37 1 9$	$\{ D_2 = 22 23 22$
	$31 48 29$	$31 38 34$	"	$20 49 53$	$29 46 10$	$\{ \eta_1' = 40 48 44$
$20^\circ 0' 58''$	$29 0 0$	$13 28 53$	$24 45 40$			$\epsilon_0 = 24 7 35$
	$34 0 0$	$17 9 20$	$31 27 56$			$\{ D_1 = 49 6 5$
	$39 0 0$	$18 56 27$	$34 41 50$			$\lambda = 19 59 38$
	$44 0 0$	$19 45 48$	$36 10 47$			$\{ D_2 = 49 6 5$
$43^\circ 37' 58''$	$43 6 49$	$8 14 52$	$18 56 15$			$\epsilon_0 = 37 57 20$
	$48 16 18$	$10 19 49$	$24 29 5$			$\{ D_1 = 58 24 43$
	$53 25 46$	$11 12 7$	$27 29 35$			$\lambda = 11 24 48$
	$58 35 15$	beyond boundary	$29 9 38$	F; $8 3 21$	$17 51 57$	$\{ D_2 = 68 54 13$
	$63 44 44$	"	$29 58 50$	E; $0 39 7$	$29 32 28$	$\{ \eta_1' = 30 12 55$
$60^\circ 47' 39''$	$61 42 9$	$3 28 55$	$12 12 16$			$\epsilon_0 = 56 53 11$
	$66 31 8$	$4 11 20$	$15 57 9$			$\{ D_1 = 70 11 49$
	$71 20 6$	beyond boundary	$18 3 17$	J; $3 17 58$	$10 3 41$	$\lambda = 4 18 37$
	$76 9 4$	"	$19 15 10$	H; $1 0 30$	$17 43 39$	$\{ D_2 = 85 47 1$
	$80 58 3$	"	$19 51 23$	G; $0 11 27$	$19 37 25$	$\{ \eta_1' = 20 1 56$

stant deviation to intersect the boundary in two discrete points. Thus for $D = 22^\circ 13' 2''$ the close intersections with the boundary locus are given by $\lambda = 0^\circ 57' 19''$, $\eta_1' = 40^\circ 36' 25''$, and $\lambda = 0^\circ 1' 34''$, $\eta_1' = 40^\circ 48' 25''$. In order to show the distribution of deviation along the boundary locus of figure 3 the following numerical data have been calculated. [This distribution is shown in another way in figure 6.] At M the deviation has the minimum value $45^\circ 22' 46''$, with the coördinates $\lambda = 3^\circ 40' 13''$, $\eta_1' = 34^\circ 20' 22''$. At the points AA, $D = 46^\circ$, $\lambda = 8^\circ 12' 47''$ with $\eta_1' = 30^\circ 34' 48''$ and $\lambda = 1^\circ 9' 18''$ with $\eta_1' = 35^\circ 56' 52''$. At BB, $D = 47^\circ$, $\lambda = 12^\circ 21' 20''$ with $\eta_1' = 25^\circ 43' 29''$, and $\lambda = 0^\circ 21' 36''$ with $\eta_1' = 36^\circ 23' 59''$. At CC, $D = 48^\circ$, $\lambda = 16^\circ 4' 5''$ with $\eta_1' = 19^\circ 11' 55''$ and $\lambda = 0^\circ 4' 39''$ with $\eta_1' = 36^\circ 33' 16''$. For $D = 48^\circ 40'$ the point D is given by $\lambda = 18^\circ 27' 26''$ with $\eta_1' = 12^\circ 20' 3''$. The upper intersection, which could not be plotted, has the coördinates $\lambda = 0^\circ 0' 38''$ and $\eta_1' = 36^\circ 35' 26''$.

Figure 6 pertains to the γ, η_1, D surface in that it shows the dependence of the deviation at grazing incidence upon the

altitude η , and the prism angle β . Each of the curves **a**, **b**, **c**, **d**, and **e** is characterized by a different value of β , and also each curve represents the intersection of the plane $\gamma_1 = 90^\circ$ with the corresponding γ, η, D surface. In other words, for the chosen values of the prism angle, figure 6 shows the distribution of deviation along the great-circle arc KT of figure 1. As before, $n = 1.5$. The values of β for the curves **a**, **b**, **c**, **d**, and **e** are $9^\circ 42'$, 20° , $43^\circ 37' 58''$, $60^\circ 47' 39''$, and $77^\circ 42'$, in the order named. The ordinate of each point along the coördinate axis OD signifies the deviation at grazing incidence in a principal plane corresponding to the associated value of the prism angle. That is, the line OD is the locus of D_1 when β varies. From the same point of view, the complete curve **t** passes through all points pertaining to simultaneous grazing incidence and emergence, or, it is the locus of D_2 when β assumes all possible values. The greatest value of the prism angle consistent with transmission is given by $\beta = 2c$. In this limiting case the only ray possible corresponds to simultaneous grazing incidence and emergence in a principal plane, $\epsilon_0 = D_1 = D_2 = 180^\circ - 2c$, the $\lambda\eta, D$ and γ, η, D surfaces degenerate into single points, and the small-circle arcs of figure 1 become tangent, so that A and B coincide. For $n = 1.5$ the limiting values are $\beta = 83^\circ 37' 14''$ and $D = 96^\circ 22' 46''$. Figure 6 shows very clearly that the greatest deviation which can be produced by a prism having a given index of refraction is obtained when $\beta = 2c$. This follows from the fact that when both β and c are fixed the greatest deviation corresponds to grazing incidence, and figure 6 has $\gamma_1 = 90^\circ$ and $c = \text{constant}$ throughout. [See pages 393 and 405.] For the curves **a**, **b**, and **c**, β is less than $\tan^{-1}(2 \tan c)$, hence each complete locus has two discrete minima and one maximum which corresponds to grazing incidence in a principal plane. [Only one-half of each curve is shown in the figure since OD is the axis of symmetry.] The line **c** satisfies condition (50), namely,

$$\cos \frac{3}{2}\beta - \cot c \sin \frac{1}{2}\beta = 0,$$

hence the ordinates at the extremities of the plotted curve are equal, ($D_1 = D_2$). The condition $\tan \beta = 2 \tan c$ is fulfilled by curve **d**, hence the discrete minima coincide with the maximum and produce a resultant minimum in the principal plane. Thus, as β increases from 0, the curve **d** is the first one to have a minimum at grazing incidence and zero altitude. One-half of the locus of discrete minima, as β varies from 0 to $\tan^{-1}(2 \tan c)$, is shown by curve **m**. Line **e** shows the type of curve when β lies within the limits $\tan^{-1}(2 \tan c)$ and $2c$. At the points of intersection of the constant- β curves with the

locus **t** the tangents to the former are perpendicular to the η_1 coördinate axis. For, at these points $\cos \eta_1 = \cot c \tan \frac{1}{2}\beta$ so that the radical symbolized by R_2 on page 403 vanishes.

Consequently, the expression for $\frac{dD}{d\eta_1}$ becomes infinite. The theoretical values of the most important quantities pertaining to figure 6 are collected in Table II.

TABLE II.

β	γ_1	D_1	D_m discrete minima	η_1 for D_m	D_2	η_1 for D_2
a ; 9° 42' 0"	90°	27° 25' 27"	10° 48' 33"	84° 31' 0"	10° 50' 53"	84° 33' 23"
b ; 20 0 0	"	36 7 50	22 2 42	78 15 37	22 23 22	78 37 49
c ; 43 37 58	"	49 6 5	45 22 46	57 47 41	49 6 5	63 24 48
d ; 60 47 39	"	58 24 43	58 24 43	0 0 0	68 54 13	49 0 52
e ; 77 42 0	"	73 51 46			89 4 0	25 46 20

The discussion of the mathematical properties of the curves **m** and **t** has been deferred to this place because the relevancy of these loci might not have been evident before the presentation of figure 6. The parametric equations of the curve **m** are given by formulæ (27) and (28), namely,

$$\begin{aligned}\sin \frac{1}{2}D &= \frac{1}{2} \cot c \sin \beta, \\ \cos \eta_1 &= \frac{1}{2} \cot c \tan \beta.\end{aligned}$$

Elimination of the parameter β leads to

$$\sin \frac{1}{2}D = \frac{\cos \eta_1}{\sqrt{1 + 4 \tan^2 c \cos^2 \eta_1}}. \quad (56)$$

In general, the rationalized equation of the curve **m** may be written

$$\tan^2 \eta_1 - \cot^2 \frac{1}{2}D + 4 \tan^2 c = 0 \quad (57)$$

$$\text{Now, } \frac{dD}{d\eta_1} = \frac{-2 \sin \eta_1}{(1 + 4 \tan^2 c \cos^2 \eta_1) \sqrt{\sin^2 \eta_1 + 4 \tan^2 c \cos^2 \eta_1}}$$

hence, when η_1 equals 0 and $\frac{1}{2}\pi$, $\frac{dD}{d\eta_1}$ becomes 0 and -2 respectively. Thus, the locus of discrete minima of fig. 6 intersects the η_1 and D coördinate axes at angles of $116^\circ 33' 54''$ and 90° , in the order named. Both angles are independent of c .

The equations of the curve **t** are given by formulæ (15) and (40), namely,

$$\begin{aligned}\sin \tfrac{1}{2}D &= \cot c \sin \tfrac{1}{2}\beta, \\ \cos \eta_1 &= \cot c \tan \tfrac{1}{2}\beta.\end{aligned}$$

Elimination of the parameter β leads to

$$\sin \tfrac{1}{2}D = \frac{\cos \eta_1}{\sqrt{1 + \tan^2 c \cos^2 \eta_1}} \quad (58)$$

In general, the rationalized equation of the curve $\mathbf{t}$ may be written

$$\tan^2 \eta_1 - \cot^2 \tfrac{1}{2}D + \tan^2 c = 0 \quad (59)$$

Since
$$\frac{dD}{d\eta_1} = \frac{-2 \sin \eta_1}{(1 + \tan^2 c \cos^2 \eta_1) \sqrt{\sin^2 \eta_1 + \tan^2 c \cos^2 \eta_1}}$$

it follows that the locus $\mathbf{t}$ makes the same angles with the coördinate axes as the locus $\mathbf{m}$. Therefore these curves have common tangents for $\eta_1 = \pm \tfrac{1}{2}\pi$. Although it can be shown analytically that the curves $\mathbf{m}$ and $\mathbf{t}$ have points of inflection for $D=0$, it is easier to obtain this result from inspection of the irrational equations of these loci, (56) and (58). The straight lines $D=0$, $\eta_1 = \tfrac{1}{2}\pi$, and $\eta_1 = -\tfrac{1}{2}\pi$ are axes of symmetry.

Fig. 7 bears the same relation to the general $\gamma_1 \eta_1 D$ surface that figures 2, 3, 4, and 5 bear to the $\lambda \eta_1' D$ surface. As before, $n=1.5$. In order to show the most complicated forms of the curves of constant deviation, and also to have negative values of the azimuth γ_1 , ($\beta < c$), the comparatively small value 20° has been taken for the prism angle β . The values of D along four constant deviation loci, together with certain other theoretical data, are given in Table III.

TABLE III.

D	Intercepts, γ_1 -axis.	Intercepts, η_1 -axis.	Maxima and minima, $\lambda=0$.
$14^\circ \ 31' \ 7''$	$+52^\circ \ 49' \ 12''$ $-18 \ 18 \ 5$	$\pm 48^\circ \ 2' \ 12''$	$\gamma_1 = 23^\circ \ 32' \ 1''$ $\eta_1 = \pm 57 \ 19 \ 7$
$18 \ 22 \ 4.5$	$+64 \ 11 \ 30$ $-25 \ 49 \ 26$	$\pm 60 \ 18 \ 1$	$\gamma_1 = 42 \ 21 \ 4$ $\eta_1 = \pm 72 \ 38 \ 47$
$21 \ 0 \ 0$	$+69 \ 47 \ 28$ $-28 \ 47 \ 28$	$\pm 63 \ 51 \ 3$	$\gamma_1 = 70 \ 16 \ 32$ $\eta_1 = \pm 77 \ 53 \ 12$
$22 \ 13 \ 2$	$+72 \ 1 \ 49$ $-29 \ 48 \ 47$	$\pm 64 \ 46 \ 2$	$\gamma_1 = 87 \ 31 \ 29$ $\eta_1 = \pm 78 \ 37 \ 9$
Boundary	$+90 \ 0 \ 0$ $-33 \ 52 \ 10$	$\pm 65 \ 59 \ 16$	$\gamma_1 = 90 \ 0 \ 0$ $\eta_1 = \pm 78 \ 37 \ 49$

FIG. 7.

ϵ , D_1 , and D_2 have already been given in Table I. The points A and H indicate the intersections of the curve for $D=22^\circ 13' 2''$ with the left boundary locus. The abscissæ of these points are $\gamma_1=55^\circ 48' 52''$ and $\gamma_1=84^\circ 11' 8''$ respectively. The corresponding ordinates, which also pertain to the intersections with the right boundary locus, are $\eta_1=77^\circ 30' 5''$ and $\eta_1=78^\circ 35' 59''$, in the order named. This locus of constant deviation has been plotted beyond the right boundary. The equation of the locus of the algebraic maxima and minima of η_1 , on the curves of constant deviation, has been designated as (44). This locus is marked " $\lambda=0$ " in fig. 7. It crosses the γ_1 -axis at the abscissa $\gamma_1=15^\circ 5' 53''$. When the numerical value of the deviation is given, the coördinates of the points of intersection of the constant deviation locus with the curve $\lambda=0$ can be obtained from equations (30) and (31). As soon as η_1' has been evaluated, η_1 can be found at once by using (4). Then γ_1 may be obtained from (5) since, for $\lambda=0$, $\gamma_1'=\frac{1}{2}\beta$. In general, at the intersections of the left boundary locus with the axis of η_1 ,

$$\frac{d\gamma_1}{d\eta_1} = \pm \sec c \cot \beta \sqrt{\sin (c+\beta) \sin (c-\beta)},$$

which requires $\beta < c$, of course. For the special case of fig. 7 the acute angle obtained from the above derivative equals $64^\circ 38' 8''$. The deviation at these points of intersection may be obtained from (42), and it equals $26^\circ 59' 43''$. At the origin of coördinates of the $\gamma_1 \eta_1$ diagram the deviation is given by (43). For $\beta=20^\circ$ and $n=1.5$ the deviation equals $10^\circ 51' 57''$.

Returning to the loci of constant deviation, and inspecting fig. 7, it is seen that the curves which pertain to the smaller values of D only have algebraic maxima and minima of γ_1 for $\eta_1=0$, that is, in a principal plane. On the contrary, for relatively large deviations the curves have additional maxima and minima of γ_1 . More specifically, each complete locus of large constant deviation has six stationary values of γ_1 , namely, one maximum and one minimum on the γ_1 -axis, two minima for numerically equal and finite ordinates, and two maxima for still greater (arithmetical) values of η_1 . Stated in a slightly different manner, it is evident that a straight line parallel to the axis of ordinates, and lying wholly within one quadrant, may intersect a curve of constant deviation in one real point, or in two separate points, or in two coincident points and one other point, or in three discrete points, etc. The quantitative investigation of these points of intersection is equivalent to the discussion of the roots of the equation obtained by expressing η_1 as an explicit function of γ_1 and D . Rationalization of formula (35) leads to the following cubic in $\cos^2 \eta_1$, namely,

$$a_0 \cos^6 \eta_1 + a_2 \cos^4 \eta_1 + a_4 \cos^2 \eta_1 + a_6 = 0 \quad (60)$$

where

$$\begin{aligned} a_0 &= 16 \sin^4 c \sin^2 \beta \sin^2 \frac{1}{2} D \cos^2 \gamma_1 \cos^2 (\gamma_1 - \beta), \\ a_2 &= 16 \sin^2 c \sin^2 \beta \sin^2 \frac{1}{2} D \cos \gamma_1 [\cos^2 c \sin \beta \sin (\gamma_1 - \beta) \\ &\quad - \sin^2 c \sin^2 \frac{1}{2} D \cos \gamma_1 \cos^2 (\gamma_1 - \beta)] - [\cos^2 c \sin^2 \beta \\ &\quad - 4 \sin^2 c \sin^2 \frac{1}{2} D + 4 \sin^2 c \cos \beta \sin^2 \frac{1}{2} D \sin \gamma_1 \sin (\gamma_1 - \beta)]^2, \\ a_4 &= 8 \sin^2 c \sin^4 \frac{1}{2} D [2 \cos^2 c \sin^2 \beta \sin^2 (\gamma_1 - \beta) - \cos^2 c \sin^2 \beta \\ &\quad + 4 \sin^2 c \sin^2 \frac{1}{2} D - 4 \sin^2 c \cos \beta \sin^2 \frac{1}{2} D \sin \gamma_1 \sin (\gamma_1 - \beta)], \\ a_6 &= -16 \sin^4 c \sin^8 \frac{1}{2} D. \end{aligned}$$

Obviously, equation (60) is altogether unmanageable. For this reason the conditions under which stationary values of γ_1 may arise will not be investigated. Suffice it to say that the constant deviation loci of fig. 7 are quite consistent with the above sextic in $\cos \eta_1$. There are several ways in which the cubic can give two and only two values of η_1 which pertain to the $\gamma_1 \eta_1$ diagram. One way is for the equation to have three real roots in $\cos^2 \eta_1$, two of which are positive and less than unity, while the third root is either negative or else positive but greater than unity. Another way is for equation (60) to degenerate. Thus for $\gamma_1 = \frac{1}{2}\pi$, $a_0 = 0$ and a biquadratic results. In order to test the correctness of the coefficients of the sextic, γ_1 was put equal to $\frac{1}{2}\pi$ and an attempt was made to factor the quartic which was obtained. The factors were found to be

$$\begin{aligned} &(\cos^2 c - 4 \sin^2 c \sin^2 \frac{1}{2} D) \sin^2 \beta \cos^2 \eta_1 \\ &\mp \sin 2c \sin 2\beta \sin^2 \frac{1}{2} D \cos \eta_1 + 4 \sin^2 c \sin^4 \frac{1}{2} D = 0. \end{aligned}$$

The quadratic with the upper sign is the equation which, when solved for $\cos \eta_1$, gives formula (26). [The sextic was also found to be satisfied by $\cos \eta_1 = \cot c \tan \beta$ when $\gamma_1 = 0$ and $\sin \frac{1}{2} D$ was taken from (42).]

Inspection of the constant deviation loci of fig. 7 throws some light on the most general form of intersection which a plane given by $\gamma_1 = \text{constant}$ can have with the $\gamma_1 \eta_1 D$ surface. [A discussion of equation (60), in this connection, is entirely out of the question.] In other words, a generalized constant- β curve of fig. 6, (for which $\gamma_1 = \frac{1}{2}\pi$), can have the following course. The deviation will have a minimum value for $\eta_1 = 0$, then as η_1 increases so also will D increase until a maximum value is reached. After this the deviation will decrease to a minimum and then increase up to, and beyond, the intersection with the boundary locus. The coördinates of some points of the $\eta_1 D$ curve corresponding to $\gamma_1 = 75^\circ$ are given in Table IV. These points are not presented graphically because the deviation is roughly constant so that the wavy form of the curve would not be sufficiently marked on a small diagram.

[The possible bearing of the remarkably straight course of certain plane sections of the $\gamma_1 \eta_1 D$ surface on the problem of diminishing the curvature of spectral lines has not been considered because it is not relevant to this paper.]

TABLE IV.

η_1	D	η_1	D
0° 0' 0''	23° 59' 28''	70° 43' 52''	21° 45' 1''
10 30 0	23 59 58	72 34 35	21 27 13
18 10 19	24 0 38	76 40 0	20 55 1
21 16 38	24 0 48	76 53 0	20 54 57
22 50 40	24 0 50	76 55 0	20 54 59
26 51 18	24 0 39	76 56 16	20 55 0
30 51 57	23 59 54	77 0 0	20 55 7
41 6 50	23 53 29	77 30 5	20 58 19
50 35 3	23 37 3	78 15 37	21 24 43
60 30 0	22 59 43	78 25 27	22 3 59

It may not be superfluous to follow the successive positions of the incident and emergent rays around the boundary loci in the actual case of a prism, that is, without reference to a sphere of projection such as is shown in fig. 1. The point of incidence will be considered fixed, so that the incident ray will generate a surface which consists of portions of a cone and of a plane, the incidence face of the prism. Commencing with grazing incidence in a principal plane, the emergent ray will also lie in the principal section and will have the azimuth $\gamma_2' = \sin^{-1}[n \sin(c - \beta)]$. This azimuth will be positive, zero, or negative, according as the angle of the prism is less than, or equal to, or greater than, the critical angle respectively. Keeping the incident ray in a grazing position and increasing the altitude of incidence, the emergent ray will have an azimuth which decreases algebraically, the altitudes of the incident and emergent rays being always equal. [See page 391.] At length, when the incidence altitude equals $\cos^{-1}[\cot c \tan \frac{1}{2}\beta]$ the emergent ray will begin to graze the second face of the prism. This is the position of simultaneous grazing incidence and emergence. Also, all of the positive half of the right boundary locus has been passed over. To continue farther, the incident ray must now leave the first face of the prism. As the azimuth of incidence decreases, the corresponding altitude must also decrease consistently with the equation

$$\eta_1 = \cos^{-1} \left\{ \frac{\cot c \sin \beta}{\cos \beta + \sin \gamma_1} \right\}$$

While the incident ray is changing its position in the manner just indicated, the emergent ray will persist in a grazing position with ever decreasing altitude. Finally, when the azimuth of incidence attains the value $\sin^{-1} [n \sin (\beta - c)]$ both rays will again lie in the principal plane and the positive half of the left boundary locus will have been generated. This process may be continued for all possible negative altitudes until the entire boundary surface shall have been traced out. However, it is not necessary to consider negative altitudes in detail because a principal section constitutes a plane of symmetry. The preceding results are made concrete by the data in Table V, for which $\beta = 20^\circ$ and $n = 1.5$. Throughout this table the values of γ_2' correspond to grazing incidence, $\gamma_1 = 90^\circ$. The left boundary values can be read off by reversing the ray; that is, by considering γ_2' as -90° throughout and by changing the signs of the angles in the second column. The values thus obtained represent γ_1 when grazing emergence is maintained. Incidentally, the corresponding data for D and E are given.

TABLE V.

$\pm \gamma_1$	γ_2'	D	E
$0^\circ \ 0' \ 0''$	$33^\circ \ 52' \ 10''$	$36^\circ \ 7' \ 50''$	$36^\circ \ 7' \ 50''$
6 0 22	33 43 26	36 4 12	36 16 34
16 37 53	32 43 30	35 39 43	37 16 30
26 0 43	30 56 39	34 57 51	39 3 21
35 52 50	27 53 17	33 50 53	42 6 43
46 39 12	22 29 43	32 6 0	47 30 17
57 0 45	13 43 50	29 45 11	56 16 10
65 59 16	0 0 0	26 59 43	70 0 0
72 38 31	-20 0 0	24 21 26	90 0 0
74 35 49	-30 0 0	23 28 46	100 0 0
76 1 0	-40 0 0	22 49 59	110 0 0
77 2 43	-50 0 0	22 23 22	120 0 0
76 46 27	-60 0 0	22 7 49	130 0 0
78 15 37	-70 0 0	22 2 42	140 0 0
78 32 21	-80 0 0	22 7 49	150 0 0
78 37 49	-90 0 0	22 23 22	160 0 0

The properties of the lines which bound the domains characterized by the conditions

$$\beta = 2c, \cos \frac{3}{2}\beta = \cot c \sin \frac{1}{2}\beta, \text{ and } \tan \beta = 2 \tan c$$

will now be discussed. The most important portions of these lines are shown in fig. 8. Unlike the preceding diagrams, this figure is perfectly general, that is, it does not depend upon special values of the prism angle or of the index of refraction.

In order to plot quantities which are measured in terms of the same unit, the critical angle instead of the index of refraction is taken as ordinate with the prism angle as abscissa. For practical convenience a few arbitrary values of the index of

FIG. 8.

refraction are indicated on the line parallel to the axis of ordinates. As the critical angle increases from 0 to $\frac{1}{2}\pi$ the index of refraction decreases from ∞ to 1. In order to avoid total reflection the condition $\beta < 2c$ must be fulfilled. Hence, for

transmission of light, the point characterizing the prism, (β, c) , must lie within the angle AOP, since the equation of the straight line $\overline{OA}$ is $\beta - 2c = 0$. For points along this line only one ray can be transmitted and the small-circle arcs $C'AC$ and $C'BC$ of fig. 1 are tangent to each other. Again, by hypothesis, the relative index of refraction, n , must exceed unity, in other words, the critical angle must be less than $\frac{1}{2}\pi$. The equation of the straight line $\overline{PH}$ is $2c - \pi = 0$, hence, for transmission of plane waves, the characterizing point must fall inside the right triangle OQP, where Q denotes the point of intersection of the lines $\overline{OA}$ and $\overline{PH}$. The coördinates of Q are $\beta = \pi$ and $c = \frac{1}{2}\pi$.

It has been shown that the necessary and sufficient condition for equality between the deviations at grazing incidence in a principal plane and at simultaneous grazing incidence and emergence is

$$\cos \frac{3}{2}\beta - \cot c \sin \frac{1}{2}\beta = 0. \quad [\text{See (49) and (50).}]$$

A portion of the curve representing this equation is shown as the line OV in fig. 8. [The reason for not plotting more of the curve OV is that the arc OIH alone has physical importance, and by drawing this arc on as large a scale as possible the practical value of the diagram is increased.] Hence, if the characterizing point (β, c) falls on this curve the relation $D_1 = D_2$ will hold. The straight line $\overline{UL}$ has the equation $3\beta - \pi = 0$, and it is seen that so much of the locus OV as lies within the domain of physical reality falls to the left of $\overline{UL}$. This shows very clearly the truth of the theorem that in order to have $D_1 = D_2$ it is necessary for the prism angle to have a value less than $\frac{1}{3}\pi$. The function

$$\cos \frac{3}{2}\beta - \cot c \sin \frac{1}{2}\beta$$

is positive on the left side of the partition OV and negative on the right. Therefore, the domain bounded by OIHPO contains the characterizing points of all prisms having the deviation at grazing incidence in a principal plane greater than the deviation at simultaneous grazing incidence and emergence. On the other hand, the region having the periphery OAQHIO includes all cases where $D_1 < D_2$. Since this area includes all points for which $\beta = \frac{1}{3}\pi$ it follows that the 60° prisms usually employed in spectroscopes always have $D_1 < D_2$.

The course of the curve OV will now be considered more in detail. In the first place

$$\frac{dc}{d\beta} = \frac{1 + 2 \cos \beta - 2 \cos^2 \beta}{2(1 - 2 \cos \beta + 2 \cos^3 \beta)}.$$

Hence, when $\beta = 0$, $\frac{dc}{d\beta} = \frac{1}{2}$, so that the line $\overline{OA}$ is tangent to

the curve OV at the origin. At H, ($\beta = \frac{1}{3}\pi$), $\frac{dc}{d\beta} = 3$ and so the tangent to the curve makes an angle with the positive direction of the axis of abscissas equal to $\tan^{-1} 3 = 71^\circ 33' 54''$. When $\beta = \cos^{-1} \frac{1}{2}(1 - \sqrt{3}) = 111^\circ 28' 15''$, $\frac{dc}{d\beta} = 0$. It will be seen later that this stationary value pertains to a maximum of c . The curve OV passes through the point Q, which was defined as the intersection of the straight lines $\overline{OA}$ and $\overline{PH}$, since the coordinates $\beta = \pi$ and $c = \frac{1}{2}\pi$ satisfy equation (50). At this point $\frac{dc}{d\beta} = -1.5$, and $\tan^{-1}(-1.5) = 123^\circ 41' 24''$. When the acute angle at H, the obtuse angle at Q, and the single stationary value of c between H and Q are taken into account, it becomes evident that the arc OIH is the only portion of the curve OV which lies within the triangle OQP, that is, within the domain of physical significance. Now

$$\frac{d^2c}{d\beta^2} = - \frac{(2 - 2 \cos \beta - \cos^2 \beta - 4 \cos^3 \beta + 2 \cos^4 \beta) \sin \beta}{(1 - 2 \cos \beta + 2 \cos^2 \beta)^2},$$

$$\text{hence for } \cos \beta = -\frac{1}{2}(\sqrt{3} - 1), \quad \frac{d^2c}{d\beta^2} = -\frac{1}{2}(5 + \sqrt{3})\sqrt[4]{108}$$

so that the existence of the maximum mentioned above is established. At this point, c has the impossible value given by $\tan^{-1}[-\frac{1}{6}\sqrt[4]{12}(\sqrt{3} + 1)] = 139^\circ 43' 9''$. The expression for $\frac{d^2c}{d\beta^2}$ vanishes for $\beta = 0$, $\beta = \cos^{-1}(0.5709797)$, and $\beta = \pi$, thereby showing that there may be one point of inflection at the origin, another at $\beta = 55^\circ 10' 53''$ with $c = 74^\circ 48' 8''$ or $n = 1.03624$, and a third at Q. That these points are truly inflectional is easily seen without proving that $\frac{d^3c}{d\beta^3}$ does not vanish. The second point of inflection lies at the center of the circle marked "I" in fig. 8. At this point the locus OV makes an angle of $72^\circ 49' 7''$ with the axis of abscissas.

The equation

$$\tan \beta = 2 \tan c$$

has been shown to be the necessary and sufficient condition that the discrete minima of deviation at grazing incidence or emergence shall coincide with their respective maximum, which occurs in a principal plane. [See (48).] Although the curve representing the above equation has not been plotted in figure 8, nevertheless, its course can be readily followed by the aid of the following data. In the first place

$$\frac{dc}{d\beta} = \frac{2}{1 + 3 \cos^2 \beta},$$

$$\text{and} \quad \frac{d^2c}{d\beta^2} = \frac{6 \sin 2\beta}{(1 + 3 \cos^2 \beta)^2}.$$

The first derivative shows that maxima and minima cannot exist on the locus. At the origin $\frac{dc}{d\beta} = \frac{1}{2}$ and $\frac{d^2c}{d\beta^2} = 0$, hence the

curve has the line $\overline{OA}$ as the inflectional tangent at this point. For $\beta = 60^\circ$, $c = \frac{1}{2} \cos^{-1} \frac{1}{7} = 40^\circ 53' 36''$ or $n = \frac{1}{3} \sqrt{21} = 1.527525$, and $\frac{dc}{d\beta} = \frac{2}{7}$; $\tan^{-1} \frac{2}{7} = 48^\circ 48' 51''$. For $\beta = 90^\circ$,

$c = 90^\circ$ or $n = 1$, and $\frac{dc}{d\beta} = 2$; $\tan^{-1} 2 = 63^\circ 26' 6''$. Also

at this point $\frac{d^2c}{d\beta^2}$ vanishes and inflection takes place. For all

points between the origin and the point $\beta = c = 90^\circ$ the locus is convex towards the axis of abscissas. Thus far the curve in question has properties like the line OV . Besides the origin, these two loci also intersect at the point $\beta = 120^\circ$, $c = \tan^{-1}(-\frac{1}{2}\sqrt{3}) = 139^\circ 6' 24''$, which is to the right of the maxi-

mum on the locus OV . At this point $\frac{dc}{d\beta} = \frac{2}{7}$ and hence the

tangent is parallel to the tangent at the point whose abscissa

is 60° . For $\beta = 180^\circ$, $c = 180^\circ$, $\frac{dc}{d\beta} = \frac{1}{2}$, and $\frac{d^2c}{d\beta^2} = 0$.

Thus, there is an inflectional tangent at the last point which is parallel to the line $\overline{OA}$.

Let the point where the curve whose equation is $\tan \beta = 2 \tan c$ crosses the line $\overline{PH}$, of figure 8, be denoted by R . Then in the domain $OAQRO$ there are no discrete minima at grazing incidence or emergence. In the region $ORHPO$ there are two discrete minima, with one maximum in a principal plane, when $\gamma_1 = 90^\circ$ or $\gamma_2' = -90^\circ$. The last area is subdivided by the curve OV so that a still more detailed classification is possible. Since a segment of the line $\overline{UH}$ lies within the region $ORHIO$ it is clear that 60° prisms can have discrete minima at grazing incidence, etc. However, for this to be the case, the critical angle must exceed $\frac{1}{2} \cos^{-1} \frac{1}{7}$ or the index of refraction must be less than $\frac{1}{3} \sqrt{21}$.

In conclusion, the author desires to thank Mr. Leigh Page for having verified the analytical work in this paper.

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New Haven, Conn., December, 1912.